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ON THE NUMBER OF GALOIS POINTS FOR PLANE CURVES OF PRIME DEGREE

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ABSTRACT. In this note we estimate an upper bound of the number of Galois points for plane curves of prime degree.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero. We fix k as the ground field of our discussion. Let C be an irreducible (possibly singular) plane curve of degree d ($d \geq 3$). The concept of Galois points for C was introduced in [3], in order to study the structure of the field extension of the function field $k(C)/k$. First, we recall several definitions in brief (cf. [2], [3]).

Choose a point $P \in \mathbb{P}^2 \setminus C$. Then we have a projection $\pi_P : C \to l$ with the center P, where l is a line not passing through P. This projection induces a field extension $\pi_P^* : k(l) \hookrightarrow k(C)$. Clearly $[k(C) : k(l)] = d$. Since this extension does not depend on the choice of l, but on P, we put $K_P = \pi_P^*(k(l))$.

Definition 1. A point P is called a *Galois point* for C if $k(C)/K_P$ is a Galois extension.

That is to say, the point P is a Galois point if and only if the projection with the center P determines a Galois covering $\pi_P : X \to \mathbb{P}^1$, where X is the smooth model of C. When P is a Galois point, we denote by G_P the *Galois group* $\mathrm{Gal}(k(C)/K_P)$. We call G_P the Galois group at P.

Suppose that P is a Galois point for C. Then an element $\sigma_P \in G_P$ induces a birational transformation of C over l. Moreover, σ_P induces an automorphism of the smooth model of C. We denote it by the same notation.

In the case where C is smooth, we have studied Galois points for C in detail (cf. [3], [5], etc.). The purpose of this note is to study Galois points for plane singular curve C. In particular, we estimate an upper bound of the number of Galois points for plane curves of prime degree. So hereafter, we assume that the degree of C is an odd prime number p.

Remark 1. When P is a Galois point, clearly G_P is isomorphic to the cyclic group of order p.

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Definition 2. Since the number of Galois points will turn out to be finitely many, we denote it by $\delta(C)$.

Under the situation above, we prove the following.

Theorem 1. *Let C be a plane curve of prime degree p $(p \geq 3)$. Assume that C is not rational. If C has no cusps as its singular points, then $\delta(C) \leq 3$. If C has at least one cusp, then $\delta(C) \leq 1$.*

Corollary 2. *The curve C has the maximal number of Galois points if and only if it is the Fermat curve : $x^p + y^p = 1$.*

2. Proofs

We use the following notation:

$\varepsilon : X \to C$: the birational morphism from the smooth model X onto C.
$g = g(X)$: the genus of X.
$m_Q = m_Q(C)$: the multiplicity of C at Q.
$s_Q = s_Q(C)$: the number of the analytic branches of C at Q.
$I_Q(C_1, C_2)$: the intersection number of C_1 and C_2 at Q.
$T_Q = T_Q(C)$: the tangent line to C at Q.
$\mathrm{Reg}(C)$: the open subset of C of all non-singular points.
$W(C)$: the sum of order of flex of C, that is,

$$W(C) = \sum_{Q \in \mathrm{Reg}(C)} \{I_Q(C, T_Q) - 2\}.$$

Definition 3. The point $Q \in \mathrm{Reg}(C)$ is called an m-flex, if $m = I_Q(C, T_Q) - 2$.

First, we note that $\pi_P : X \to \mathbb{P}^1$ is a branched covering of prime degree p. Hence we infer the following lemma.

Lemma 1. *Suppose P is a Galois point. Then π_P is totally ramified, namely, for any branch point $\alpha \in \mathbb{P}^1$, $\pi_P^{-1}(\alpha)$ consists of one point.*

On the number of the ramification points, we have the following.

Lemma 2. *Suppose P is a Galois point for C. Then the number of ramification points of π_P is equal to*

$$\frac{2g + 2p - 2}{p - 1}.$$

Proof. From the Riemann-Hurwitz formula for π_P, we have

$$\sum_{R \in X} (e_R - 1) = 2g + 2p - 2,$$

where e_R is the ramification index of π_P at $R \in X$. Furthermore, by Lemma 1, if P is a Galois point, then π_P is totally ramified. That is, $e_R = p$. Hence we have the lemma. □

Remark 2. From Lemma 2, if $g = 1$, then we have $p = 3$.

Next, we recall the ramification points of π_P (cf. [2]). From the definition of π_P, we infer the following assertions.

(i) The case where Q is a smooth point of C:
Then there exist a $\tilde{Q} \in X$ such that $\varepsilon(\tilde{Q}) = Q$. Hence we have $e_{\tilde{Q}} = I_Q(C, \overline{PQ})$, where $\overline{PQ}$ is the line passing through P and Q.

(ii) The case where Q is a singular point of C:
Let $C_1, C_2, \cdots, C_s$ be the analytic branches at Q, and $\varepsilon^{-1}(Q) = \tilde{Q}_1, \cdots, \tilde{Q}_s$, where $s = s_Q(C)$. Then we have $e_{\tilde{Q}_k} = I_Q(C_k, \overline{PQ})$.

Therefore, we infer the following.

Lemma 3. *Let Q be a point of C. The covering π_P is totally ramified at Q if and only if*

(i) $s_Q(C) = 1$ *and*
(ii) $I_Q(C, \overline{PQ}) = p$.

Note that if Q satisfies this condition, then Q must be a flex or cusp.

Now we state a result of $W(C)$. In [1], [4], we have a generalization of Plücker-type relations to arbitrary curves. For a point $Q \in C$, we put as before: let $C_1, C_2, \cdots, C_s$ be the analytic branches at Q. Putting $\lambda_{Q_k} = I_Q(C_k, T_Q(C_k))$ and $|\lambda_Q| = \lambda_{Q_1} + \lambda_{Q_2} + \cdots + \lambda_{Q_s}$, we have the following formula.

The flex formula (cf. [4])

$$W(C) = 6g - 6 + 3p - \sum_Q (|\lambda_Q| + m_Q - 3s_Q),$$

where $\sum$ extended over all singular points Q on C.

By using this formula, we prove Theorem 1 separately according to the cases C has at least one cusp or not.

2.1. **C has no cusp.** Suppose P is a Galois point for C. Then we infer that the ramification points of π_P must be the inverse image of flexes of C by ε. Indeed, let $\alpha \in \mathbb{P}^1$ be a branch point of π_P, then $\pi_P^{-1}(\alpha)$ must consists of one point $R \in X$. Namely, the point $\varepsilon(R) \in C$ satisfies that $s_{\varepsilon(R)}(C) = 1$ and $I_{\varepsilon(R)}(C, \overline{P\varepsilon(R)}) = p$. Since C has no cusp, we conclude that $\varepsilon(R)$ must be a $(p-2)$-flex.

Lemma 4. *Suppose $Q \in C$ is a $(p-2)$-flex. Then there exists at most one Galois point on T_Q.*

Proof. Suppose there exist two Galois points P and P' on T_Q. Then, we first note that σ_P and $\sigma_{P'}$ have the same order p, furthermore, $\sigma_P \neq \sigma_{P'}$ since $\pi_P \neq \pi_{P'}$. Since the stabilizer of any point is a cyclic group of $\mathrm{Aut}(X)$, σ_P and $\sigma_{P'}$ cannot have fixed points in common for otherwise we would have $\sigma_P = \sigma_{P'}$. However, by our assumption, we see that $\sigma_P(Q) = \sigma_{P'}(Q) = Q$. This is a contradiction. Hence we obtain the lemma. □

Suppose P is a Galois point. Then there must exist $(2g+2p-2)/(p-1)$ pieces of $(p-2)$-flex and the tangent lines at that flexes must pass through P. By the above lemma, we see that one flex contributes to one Galois point. Namely, one Galois point needs $(p-2)(2g+2p-2)/(p-1)$ of $W(C)$. That is,

$$\delta(C) \times (p-2)\left(\frac{2g+2p-2}{p-1}\right) \leq W(C),$$

$$\delta(C) \leq \frac{(p-1)W(C)}{(p-2)(2g+2p-2)}.$$

Claim 1. $\delta(C) \leq 3$.

Proof. From the genus formula, we have

$$\frac{(p-1)(p-2)}{2} - g \geq 0.$$

That is,

$$3(p-1)(p-2) - 6g \geq 0.$$

This implies

$$6(p-2)g + 6(p-1)(p-2) \geq 6(p-1)g + 3(p-1)(p-2).$$

Since $W(C) \leq 6g + 3p - 6$ by the flex formula, we obtain

$$3 \geq \frac{6(p-1)g + 3(p-1)(p-2)}{2(p-2)g + 2(p-1)(p-2)} \geq \frac{(p-1)W(C)}{(p-2)(2g+2p-2)}.$$

□

In the claim, we see that if the equality holds then

$$\frac{(p-1)(p-2)}{2} = g,$$

i.e., C is smooth. In the case where C is smooth, Yoshihara studied the number of Galois points in [5]. Indeed, Corollary 2 follows from a result in [5].

Thus we complete the proof of the former part of Theorem 1 and Corollary 2.

2.2. C **has at least one cusp.** We first note the following.

Lemma 5. *Suppose P is a Galois point for C and Q is a cusp of C. Then P must lie on the tangent line T_Q, furthermore, the cusp Q must satisfy $I_Q(C, \overline{PQ}) = p$.*

Proof. Suppose P does not lie on T_Q. Then we see that the line passing through P meets C at Q with the intersection number at most $p-1$, and it intersects C at other point. That is, π_P is not totally ramified. Therefore P must lie on T_Q. The latter part follows from Lemma 3. □

From the lemma, if C has two cusps Q and Q', then the Galois point must lie on $T_Q \cap T_{Q'}$. Hence we conclude $\delta(C) \leq 1$. If C has more than two cusps, then we obtain $\delta(C) \leq 1$ by an argument similar to the above.

Finally we consider the case when C has just one cusp Q. Then we obtain the following claim.

Claim 2. $\delta(C) \leq 1$.

Proof. Suppose that there exist two Galois points P and P'. Then, by Lemma 5, we see that $P, P' \in T_Q$. We note that $\sigma_P \neq \sigma_{P'}$ since $\pi_P \neq \pi_{P'}$. Furthermore, σ_P and $\sigma_{P'}$ fix the cusp. Correctly speaking, $\sigma_P(\varepsilon^{-1}(Q)) = \sigma_{P'}(\varepsilon^{-1}(Q)) = \varepsilon^{-1}(Q)$. Therefore, by an argument similar to Lemma 4, we obtain the claim. □

Thus we complete the proof of Theorem 1. Finally we give an example.

Example. Let α, β, γ be mutually distinct elements of $k \setminus \{0\}$. If C is the curve $(y-\alpha x)^{p-2}(y-\beta x)(y-\gamma x)+1=0$ and $P=(0,0)$, then we can check easily that P is a Galois point for C and C has one cusp.

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