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SOME NEW CRITERIA FOR P-VALENT MEROMORPHIC STARLIKE FUNCTIONS

Nak Eun Cho

Abstract

Let $\sum_{n,p}(\alpha)$ be the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty}a_kz^k \quad (a_{-p}\neq 0, p\in N=\{1,2,...\})$$

which are regular in the punctured disk $E=\{z:0<|z|<1\}$ and satisfying

$$Re\left\{\frac{z(D^nf(z))'}{D^nf(z)}\right\}<-p\frac{n+\alpha}{n+1} \quad (n\in N_0=\{0,1,2,...\},|z|<1,0\leq\alpha<1),$$

where

$$D^nf(z)=\frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^na_{m-1}z^{m-1}.$$

It is proved that $\sum_{n+1,p}(\alpha)\subset\sum_{n,p}(\alpha)$. Since $\sum_{0,p}(\alpha)$ is the class of p-valent meromorphic starlike functions of order α, all functions in $\sum_{n,p}(\alpha)$ are p-valent starlike. Futher property preserving integrals are considered.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

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$$(1.1) \qquad f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in N = \{1, 2, ...\})$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Define

$$(1.2) \qquad D^0 f(z) = f(z),$$

$$(1.3) \qquad D^1 f(z) = \frac{a_{-p}}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + ...$$
$$= \frac{(z^{p+1} f(z))'}{z^p},$$

$$(1.4) \qquad D^2 f(z) = D(D^1 f(z)),$$

and for $n = 1, 2, ...,$

$$(1.5) \qquad D^n f(z) = D(D^{n-1} f(z)$$
$$= \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}$$
$$= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}.$$

In this paper, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$(1.6) \qquad Re\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\} < -p\frac{n+\alpha}{n+1}, \quad (z \in U = \{z : |z| < 1\}),$$

for some α $(0 \leq \alpha < 1)$ and $n \in N_0 = \{0, 1, 2, ...\}$, is p-valent meromorphic starlike in E. More precisely, it is proved that, for the classes $\sum_{n,p}(\alpha)$ of functions in $\sum_p$ satisfying (1.6), $\sum_{n+1,p}(\alpha) \subset \sum_{n,p}(\alpha)$ holds. Since $\sum_{0,p}(\alpha)$ equals the class of

p-valent meromorphic starlike functions of order α [4], the starlikeness of members of $\sum_{n,p}(\alpha)$ is a consequence of (1.7). Further for $c > 0$, let

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt, \tag{1.7}$$

it is shown that $F(z) \in \sum_{n,p}(\alpha)$ whenever $f(z) \in \sum_{n,p}(\alpha)$. Some known results of Bajpai [1], Goel and Sohi [2] are extended.

2. Properties of the class $\sum_{n,p}(\alpha)$

In proving our main results, we shall need the following lemma due to Jack [3].

Lemma. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.*

Theorem 1. $\sum_{n+1,p}(\alpha) \subset \sum_{n,p}(\alpha)$ *for each integer $n \in N_0$.*

Proof. Let $f(z) \in \sum_{n+1,p}(\alpha)$. Then

$$Re\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} < -p\frac{n+1+\alpha}{n+2}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$Re\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\} < -p\frac{n+\alpha}{n+1}. \tag{2.2}$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$\frac{z(D^n f(z))'}{D^n f(z)} = -p\Big[\frac{n+\alpha}{n+1} + \frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\Big]. \tag{2.3}$$

Clearly $w(z)$ is regular and $w(0) = 0$. Using the identity

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z), \tag{2.4}$$

the equation (2.3) may be written as

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{(n+1) + (n+1+2p(1-\alpha))w(z)}{(n+1)(1+w(z))}. \tag{2.5}$$

Differentiating (2.5) logarithmically, we obtain

$$\begin{aligned} \frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} &= -p\Big[\frac{n+\alpha}{n+1} + \frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\Big] \\ &\quad + \frac{2p(1-\alpha)zw'(z)}{(1+w(z))(n+1+(n+1+2p(1-\alpha))w(z))}. \end{aligned} \tag{2.6}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z_0) = k w(z_0) \tag{2.7}$$

where $|w(z_0)| = 1$ and $k \geq 1$. From (2.6) and (2.7), we obtain

$$\begin{aligned} \frac{z_0(D^{n+1} f(z_0))'}{D^{n+1} f(z_0)} &= -p\Big[\frac{n+\alpha}{n+1} + \frac{(1-\alpha)(1-w(z_0))}{(n+1)(1+w(z_0))}\Big] \\ &\quad + \frac{2p(1-\alpha)kw(z_0)}{(1+w(z_0))(n+1+(n+1+2p(1-\alpha))w(z_0))}. \end{aligned} \tag{2.8}$$

Thus

$$(2.9) \qquad Re\left\{\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}f(z_0)}\right\} \geq -p\frac{n+\alpha}{n+1}+\frac{p(1-\alpha)}{2(n+1+p(1-\alpha))} > -p\frac{n+\alpha}{n+1},$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in \sum_{n,p}(\alpha)$.

Theorem 2. *Let $f(z) \in \sum_p$ satisfy the condition*

$$(2.10) \qquad Re\left\{\frac{(D^n f(z))'}{D^n f(z)}\right\} < -p\frac{n+\alpha}{n+1}+\frac{p(1-\alpha)}{2(c(n+1)+p(1-\alpha))} \quad (z \in U)$$

for a given $n \in N_0$ and $c > 0$. Then

$$(2.11) \qquad F(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt$$

belongs to $\sum_{n,p}(\alpha)$.

Proof. Let $f(z) \in \sum_{n,p}(\alpha)$. Define $w(z)$ in U by

$$(2.12) \qquad \frac{(D^n f(z))'}{D^n f(z)} = -p\left[\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\right].$$

Then $w(z)$ is regular and $w(0) = 0$. Using the identity

$$(2.13) \qquad z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z),$$

after simple computation, the equation (2.12) may be written as

$$(2.14) \qquad \frac{z(D^n f(z))'}{D^n f(z)} = -p\left[\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\right] + \frac{2p(1-\alpha)zw'(z)}{(1+w(z))(c(n+1)+(c(n+1)+2p(1-\alpha))w(z))}.$$

We claim that $|w(z)| < 1$ in U. The remaining part of the proof is similar to that of Theorem 1.

Remarks. (1). A result of Bajpai [1] turns out to be a particular case of the above Theorem 2 when $p = 1, a_{-1} = 1, n = 0, \alpha = 0$ and $c = 1$.
(2). For $p = 1, a_{-1} = 1, n = 0$, and $\alpha = 0$, the above Theorem 2 extends a result of Goel and Sohi [2].

Theorem 3. *Let $f(z) \in \sum_{n,p}(\alpha)$ if and only if*

$$F(z) = \frac{1}{z^{1+p}} \int_0^z t^p f(t)dt \tag{2.15}$$

belongs to $\sum_{n+1,p}(\alpha)$.

Proof. From the definition of $F(z)$, we have

$$D^n(zF'(z) + (p+1)D^n F(z) = D^n f(z). \tag{2.16}$$

That is,

$$z(D^n F(z)' + (p+1)D^n F(z) = D^n f(z). \tag{2.17}$$

By using the identity (2.4), equation (2.17) reduces to $D^n f(z) = D^{n+1} F(z)$. Hence

$$\frac{z(D^n f(z))'}{D^n f(z)} = \frac{z(D^{n+1} F(z))'}{D^{n+1} F(z)}. \tag{2.18}$$

and the result follows.

Theorem 4. *Let $F(z) \in \sum_{n,p}(\alpha)$ and let $f(z)$ be defined as* (2.1). *Then $f(z) \in \sum_{n,p}(\alpha)$ in $|z| < R_c$, where*

(2.19)

$$R_c = \frac{-(n+1+p(1-\alpha)) + \sqrt{(n+1+p(1-\alpha))^2 + c(n+1)(c(n+1)+2p(1-\alpha))}}{c(n+1)+2p(1-\alpha)}.$$

Proof. Since $F(z) \in \sum_{n,p}(\alpha)$, we can write

$$\frac{z(D^n f(z))'}{D^n f(z)} = -p\left(\frac{n+\alpha}{n+1} + \left(\frac{1-\alpha}{n+1}\right)u(z)\right), \tag{2.20}$$

where $u(z) \in P$, the class of functions with positive real part in U and normalized by $u(0) = 1$. Using the equation (2.13) and differentiating (2.20), we obtain

$$-\frac{\frac{z(D^n f(z))'}{D^n f(z)} + p\left(\frac{n+\alpha}{n+1}\right)}{\frac{p(1-\alpha)}{n+1}} = u(z) + \frac{zu'(z)}{(c+p) - p\left(\frac{n+\alpha}{n+1} + \frac{1-\alpha}{n+1}u(z)\right)}. \tag{2.21}$$

Using the well known estimates, $\frac{|zu'(z)|}{Reu(z)} \leq \frac{2r}{1-r^2}$ $(|z| = r)$ and $Reu(z) \leq \frac{1+r}{1-r}$ $(|z| = r)$, the equation (2.21) yields

(2.22)

$$Re\left\{-\frac{\frac{z(D^n f(z))'}{D^n f(z)} + p\left(\frac{n+\alpha}{n+1}\right)}{\frac{p(1-\alpha)}{n+1}}\right\} \geq Reu(z)\left(1 - \frac{2r}{(1-r^2)(c+p) - p\left(\frac{n+\alpha}{n+1} + \frac{1-\alpha}{n+1}u(z)\right)}\right).$$

Now the right hand side of (2.22) is positive provided $r < R_c$. Hence $f(z) \in \sum_{n,p}(\alpha)$ for $|z| < R_c$.

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Nak Eun Cho
Department of Applied Mathematics
College of Natural Sciences
National Fisheries University of Pusan
Pusan 608-737
Korea

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