Nihonkai Math.J.
vol.4 (1993),153-162

BOUNDED LINEAR OPERATORS WITH FINITE CHARACTERISTIC IN A HILBERT SPACE

by Sadoon Ibrahim Othman

ABSTRACT. Here is considered some of the properties of the bounded linear operators T on a Hilbert space H, such that for some integer $k \geq 1$, $\| T^{*}x \|^{k} \leq M \| Tx \|$, for $M > 0$ and all $x \epsilon H, \| x \| = 1$. This category of operators includes among others, the hyponormal operators (and hence normal, quasinormal and subnormal operators) and the M-paranormal operators of the unilateral weighted shift type.

1. INTRODUCTION

This article deals with the bounded linear operators T on a Hilbert space H, satisfying the condition $\| T^{*}x \|^{k} \leq M \| Tx \|$ for some $M > 0$ and all $x \epsilon H$, $\| x \| = 1$, where k is an integer ≥ 1; such operators are termed operators with finite characteristic.

The motivation for the study of such operators is as follows: An operator T on H satisfying the inequality $\| T^{*}x \|^{k} \leq M \| Tx \|$, $x \epsilon H$, $\| x \| = 1$, $M > 0$ (if $M = 1$, T is called hyponormal) has many nice properties. One of the situations where T fails to satisfy this inequality is when there exists a sequence $\{x_n\}$, $\| x_n \| = 1$ such that $\| T^{*}x_n \| \rightarrow 0$ while $\| Tx_n \|$ also tends to 0 but at a faster rate. However, many such operators, though not satisfying this inequality, preserve the essential properties of a hyponormal operator; hence the interest in introducing the operators with finite characteristic.

Clearly, all hyponormal operators (and hence many other well-known operators like normal, quasinormal and subnormal) have finite characteristic. Among other such operators, we find the M-hyponormal operators (due to J.G. Stampfli) and the M-paranormal operators (due to V.I. Istrǎtescu) of unilateral weighted shift.

In this article, we show that the operators with finite characteristic possess most of the well-known properties of hyponormal operators.

2. OPERATORS WITH FINITE CHARACTERISTIC

Let H be a Hilbert space and $B(H)$ be the space of all bounded linear operators $T : H \to H$.

Definition 1. For an operator $T \epsilon B(H)$, define the k-th characteristic of T as $\chi_k(T) = \sup_{\|x\|=1} \frac{\|T^* x\|^k}{\|Tx\|}$.

Remarks:

1) For a linear operator $T \epsilon B(H)$, with $\| T \| \leq 1$, $\chi_{k+1}(T) \leq \chi_k(T)$ for $k \geq 1$.

2) For any number $\alpha \neq 0$, the operator $T_\alpha \epsilon B(H)$ defined as $T_\alpha(u) = \alpha u$ has its k-th characteristic $\chi_k(T_\alpha) = | \alpha |^{k-1}$.

3) If T^* is surjective, then $\chi_k(T)$ is finite. In this case, there exists $c > 0$ such that $\| Tx \| \geq c$ for all $\| x \| = 1$ (Rudin [4], p. 97) and hence $\chi_k(T) \leq \frac{1}{c} \sup_{\|x\|=1}$ $\| T^* x \|^k \leq \frac{1}{c} \| T \|^k$.

Proposition 2. For any $T \epsilon B(H)$, $\chi_k(T) \geq \| T \|^{k-1}$.

Proof. For any $\epsilon > 0$, there exists $x_0 \epsilon H$, $\| x_0 \| = 1$, such that $\| T^* x_0 \| \geq \| T \| - \epsilon$.

Hence, $\chi_k(T) \geq \frac{\|T^* x_0\|^k}{\|Tx_0\|} \geq \frac{(\|T\| - \epsilon)^k}{\|T\|}$; ϵ being arbitrary, the proposition is proved.

Corollary: If T is a hyponormal operator, $\chi_k(T) = \| T \|^{k-1}$.

Example: We construct an operator $T \epsilon B(H)$ for which $\| T \|^{k-1} < \chi_k(T) < \infty$.

Take $H = l_2$ and define for $u = (u_1, u_2, \cdots) \epsilon l_2$, $Tu = (0, u_1, 2u_2, u_3, \cdots)$. Then

$T^*u = (u_2, 2u_3, u_4, \cdots)$, and for $\| x \| = 1$, $\| T^*x \|^2 = 1 + 3x_3^2 - x_1^2$ and $\| Tx \|^2 = 1 + 3x_2^2$.

Since $\| T^*x \| \leq 2$ and $\| Tx \| \geq 1$ for $\| x \| = 1$, we have

$$\begin{aligned} 2^k \geq \chi_k(T) &= \sup_{\|x\|=1} \frac{\| T^*x \|^k}{\| Tx \|} \\ &\geq \frac{\| T^*e_3 \|^k}{\| Te_3 \|} \quad \text{where } e_3 = (0,0,1,0,\cdots) \\ &= 2^k \end{aligned}$$

Hence $\chi_k(T) = 2^k$.

However, $\| T \| = 2$, since $\| Tx \| \leq 2$ for $\| x \| = 1$ and $\| Te_2 \| = 2$.

Proposition 3. With the convention that if Φ is an empty set, then $\inf \Phi = \infty$, we have for any $T \epsilon B(H)$

$$\chi_k(T) = \inf\{M \epsilon \mathbb{R}^+ : \| T^*x \|^k \leq M \| Tx \| \quad \text{for all} \quad \| x \| = 1\}.$$

Proof. Let S be the subset $\{M \epsilon \mathbb{R}^+ : \| T^*x \|^k \leq M \| Tx \|, \ \| x \| = 1\}$

Let $\chi_k(T) = \alpha$ and $\inf S = \beta$.

Since α is ∞ if and only if β is ∞, we assume that α and β are finite.

Now, by the definition of $\chi_k(T)$, $\alpha \epsilon S$ and hence $\beta \leq \alpha$. On the other hand, for any $\epsilon > 0$, $\| T^*x \|^k \leq (\beta + \epsilon) \| Tx \|$ for all $\| x \| = 1$; hence $\alpha \leq \beta + \epsilon$.

Hence the proposition follows.

Corollary: Let $T \epsilon B(H)$ and U be a unitary operator $\epsilon B(H)$ i.e. $U^*U = UU^* = I$. If $S = U^*TU$, then $\chi_k(S) = \chi_k(T)$.

For, $\| Ux \| = 1$ if and only if $\| x \| = 1$; $\| S^*x \| = \| T^*Ux \|$ and $\| Sx \| = \| TUx \|$. Consequently,

$$\chi_k(T) = \inf\{M : \| T^*x \|^k \leq M \| Tx \|, \ \| x \| = 1\}$$

$$
\begin{aligned}
&= \inf\{M : \| T^*Ux \|^k \leq M \| TUx \|, \quad \| Ux \| = 1\} \\
&= \inf\{M : \| S^*x \|^k \leq M \| Sx \|, \quad \| x \| = 1\} \\
&= \chi_k(S)
\end{aligned}
$$

Notation: Let us denote by $F_k(H)$ the set of all bounded linear operators $T \epsilon B(H)$ for which $\chi_k(T)$ is finite.

Remarks:

1) $F_s(H) \subset F_t(H)$ if $s \leq t$. This follows form the fact that since $\| T^*x \|^t \leq \| T \|^{t-s} \| T^*x \|^s$, $\chi_t(T) \leq \| T \|^{t-s} \chi_s(T)$. However, if $T \epsilon F_t(H)$ is surjective, then $T \epsilon F_s(H)$. In this case there exists $\lambda > 0$ such that $\| T^*x \| \geq \lambda$ for all $\| x \| = 1$ and consequently $\chi_s(T) \leq \lambda^{s-t} \chi_t(T)$.

2) $F_s(H) \neq F_t(H)$ if $s \neq t$. To show this, we construct in the following example a unilateral weighted shift operator $T \epsilon F_2(H) \backslash F_1(H)$; the general case when $s < t$ can be dealt with in a similar fashion.

Recall that if H is a Hilbert space with an orthonormal basis $\{e_0, e_1, e_2, \cdots\}$ and if $\alpha_n \epsilon \mathbb{C}$, $\quad n = 0, 1, 2, \cdots$ with $\sup \mid \alpha_n \mid < \beta$, the linear operator T defined on H by $Te_n = \alpha_n e_{n+1}$ is called a unilateral weighted shift with weight sequence $\{\alpha_n\}$ (J. Conway [1], p. 154). We can assume $\alpha_n \neq 0$ for every n.

Proposition 4 Let T be a weighted shift operator with $\{\alpha_n\}$ as the weight sequence. Then $T \epsilon F_2(H)$ if and only if $\beta = \sup_n \frac{|\alpha_{n-1}|^2}{|\alpha_n|}$ is finite; in this case , $\chi_2(T) = \beta$.

Proof: Suppose $T \epsilon F_2(H)$. Then, for $M > \chi_2(T)$

$$\| T^*x \|^2 \leq M \| Tx \|, \quad x \epsilon H, \quad \| x \| = 1$$

Since $T^*e_n = \alpha_{n-1}$ for $n \geq 1$ and $T^*e_0 = 0$, the assumption that $T \epsilon F_2(H)$ would imply $\mid \alpha_{n-1} \mid^2 \leq M \mid \alpha_n \mid$; consequently, $\beta = \sup_n \frac{|\alpha_{n-1}|^2}{|\alpha_n|} \leq \chi_2(T)$.

Conversely, suppose β is finite. Then for $\epsilon > 0$, $\quad \mid \alpha_{n-1} \mid^2 \leq (\beta + \epsilon) \mid \alpha_n \mid$.

Further, if $x = \sum \beta_n e_n$ with $\| x \|^2 = \sum \beta_n^2 = 1$,

$$\begin{aligned} \| Tx \| &= \| \sum \beta_n \alpha_n e_{n+1} \| = (\sum | \beta_n |^2 | \alpha_n |^2)^{\frac{1}{2}}, \quad \text{and} \\ \| T^* x \|^2 &= \sum | \beta_n |^2 | \alpha_{n-1} |^2 \\ &= \sum (| \beta_n |)(| \beta_n || \alpha_{n-1} |^2) \\ &\leq (\sum | \beta_n |^2)^{\frac{1}{2}} (\sum | \beta_n |^2 | \alpha_{n-1} |^4)^{\frac{1}{2}} \\ &\leq 1 \times (\sum | \beta_n |^2 (\beta + \epsilon)^2 | \alpha_n |^2)^{\frac{1}{2}} \\ &= (\beta + \epsilon) \| Tx \| \end{aligned}$$

Hence, $T \epsilon F_2(H)$ and $\chi_2(T) \leq \beta$. This completes the proof of the proposition.

Example of an operator $T \epsilon F_2(H) \backslash F_1(H)$. Consider the unilateral weighted shift operator with the weighted sequence $\alpha_n = 2^{1-2^n}$, $n \geq 0$. Then, $| \alpha_{n-1} |^2 = 2 | \alpha_n |$; hence $T \epsilon F_2(H)$ with $\chi_2(T) = 2$ (a consequence of the above proposition).

But $T \notin F_1(H)$; for, otherwise, $| \alpha_{n-1} | \leq M | \alpha_n |$ for some M which implies that $2^{2^{n-1}} \leq M$ for all n, a contradiction.

Proposition 5: Let $S \epsilon B(H)$. Then $S \epsilon F_1(H)$ if and only if $SS^* \leq \lambda S^* S$ for some $\lambda > 0$.

Proof.

$$\begin{aligned} \| S^* x \|^2 &= < SS^* x, x > \\ &\leq \lambda < S^* S x, x > \\ &= \lambda \| Sx \|^2 \end{aligned}$$

Hence, $\chi_1(S) \leq \sqrt{\lambda}$ and $S \epsilon F_1(H)$.

Proposition 6. Let $S \epsilon F_1(H)$. If S^* commutes with any $T \epsilon F_k(H)$, then both ST and $TS \epsilon F_k(H)$; in fact, in this case

$$\max(\chi_k(ST), \chi_k(TS)) \leq \chi_k(T) \chi_k(S) \max(\sqrt{\lambda}, \sqrt{\lambda^k}).$$

Proof. Since $S \epsilon F_1(H)$, $SS^* \leq \lambda S^* S$ for some $\lambda > 0$. Let $x \epsilon H$, $\| x \| = 1$ and

$M = \chi_k(T) + \epsilon$, ϵ arbitrary,

1)

$$\begin{aligned}
\| (ST)^* x \|^k &= \| T^*(S^* x) \|^k \\
&\leq M \| T(S^* x) \| \| S^* x \|^{k-1} \quad \text{since} \quad T \epsilon F_k(H) \\
&= M \| S^*(Tx) \| \| S^* x \|^{k-1} \quad \text{since} \quad S^*T = TS^* \quad \text{by hypothesis} \\
&\leq M\sqrt{\lambda} \| S(Tx) \| \| S \|^{k-1} \\
&\leq M\sqrt{\lambda} \| S(Tx) \| \chi_k(S) \quad \text{by Proposition 2.}
\end{aligned}$$

This implies that $\chi_k(ST) \leq \sqrt{\lambda}\chi_k(T)\chi_k(S)$

2) Since $S^*T = TS^*$ implies that $T^*S = ST^*$,

$$\begin{aligned}
\| (TS)^* x \|^k = \| S^*(T^* x) \|^k &\leq \sqrt{\lambda^k} \| S(T^* x) \|^k \\
&= \sqrt{\lambda^k} \| T^*(Sx) \|^k \\
&\leq \sqrt{\lambda^k} M \| T(Sx) \| \| Sx \|^{k-1} \\
&\leq \sqrt{\lambda^k} M \| S \|^{k-1} \| TSx \|
\end{aligned}$$

This implies that $\chi_k(TS) \leq \sqrt{\lambda^k}\chi_k(T)\chi_k(S)$. Hence the proposition follows.

3. THE CLASS OF OPERATORS $F_k(H)$

Let us denote $F(H) = U_{k=1}^{\infty} F_k(H)$. That $F(H) \neq B(H)$ can be seen from the following example:

For

$$u = (u_1, u_2, \cdots\cdots) \epsilon l_2 = H,$$

let

$$Tu = (u_2, u_3, \cdots\cdots).$$

Then $T^* u = (0, u_1, u_2, \cdots\cdots)$.

This operator $T\epsilon B(H)\backslash F(H)$ since $\| Te_1 \|= 0$ and $\| T^*e_1 \|= 1$.

Notation: For $\alpha > 0$, denote $S_k^\alpha = S_k^\alpha(H) = \{T : \chi_k(T) \leq \alpha\}$. Remark that if $\alpha < \beta$, then $S_k^\alpha \overset{\subset}{\neq} S_k^\beta$. For clearly, $S_k^\alpha \subset S_k^\beta$. To verify the strict inclusion, consider the bounded linear operator $T_r(u) = ru$ where $\alpha < r^{k-1} < \beta$. Then $\chi_k(T_r) = r^{k-1}$ so that $T_r\epsilon S_k^\beta\backslash S_k^\alpha$.

Proposition 7: $T\epsilon S_k^\alpha$ if and only if for any $M > \alpha$, any real λ and any $u\epsilon H$,

$$\lambda^2 \| u \|^{k-1} -2\lambda\sqrt{\| T^*u \|^k} + M \| Tu \|\geq 0.$$

Proof: Let $T\epsilon S_k^\alpha$ i.e. $\| T^*u \|^k -M \| Tu \|\| u \|^{k-1}\leq 0$ for $M > \alpha$. This means that the quadratic expression in $\lambda : \lambda^2 \| u \|^{k-1} -2\lambda\sqrt{\| T^*u \|^k} + M \| Tu \|\geq 0$, by considering its discriminant.

Conversely, if the given condition is satisfied, take $\lambda = \frac{\sqrt{\|T^*u\|^k}}{\|u\|^{k-1}}$ $(u \neq 0)$, which leads to the inequality $\| T^*u \|^k\leq M \| Tu \|\| u \|^{k-1}$ for all $u\epsilon H$. Hence $T\epsilon S_k^\alpha(H)$.

Proposition 8. S_k^α is a closed subset of $B(H)$ with norm topology. Consequently, $F_k(H)$ is a F_σ- set in $B(H)$ with norm topology.

Proof. Let $T\epsilon\overline{S}_k^\alpha(H)$, the closure of S_k^α in the norm topology of $B(H)$.

Let $T_n\epsilon S_k^\alpha$ be a sequence such that $\| T_n - T \|\rightarrow 0$.

Then $\| T_n^* - T^* \|=\| T_n - T \|\rightarrow 0$ and for any $u\epsilon H$,

$$\| T_n^*u - T^*u \|\leq\| T_n^* - T^* \|\| u \|\rightarrow 0.$$

Hence, $|\| T_n^*u \| - \| T^*u \||\leq\| T_n^*u - T^*u \|\rightarrow 0$. Also $\| T_nu \|\rightarrow\| Tu \|$.

Since $T_n\epsilon S_k^\alpha$, we have for any $M > \alpha$

$$\| T_n^*x \|^k\leq M \| T_nx \|$$

Hence, taking limits, $\| T^*x \|^k\leq M \| Tx \|$, i.e. $T\epsilon S_k^\alpha(H)$.

Consequently, S_k^α is closed and $F_k = U_{n=1}^{\infty} S_k^n$. Hence the proposition.

Corollary: Let T_n be a sequence converging to T in the norm topology of $B(H)$. Suppose, for some $k \geq 1$, $\lim_{n\to\infty} \sup \chi_k(T_n)$ is finite. Then $T \epsilon F_k(H)$.

If $\chi_k(T_n) \leq \alpha$ for all n, then $T_n \epsilon S_k^\alpha$ which is a closed subset of $B(H)$. Hence $T \epsilon S_k^\alpha \subset F_k(H)$.

Recall that $T \epsilon B(H)$ is said to be a partial isometry (section 98, Halmos [2]) if $T : N(T)^\perp \to R(T)$ is such that $\| Tx \| = \| x \|$ for every $x \epsilon N(T)^\perp$. A bounded linear operator T is a partial isometry if and only if $T = TT^*T$.

Recall also that $T \epsilon B(H)$ is said to be quasinormal if T commutes with T^*T (Section 108, Halmos [2]). Since every quasinormal operator is hyponormal (Section 160, Halmos [2]). Following Proposition 5, every quasinormal operator has finite characteristic. In the converse direction, we have

Proposition 9: Let $T \epsilon F(H) = U_{k=1}^{\infty} F_k(H)$ be a partial isometry . Then T is quasinormal.

Proof. Since T is a partial isometry, $T^*T = I$ on $N(T)^\perp$. Now if $T \epsilon F_k(H)$, i.e. $\| T^*u \|^k \leq M \| Tu \| \| u \|^{k-1}$,

$$N(T) \subset N(T^*) = R(T)^\perp.$$

Hence, $R(T) \subset R(T)^{\perp\perp} \subset N(T)^\perp$.

Consequently, for any $u \epsilon H$, $Tu \epsilon R(T) \subset N(T)^\perp$ which implies that $(T^*T)Tu = Tu$.

But, T being a partial isometry $T = TT^*T$.

Thus $(T^*T)T = T(T^*T)$ i.e. T is quasinormal.

Proposition 10: Let $T \epsilon F(H)$. Suppose T^n is a compact operator for some $n \geq 1$. Then T itself is a compact operator.

Proof: The argument is familiar; if $n > 1$, we show that the hypothesis implies that T^{n-1} is compact which is sufficient to prove the proposition.

Let $T\epsilon F_k(H)$; then, $\| T^*u \|^k \leq M \| Tu \| \| u \|^{k-1}$, for $u\epsilon H$.

Then, $\| T^*T^{n-1}u \|^k \leq M \| T^n u \| \| T^{n-1}u \|^{k-1}$.

Since T^n is a compact operator, T^*T^{n-1} is compact. Hence, $(T^{n-1})^*T^{n-1} = (T^*)^{n-2}(T^*T^{n-1})$ is compact, which implies that T^{n-1} is compact.

Hence the proposition.

V.I. Istrătescu [3] has defined an operator $T\epsilon B(H)$ as <u>M-paranormal</u> if $\| Tx \|^2 \leq M \| T^2x \|$, for all $\| x \| = 1$. Let us denote by $P_M(H)$ the family of all M-paranormal operators in $B(H)$. Let $P(H) = U_{M>0}P_M(H)$.

Proposition 11: $F_1(H) \subset P(H)$ i.e. every $T\epsilon F_1(H)$ is M-paranormal.

Proof. Since $T\epsilon F_1(H)$, for some $M > 0$.

$$\| T^*x \| \leq M \| Tx \|, \quad \text{for} \quad \| x \| = 1.$$

Hence

$$\begin{aligned} \| Tx \|^2 &= < T^*Tx, x > \\ &\leq \| T^*(Tx) \| \\ &\leq M \| T^2x \| . \end{aligned}$$

Hence T is M-paranormal.

Corollary: If $T\epsilon F(H) = U_{k=1}^{\infty}F_k(H)$ is surjective, then $T\epsilon P(H)$.

Since T is surjective, we have $T\epsilon F_1(H)$ from Remarks of Proposition 3.

Proposition 12. Suppose $T\epsilon P(H)$ is a unilateral weighted shift operator. Then $T\epsilon F_2(H)$.

Proof: Suppose $\{\alpha_n\}$ is the weight sequence corresponding to T. Note that $\{| \alpha_n |\}$ is a bounded sequence by definition.

Then Proposition 4 states that $T \epsilon F_2(H)$ if and only if

$$| \alpha_{n-1} |^2 \leq \beta | \alpha_n | \quad \text{for some} \quad \beta > 0.$$

In the same way, $T \epsilon P(H)$ if and only if

$$| \alpha_n | \leq M | \alpha_{n+1} | \quad \text{for some} \quad M > 0.$$

Now suppose $T \epsilon P(H)$ is a unilateral weighted shift operator with $\{\alpha_n\}$ as its associated weight sequence. Then we have $| \alpha_n | \leq c$ for all n and $| \alpha_n | \leq M | \alpha_{n+1} |$.

Consequently, $\sup_n \frac{|\alpha_{n-1}|^2}{|\alpha_n|} \leq cM$. Hence $T \epsilon F_2(H)$

ACKNOWLEDGMENT

The author thanks the referee for his very useful comments.

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Department of Mathematics, King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia.

Received Mar. 22,1993