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Einstein H-umbilical submanifolds with parallel mean curvatures in complex space forms

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Abstract

In this paper we determine H-umbilical Einstein submanifolds with parallel mean curvatures in complex space forms with non-negative holomorphic sectional curvatures.

1 Introduction

In Riemannian Geometry, Einstein manifolds are very important subject. When we focus our attention to submanifolds in complex space forms, there are many interesting results (cf. [1]). There are two important classes of submanifolds of a complex space form. One is the class of holomorphic submanifolds and another is the class of totally real submanifolds. A submanifold in a complex space form is said to be totally real if the complex structure of the ambient space carries each tangent vector to a normal vector. A totally real submanifold is called a Lagrangian submanifold if its real dimension is equal to the complex dimension of the ambient space. The classification of Lagrangian Einstein submanifolds of a complex space form is still open. We know the fact that a non-flat complex space form of complex dimension ≥ 2 admits no totally umbilical Lagrangian submanifolds except the totally geodesic ones. So, B. Y. Chen [3] introduced the notion of H-umbilical submanifolds which are the simplest Lagrangian submanifolds next to the totally geodesic ones in a complex space form (for the definition see §2).

In this paper we investigate H-umbilical Einstein submanifolds of complex space forms with non-negative holomorphic sectional curvatures and give the following theorem:

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Theorem 1.1 *Let M^n be a complete n (≥ 3) dimensional Einstein H-umbilical submanifold with parallel mean curvature in an n-dimensional complex space form $\widetilde{M}_n(\tilde{c})$ with constant holomorphic sectional curvature $\tilde{c} \geq 0$. Then M^n is congruent to a totally geodesic Lagrangian submanifold of $\widetilde{M}_n(\tilde{c})$ or $S^1(\frac{1}{\sqrt{\lambda}}) \times \mathbb{R}^{n-1}$ in $\mathbb{C}^n$, where we denote the radius of sphere in the parentheses (for the definition of λ see §2).*

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2 Preliminaries

In this section we explain preliminary results concerning Riemannian submanifolds and give some definitions.

Let $\widetilde{M}_n(\tilde{c})$ be an n-dimensional complex space form with constant holomorphic sectional curvature $\tilde{c}$. Let $\langle \cdot, \cdot \rangle$ and J be its Riemannian metric and complex structure, respectively. Then, the curvature tensor $\overline{R}$ of $\widetilde{M}_n(\tilde{c})$ is obtained by

$$\begin{aligned} \overline{R}(X,Y)Z = \ & \tfrac{\tilde{c}}{4}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY \\ & -2\langle JX,Y\rangle JZ\}, \quad X,\ Y,\ Z \in T\widetilde{M}, \end{aligned} \tag{2.1}$$

where $T\widetilde{M}$ denotes the tangent bundle of $\widetilde{M}$.

Let M^n be a connected n-dimensional real submanifold of $\widetilde{M}_n(\tilde{c})$. Then, M^n is said to be a Lagrangian submanifold if the complex structure J of $\widetilde{M}_n(\tilde{c})$ carries each tangent space of M^n into its corresponding normal space.

For a Lagrangian submanifold M^n of $\widetilde{M}_n(\tilde{c})$, the Gauss and Ricci equations become

$$\begin{aligned} \langle R(X,Y)Z, W\rangle = \ & \tfrac{\tilde{c}}{4}(\langle Y,Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle) \\ & +\langle\sigma(Y, Z), \sigma(X,W)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle, \\ & X, Y, Z, W \in TM, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \langle R^{\perp}(X,Y)JZ, JW\rangle = \ & \langle [A_{JZ}, A_{JW}]X, Y\rangle + \tfrac{\tilde{c}}{4}(\langle Y, Z\rangle\langle X, W\rangle \\ & -\langle X, Z\rangle\langle Y, W\rangle), \\ & X, Y, Z, W \in TM, \end{aligned} \tag{2.3}$$

where σ and A_ν denote the second fundamental form of M^n and the shape operator in the direction ν, respectively, and $R^{\perp}$ is the curvature tensor of the normal bundle $T^{\perp}M^n$ of M^n.

The Gauss and Weingarten formulas are the following:

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \; X, \, Y \in TM^n,$$

$$\overline{\nabla}_X \nu = -A_\nu X + \nabla^\perp_X \nu, \; X \in TM^n, \; \nu \in T^\perp M^n,$$

where $\overline{\nabla}$, ∇ and $\nabla^\perp$ denote the Levi Civita connection of $\widetilde{M_n}(\tilde{c})$, that of M^n and the normal connection of $T^\perp M^n$.

Lagrangian submanifold M^n of $\widetilde{M_n}(\tilde{c})$ is called H-umbilical if the second fundamental form σ of M^n takes the following form for some functions λ and μ with respect to some local orthnormal frame field $e_1, \ldots, e_n$ on M^n:

$$\begin{array}{ll} (2.4) \quad & \sigma(e_1, e_1) = \lambda J e_1, \; \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1, \\ & \sigma(e_1, e_j) = \mu J e_j, \; \sigma(e_j, e_k) = 0, \; j \neq k, \; j, \, k = 2, \ldots, n. \end{array}$$

The mean curvature vector $\mathfrak{H}$ of M^n is defined by

$$\mathfrak{H} = \sum_{i=1}^{n} \sigma(e_i, e_i), \tag{2.5}$$

where $e_1, \ldots, e_n$ is a orthonormal frame field of M^n.

M^n is said to be of parallel mean curvature if $\nabla^\perp \mathfrak{H} = 0$ is satisfied. And M^n is said to be of constant mean curvature if $\langle \mathfrak{H}, \mathfrak{H} \rangle$ is constant on M^n. If M^n has the parallel mean curvature, then $\langle \mathfrak{H}, \mathfrak{H} \rangle$ is always constant.

According to (2.2), the Ricci tensor S of M^n is given by

$$\begin{array}{rl} S(X, Y) = & \frac{\tilde{c}}{4}(n-1)\langle X, Y \rangle + \langle \sigma(X, Y), \mathfrak{H} \rangle \\ & - \sum_{i=1}^{n} \langle \sigma(X, e_i), \sigma(Y, e_i) \rangle, \; X, \, Y \in TM^n. \end{array} \tag{2.6}$$

For the scalar curvature ρ of M^n, using (2.5) and (2.6), we obtain

$$\rho = \frac{\tilde{c}}{4} n(n-1) + \langle \mathfrak{H}, \mathfrak{H} \rangle - \|\sigma\|^2, \tag{2.7}$$

where $\|\sigma\|^2$ denotes the square of the length of σ.

3 Lemmas

In this section we present two lemmas to prove our main theorem.

We explain the equation of the Laplacian $\triangle \|\sigma\|^2$ of the function $\|\sigma\|^2$ given by J. Simons [8]. Let $E_1, \ldots, E_n$ be a local orthonormal frame field of M^n around a point $p \in M^n$ which satisfy $E_i(p) = e_i$ and $\nabla E_i(p) = 0$ ($i = 1, \ldots, n$). Then, we have

Lemma 3.1 ([8], [4]) *Let $\iota : M^n \to \overline{M}^{n+k}$ be an isometric immersion of an n-dimensional Riemannian manifold into an $(n+k)$-dimensional Riemannian manifold $\overline{M}^{n+k}$. Then, the following equation holds:*

$$(3.1)\qquad \begin{aligned} \tfrac{1}{2}\triangle\|\sigma\|^2 = \; & \|\nabla'\sigma\|^2 + \textstyle\sum_{i,j,k}\langle\overline{\nabla}_{e_j}((\nabla'_{E_k}\sigma)(E_i,E_i)),\, \sigma(e_j,e_k)\rangle \\ & + \textstyle\sum_{i,j,k}\langle\overline{\nabla}_{e_i}(\overline{R}(E_j,E_k)E_i)^N,\, \sigma(e_j,e_k)\rangle \\ & + \textstyle\sum_{i,j,k}\langle R^{\perp}(e_i,e_j)\sigma(e_i,e_k),\, \sigma(e_j,e_k)\rangle \\ & - \textstyle\sum_{i,j,k}\langle\sigma(R(e_i,e_j)e_i,e_k),\, \sigma(e_j,e_k)\rangle \\ & - \textstyle\sum_{i,j,k}\langle\sigma(e_i,R(e_i,e_j)e_k),\, \sigma(e_j,e_k)\rangle \\ & + \textstyle\sum_{i,j,k}\langle\overline{\nabla}_{e_i}(\overline{R}(E_i,E_j)E_k),\, \sigma(e_j.e_k),\rangle \end{aligned}$$

where $\langle\cdot,\cdot\rangle$, R and $\overline{R}$ denote the metric tensor of $\overline{M}$, the Riemannian curvature tensor of M and the Riemannan curvature tensor of $\overline{M}$, respectively, and $(\ )^N$ denotes the normal component of a vector.

Using Lemma 3.1, (2.1), (2.2) and (2.3), we get the following:

Lemma 3.2 *Let M^n be a Lagrangian submanifold in a complex space form $\widetilde{M}_n(\tilde{c})$. Then, we have the following equation:*

$$(3.2)\qquad \begin{aligned} \tfrac{1}{2}\triangle\|\sigma\|^2 = \; & \|\nabla'\sigma\|^2 + \tfrac{\tilde{c}}{4}(n+1)\|\sigma\|^2 - \tfrac{\tilde{c}}{2}\langle\mathfrak{H},\mathfrak{H}\rangle \\ & + \textstyle\sum_{i,j,k}\langle\overline{\nabla}_{e_j}((\nabla'_{E_k}\sigma)(E_i,E_i)),\, \sigma(e_j.e_k)\rangle \\ & + \textstyle\sum_{\alpha,\beta=1}^{n} Tr\,(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta=1}^{n}(Tr\,A_\alpha A_\beta)^2 \\ & + \textstyle\sum_{j,k}\langle A_{\mathfrak{H}}e_j,\, A_{\sigma(e_j,e_k)}e_k\rangle, \end{aligned}$$

where A_α is the shape operator of M^n in the direction Je_α.

4 Proof of Theorem 1.1

Let $e_1,\ldots,e_n$ be an orthonormal basis of T_pM which satisfies (2.4). Using (2.4) and (2.6), we have the following for the Ricci tensor S of M:

$$(4.1)\qquad \begin{aligned} & S(e_1,e_1) = \tfrac{n-1}{4}\tilde{c} + (n-1)\mu(\lambda-\mu), \\ & S(e_j,e_j) = \tfrac{n-1}{4}\tilde{c} + \mu(\lambda+(n-3)\mu) \quad (j \geq 2), \\ & S(e_i,e_k) = 0 \quad (i \neq k). \end{aligned}$$

Since M^n is Einstein and $n \geq 3$, using (4.1), we are led to

$$(4.2)\qquad \mu(\lambda - 2\mu) = 0.$$

Because of the Ricci curvature of Einstein manifold is constant, we have $\mu(\lambda-\mu) = constant$. Using this fact and (4.2), we deduce that $\mu = constant$. So, either $\mu \equiv 0$ or $\lambda = 2\mu = constant \neq 0$ is satisfied on M.

We discuss dividing into the following two cases:
Case 1 $\mu \equiv 0$; Case 2 $\lambda = 2\mu \neq 0$.
Case 1 Using (2.4) and $\mu \equiv 0$, we have $\mathfrak{H} = \lambda J e_1$. Since $\mathfrak{H}$ is parallel, λ is constant. Because of the scalar curvature ρ of Einstein manifold is constant, we conclude that $\|\sigma\|^2$ is constant from (2.7). According to (2.4) and (3.2) we obtain the following:

$$0 = \frac{1}{2}\Delta\|\sigma\|^2 = \frac{\tilde{c}}{4}(n-1)\lambda^2 + \|\nabla'\sigma\|^2. \tag{4.3}$$

When $\tilde{c}$ is positive, we have λ^2=0 and $\|\nabla'\sigma\|^2 = 0$ from (4.3). So M^n is totally geodesic. When $\tilde{c} = 0$, we deduce that M^n is a parallel submanifold in $\mathbb{C}^n$ from (4.3). According to the classification in [5], we conclude that either M^n is congruent to $S^1(\frac{1}{\sqrt{\lambda}}) \times \mathbb{R}^{n-1}$ or a totally geodesic submanifold.

Case 2 In the following we shall show that this case cannot occur.

In this case, using the fact that $\lambda = 2\mu = \textit{constant}$ and (2.4), we can deduce that both $\langle \mathfrak{H}, \mathfrak{H} \rangle$ and $\|\sigma\|^2$ are constant on M. According to (3.2), we get the following equation:

$$\begin{aligned} 0 = \tfrac{1}{2}\Delta\|\sigma\|^2 = \; & (n^2-1)\mu^2(\tfrac{\tilde{c}}{4} + \mu^2) + \|\nabla'\sigma\|^2 \\ & + \textstyle\sum_{i,j,k}\langle \overline{\nabla}_{e_j}((\nabla'_{E_k}\sigma)(E_i, E_i)), \sigma(e_j, e_k)\rangle. \end{aligned}$$

From this equation, we have $\sum_i (\nabla'_{E_k}\sigma)(E_i, E_i) \neq 0$. This contradicts our assumption $\nabla^{\perp}\mathfrak{H} = 0$. So this case cannot occur. We have thus proved the theorem. ∎

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