Quasi-linear equations of evolution in a Banach space

By

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1. Introductinn and assumptions

In this paper we consider the Cauchy problem for quasi-linear evolution equations of the from

(1) $$du/dt+A(t, u)u=F(t, u),\ 0\leq t\leq T,$$

(2) $$u(0)=\varphi$$

in an arbitrary Banach space X. Here $A(t, u)$ is, for every $t\in[0, T]$ and $u\in X$, not necessarily a bounded linear operator acting in X and the given function $F(t, u)$ takes values in X.

Under some assumptions on $A(t, u)$ and $F(t, u)$ specified below, making use of the semi-group theory, we try to solve (1)-(2) by successive approximation method in the following manner:

$$\begin{cases} du_0/dt+A(t, \varphi)u_0=F(t, \varphi),\ 0\leq t\leq T,\quad u_0(0)=\varphi; \\ du_{m+1}/dt+A(t, u_m)u_{m+1}=F(t, u_m),\ 0\leq t\leq T,\ u_{m+1}(0)=\varphi,\ m=0,1,\ldots. \end{cases}$$

Assuming nothing on X itself and constructing step by step the fundamental solutions of the above equations by method of H. Tanabe [2], we can obtain a sequence of uniformly bounded functions $u_m(t)$, $m=0,1,\ldots$ whose strong limit exists and gives a desired solution* in some interval $[0, T_0]$ $(0<T_0(\varphi)\leq T)$ for an arbitrary φ in the domain $D=D(A(t, u))$ of $A(t, u)$. Next, we shall prove uniqueness and continuous dependence on the initial value of the solution $u(t)$.

Throughout this note, we shall make the following assumptions:

$(\mathrm{A})_\varphi$ For each $t\in[0, T]$ and $u\in X$, $A(t, u)$ is a closed linear operator, whose domain D is dense in X and independent of t and u, and fulfills

$$(\mathrm{I})_\varphi\quad \|\{A(t, u)-A(t, v)\}A_\varphi(t)^{-1}\|\leq f_\varphi(\|u\|+\|v\|)\|u-v\|,$$

$$(\mathrm{II})_\varphi\quad \|\{A(t, u)-A(s, u)\}A_\varphi(s)^{-1}\|\leq g_\varphi(\|u\|)|t-s|$$

* We call u a solution of (1)-(2) in $[0, T]$ if u belongs to D, is strongly continuously differentiable on $[0, T]$ and satisfies (1)-(2).

for any $u, v \in X$ and $t, s \in [0, T]$ where f_φ and f_φ are non-decreasing continuous functions on $[0, \infty)$ to $[0, \infty)$ and $A_\varphi(t) = A(t, \varphi)$ for $\varphi \in D$;

$(B)_\varphi$ For each $t \in [0, T]$ the resolvent set of $A_\varphi(t)$ contains a fixed closed sector $\Sigma_\varphi = \{\lambda;\ arg\,\lambda \notin (-\theta, \theta)\}$ $(0 < \theta < \pi/2)$ and the resolvent of $A_\varphi(t)$ satisfies $\|A_\varphi(t)(\lambda - A_\varphi(t)^{-1}\| \leq M_\varphi (1 \leq M_\varphi))$ for any $t \in [0, T]$ and $\lambda \in \Sigma_\varphi$, where $\theta = \theta_\varphi$ and M_φ are constants independent of t and λ;

(C) $F(t, u)$ is a function defined on $[0, T] \times X$ to X and satisfies

$$\text{(III)} \quad \|F(t, u) - F(t, v)\| \leq q(\|u\| + \|v\|)\|u - v\|,$$

$$\text{(IV)} \quad \|F(t, u) - F(s, u)\| \leq r(\|u\|)|t - s|$$

for any $u, v \in X$ and $t, s \in [0, T]$, where p and r are also non-decreasing continuous functions on $[0, \infty)$ to $[0, \infty)$.

In the assumptions $(A)_\varphi$ and $(B)_\varphi$, for the sake of simplicity, we shall denote $A_\varphi(t)$, f_φ, g_φ, Σ_φ, θ_φ and M_φ by $A(t)$, f, g, Σ, θ and M unless otherwise stated.

From $(A)_\varphi$ and (C) we get

$$\|\{A(t, u(t) - A(s, u(s))\} A(s)^{-1}\| \leq g(\|u(t)\|)|t - s| + f(\|u(t)\| + \|u(s)\|)\|u(t) - u(s)\|,$$

$$\|F(t, u(t)) - F(s, u(s))\| \leq r(\|u(t)\|)|t - s| + q(\|u(t)\| + \|u(s)\|)\|u(t) - u(s)\|$$

for $u(t) \in X$ and $t, s \in [0, T]$.

Noting $A(t)^{-1} - A(0)^{-1} = -A(t)^{-1}\{A(t) - A(0)\}A(0)^{-1}$ and $(\text{II})_\varphi$, we have

$$\|A(t)^{-1}\| \leq \{1 - g(\|\varphi\|)t\}^{-1}\|A(0)^{-1}\| \leq 2\|A(0)^{-1}\| = N (0 < N),\ t \in [0, T'],$$

where $T' \leq T$ is a positive number such that $g(\|\varphi\|)T' \leq \frac{1}{2}$.

Hence, with the aid of the inequalities

$$\|\{A(t) - A(s)\} A(s)^{-1}\| \leq g(\|\varphi\|)|t - s|,$$

$$\|F(t, \varphi) - F(s, \varphi)\| \leq r(\|\varphi\|)|t - s|,\ t, s \in [0, T]$$

together with $(B)_\varphi$, the fundamental solution $U(t, s)$, $0 \leq s \leq t \leq T'$ of the linear equation

$$(1)_0 \qquad du/dt + A(t)u = F(t, \varphi),\ 0 \leq t \leq T$$

is constructed as follows:

$$(2) \qquad \begin{cases} U(t, s) = \exp(-(t - s)A(s)) + W(t, s), \\ W(t, s) = \int_s^t \exp(-(t - r)A(r))R(r, s)\, dr, \\ R(t, s) = \sum_{k=1}^{\infty} R_k(t, s), \end{cases}$$

$$\begin{cases} R_1(t, s) = -(A(t)-A(s))\exp(-(t-s)A(s)), \\ R_k(t, s) = \int_s^t R_1(t, r)R_{k-1}(r, s)dr, \ k=2, 3, \ldots\ldots, \end{cases}$$

where $\exp(-tA(s))$, $0 \leq t \leq T$ is the analytic semi-group generated by $-A(s)$ for each s. The unique solution of $(1)_0$-(2) is given by

$$u_0(t) = U(t, 0)\varphi + \int_0^t U(t, r)F(r, \varphi)dr, \ t \in [0, T'],$$

which is the first approximation. For the details, see H. Tanabe [2].

2. Preprartory lemmas.

In view of (2), noting

$$\|u_0(t)\| \leq \|U(t, 0)\varphi\| + \int_0^t \|U(t, r)F(r, \varphi)\|dr$$

and

$$\begin{aligned} \|A(t)u_0(t)\| &\leq \|A(t)U(t, 0)\varphi\| + \|\{I-\exp(-tA(t))\}F(t, \varphi)\| \\ &+ \int_0^t \|A(t)\{\exp(-(t-r)A(r)) - \exp(-(t-r)A(t))\}F(r, \varphi)\|dr \\ &+ \int_0^t \|A(t)\exp(-(t-r)A(t))\{F(r, \varphi) - F(t, \varphi)\}\|dr + \int_0^t \|A(t)W(t, r)F(t, \varphi)\|dr, \end{aligned}$$

we can find positive numbers R, L_1 and L_2 depending only on φ such that if $\varphi \in D$,

$$\|U(t, s)\| \leq R; \ \|u_0(t)\| \leq L_1, \ \|A(t)u_0(t)\| \leq L_2$$

and hence that

$$\|u_0(t) - u_0(s)\| \leq \int_s^t \|F(r, \varphi) - A(r)u_0(r)\|dr \leq (H+L_2)|t-s|$$

because of strong continuity of $F(t, \varphi)$ in $t \in [0, T]$ and $\|F(t, \varphi)\| \leq H (0 < H)$. It is clear that $1 \leq R$, $\|\varphi\| \leq L_1$ and $\|A(0)\varphi\| \leq L_2$.

Moreover, if t satisfies $0 \leq Mf(L_1 + \|\varphi\|)(H+L_2)t \leq \frac{1}{2}$, then, by (I),

$$\|\{A(t, u_0) - A(t)\}\lambda - A(t))^{-1}\| \leq Mf(L_1 + \|\varphi\|)(H+L_2)t \leq \frac{1}{2}$$

and the resolvent of $A(t, u_0)$ is expressed by

$$\begin{aligned} &(\lambda - A(t, u_0))^{-1} \\ &= \sum_{k=0}^{\infty} (\lambda - A(t))^{-1}\{(A(t, u_0) - A(t))(\lambda - A(t))^{-1}\}^k, \ \lambda \in \Sigma, \end{aligned}$$

which implies

$$\|\lambda(\lambda - A(t, u_0))^{-1}\| \leq 2(M+1), \ \|A(t)(\lambda - A(t, u_0))^{-1}\| \leq 2M, \ \lambda \in \Sigma.$$

On the other hand, from $(\mathrm{II})_\varphi$ and (IV), we get

$$\|A(t, u_0(t)) - A(s, u_0(s))\}A(s)^{-1}\| \leq \{g(L_1) + f(2L_1)(H+L_2)\}|t-s|,$$

$$\|F(t, u_0(t))-F(s, u_0(s))\| \leq \{r(L_1)+q(2L_1)(H+L_2)\}|t-s|.$$

Thus, replacing $A(t)$ by $A(t, u_0)$ in (2), we can construct the fundamental solution $U_0(t, s)$, $0 \leq s \leq t$ of

$$(1)_1 \qquad du/dt+A(t, u_0)u=F(t, u_0),\ 0 \leq t \leq T$$

and get the solution of $(1)_1$-(2):

$$u_1(t)=U_0(t, 0)\varphi+\int_0^t U_0(t, r)F(r, u_0)dr.$$

Moreover, we can find positive numbers R^0, $L^0{}_1$ and $L^0{}_2$ such that

$$\|U_0(t, s)\| \leq R^0;\ \|u_1(t)\| \leq L_1{}^0,\ \|A(t)u_1(t)\| \leq L_2{}^0$$

and hence

$$\|u_1(t)-u_1(s)\| \leq \{p(L_1)+L_2{}^0 h(L_1)\}|t-s|$$

where $h(\ )=f(\cdot+\|\varphi\|)(\cdot+\|\varphi\|)+1$, $p(\ )=H+q(\cdot+\|\varphi\|)(\cdot+\|\varphi\|)$. However, more carefull estimation is necessary in the proof of Lemma 2.

If t satisfies $0 \leq Mf(L_1{}^0+\|\varphi\|)\{p(L_1)+L^0{}_2 h(L_1)\}t \leq 1/2$, then by similar argument we have

$$\|\lambda(\lambda-A(t, u_1))^{-1}\| \leq 2(M+1),\ \|A(t)(\lambda-A(t, u_1))^{-1}\| \leq 2M \text{ for } \lambda \in \Sigma.$$

Hence, using

$$\begin{cases} \|\{A(t, u_1(t))-A(s, u_1(s))\}A(s)^{-1}\| \leq \{g(L_1{}^0)+f(2L_1{}^0)(p(L_1)+L_2{}^0 h(L_1))\}|t-s|, \\ \|F(t, u_1(t))-F(s, u_1(s))\| \leq \{r(L_1{}^0)+q(2L_1{}^0)(p(L_1)+L_2{}^0 h(L_1))\}|t-s|, \end{cases}$$

we can construct the fundamental solution $U_1(t, s)$, $0 \leq s \leq t$ of

$$(1)_2 \qquad du/dt+A(t, u_1)u=F(t, u_1),\ 0 \leq t \leq T$$

and get the solution of $(1)_2$-(2) in the form

$$u_2(t)=U_1(t, 0)\varphi+\int_0^t T_1(t, r)F(r, u_1)dr.$$

It is easy to see that

$$\|U_1(t, s)\| \leq R^1;\ \|u_2(t)\| \leq L_1{}^1,\ \ \|A(t)u_1(t)\| \leq L_2{}^1$$

and hence

$$\|u_2(t)-u_2(s)\| \leq \{p(L^0{}_1)+L^1{}_2 h(L^0{}_1)\}|t-s|$$

for some positive constants R^1, $L^1{}_1$ and $L^1{}_2$.

Thus in the same manner, we can construct step by step the fundamental solutions $U_m(t, s)$, $0 \leq s \leq t$, $m=0, 1, \ldots\ldots$ of

$$(1)_{m+1} \qquad du/dt+A(t, u_m)u=F(t, u_m),\ 0 \leq t \leq T$$

and get the solutions of $(1)_{m+1}$-(2):

$$u_{m+1}(t)=U_m(t,\ 0)\varphi+\int_0^t U_m(t,\ r)\,F(r,\ u_m)dr.$$

Now we shall prepare several lemmas.

LEMMA 1. If

$$\|u_{m-1}(t)\|\leq L_1,\ \|u_m(t)\|\leq L_1;\ \|A(t)u_m(t)\|\leq L_2,$$

then

$$\|\lambda(\lambda-A(t,\ u_m))^{-1}\|\leq 2(M+1),\ \|A(t)(\lambda-A(t,\ u_m))^{-1}\|\leq 2M$$

for $\lambda\in\Sigma$, $m=1,\ 2,\ldots\ldots$ and non negative t with

$$Mf(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t\leq\frac{1}{2},\ g(\|\varphi\|)t\leq\frac{1}{2}.$$

PROOF. We have only to note

$$\|\{A(t,\ u_m)-A(t)\}(\lambda-A(t))^{-1}\|\leq Mf(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t\leq\frac{1}{2}$$

and

$$(\lambda-A(t,\ u_m))^{-1}=\sum_{k=0}^{\infty}(\lambda-A(t))^{-1}\{A(t,\ u_m)-A(t))(\lambda-A(t))^{-1}\}^k.$$

LEMMA 2. There exists a positive constant $T_0=T_0(\varphi)\leq T$ depending only on φ such that

$$\|U_m(t,\ s)\|\leq R;\ \|u_m(t)\|\leq L_1,\ \|A(t)u_m(t)\|\leq L_2$$

for $t,\ s\in[0,\ T_0]$ and $m=0,\ 1,\ldots\ldots$.

PROOF. We assume that

$$\|u_{m-1}(t)\|\leq L_1,\ \|u_m(t)\|\leq L_1;\ \|A(t)u_m(t)\|\leq L_2.$$

Then, by Lemma 1, we have, for $\lambda\in\Sigma$ and suitable t,

$$\|\lambda(\lambda-A(t,\ u_m))^{-1}\|\leq 2(M+1),\ \|A(t)(\lambda-A(t,\ u_m))^{-1}\|\leq 2M.$$

On the other hand, using $\|u_m(t)-u_m(s)\|\leq\{p(L_1)+L_2h(L_1)\}|t-s|$, we obtain

$$\begin{cases}\|\{A(t,\ u_m(t))-A(s,\ u_m(s))\}A(s)^{-1}\|\leq\{g(L_1)+f(2L_1)(p(L_1)+L_2h(L_1))\}|t-s|,\\ \|F(t,\ u_m(t))-F(s,\ u_m(s))\|\leq\{r(L_1)+q(2L_1)(p(L_1)+L_2h(L_1))\}|t-s|.\end{cases}$$

From the above inequalities and the formula (2) we get

$$\|R_m^1(t,\ s)\|=\|\{A(t,\ u_m)-A(s,\ u_m)\}\exp(-(t-s)A(s,\ u_m))\|\leq 2MCD,$$

$$\|R^m(t,\ s)\|=\|\sum_{k=0}^{\infty}R_m^k(t,\ s)\|\leq\sum_{k=0}^{\infty}(2MCD)^k(t-s)^{k-1}/(k-1)!$$

$$=2MCD\ \exp(2MCD(t-s)),\ D=g(L_1)+f(2L_1)(p(L_1)+L_2h(L_1)).$$

In this paper, we denote by C various generic constants depending only on φ.

Now we get

$$\|U_m(t, s)\| \leq 2(M+1)C\{1+\int_s^t \|R^m(r, s)\|dr\}$$
$$=2C(M+1)+2C(M+1)(\exp(2MCDt)-1)$$

and

$$\int_0^t \|U_m(t, r)F(r, u_m)\|dr \leq \int_0^t 2C(M+1)\ \exp(2MCDr)\,p(L_1)\,dr$$
$$=2C(M+1)\ \exp(2MCDt)\,p(L_1)\,t,$$

which conclude

$$\|u_{m+1}(t)\| \leq C(M+1)\|\varphi\| + 2C(M+1)\|\varphi\|(\exp(2MCDt)-1)$$
$$+2C(M+1)\exp(2MCDt)\,p(L_1)\,t.$$

Next we must estimate $\|A(t)u_{m+1}(t)\|$. It is clear that

$$\|A(t)\exp(-tA(0)\varphi\| \leq (1+g(\|\varphi\|)t)\|A(0)\exp(-tA(0))\varphi\|$$
$$\leq \frac{3}{2}C(M+1)\|A(0)\varphi\|.$$

Writing

$$A(t)W_m(t, s)=A(t, u_m)W_m(t, s)-\{A(t, u_m)-A(t)\}A(t)^{-1}A(t)W_m(t, s),$$
$$A(t, u_m)W_m(t, s)=\{I-\exp(-(t-s)A(t, u_m))\}R^m(t, s)$$
$$+\int_s^t A(t, u_m)\{\exp(-(t-r)A(r, u_m))-\exp(-(t-r)A(t, u_m))\}R^m(r, s)dr$$
$$+\int_s^t A(t, u_m)\exp(-(t-r)A(t, u_m))\{R^m(r, s)-R^m(t, s)\}dr$$
$$=I_1+I_2+I_3,$$

we obtain

$$\|I_1\| \leq \{\|I-\exp(-(t-s)A(0))\|+\|\exp(-(t-s)A(t))-\exp(-(t-s)A(0))\|$$
$$+\|\exp(-(t-s)A(t, u_m))-\exp(-(t-s)A(t))\|\}\|R^m(t, s)\|$$
$$\leq [\|I-\exp(-(t-s)A(0))\|+CM(M+1)g(\|\varphi\|)t$$
$$+2CM(M+1)f(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t]2MCD\ \exp(2MCD(t-s))$$

and

$$\|I_2\| \leq \int_s^t C(2M+3)\,2MD\ 2MD\ \exp(2MCD(r-s))dr$$
$$=C(2M)^2(2M+3)D^2\ \exp(2MCD(t-s))(t-s),$$

which is a simple consequence of

$$\|A(t, u_m)\{(\lambda-A(r, u_m))^{-1}-(\lambda-A(t, u_m))^{-1}\}\|$$
$$\leq \|A(t, u_m)(\lambda-A(t, u_m))^{-1}\{A(t, u_m)-A(r, u_m)\}A(r)^{-1}A(r)(\lambda-A(r, u_m))^{-1}\|$$
$$\leq (2M+3)2MD|t-r|,\ \lambda\in\Sigma.$$

From

$$\begin{aligned}
&\|R^m{}_1(t, s)-R^m{}_1(r, s)\| \\
&\leq \|\{A(t, u_m)-A(r, u_m)\} \exp(-(t-s)A(s, u_m))\| \\
&+\|\{A(r, u_m)-A(s, u_m)\}\{\exp(-(t-s)A(s, u_m))-\exp(-(r-s)A(s, u_m))\}\| \\
&\leq 2MCD(2+g(\|\varphi\|)t)(t-r)/(t-s)
\end{aligned}$$

and

$$\begin{aligned}
&\|R^m(t, s)-R^m(r, s)\| \\
&\leq \int_r^t \|R_m{}^1(t, x)R^m(x, s)\|dx+\int_s^r \|R_m{}^1(t, x)-R_m{}^1(r, x)\|\|R^m(x, s)\|dx \\
&\leq (2MCD)^2 \exp(2MCD(t-s))\{1+(2+g(\|\varphi\|)t)\log\frac{t-s}{t-r}\}(t-r), \\
&0\leq s\leq r<t,
\end{aligned}$$

it follows that

$$\|I_3\|\leq C(2M)^2(2M+3)D^2 \exp(2MCD(t-s))(t-s)(3+g(\|\varphi\|)t).$$

Combining the above inequalities, we obtain

$$\begin{aligned}
&\|A(t)W_m(t, s)\|\leq[1-f(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t]^{-1}\|A(t, u_m)W_m(t, s)\| \\
&\qquad \leq 2\|A(t, u_m)W_m(t, s)\| \\
&\leq 2[\|I-\exp(-(t-s)A(0))\|+CM(M+1)g(\|\varphi\|)t+2CM(M+1)f(L_1+\|\varphi\|) \\
&\times\{p(L_1)+L_2h(L_1)\}t]2MCD \exp(2MCD(t-s)) \\
&\qquad +2C(2M)^2(2M+3)D^2\exp(2MCD(t-s))(t-s)(4+g(\|\varphi\|)t)
\end{aligned}$$

and hence

$$\begin{aligned}
&\|A(t)U_m(t, 0)\varphi\| \\
&\leq \frac{3}{2}C(M+1)\|A(0)\varphi\|+2[\|I-\exp(-tA(0))\|+CM(M+1) \\
&\times g(\|\varphi\|)t+2CM(M+1)f(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t]2MCD \\
&\times\exp(2MCDt)\|\varphi\|+2C(2M)^2(2M+3)D^2\exp(2MCDt)(4+g(\|\varphi\|)t)\|\varphi\|.
\end{aligned}$$

Since

$$\begin{aligned}
&\|\int_0^t A(t)U_m(t, r)F(r, u_m)dr\|\leq 2\|\int_0^t A(t, u_m)U_m(t, r)F(r, u)dr\| \\
&\leq 2\|\{I-\exp(-tA(t, u_m))\}F(t, u_m)\| \\
&+2\int_0^t \|A(t, u_m)\{\exp(-(t-r)A(r, u_m))-\exp(-(t-r)A(t, u_m))\}F(r, u_m)\|dr \\
&+2\int_0^t \|A(t, u_m)\exp(-(t-r)A(t, u_m))\{F(r, u_m)-F(t, u_m)\}\|dr
\end{aligned}$$

$$+2\int_0^t \|A(t, u_m)W_m(t, r)F(r, u_m)\|dr,$$

we get

$$\|\int_0^t A(t)U_m(t, r)F(r, u_m)dr\|$$

$$\leq 2[\|I-\exp(-tA(0))\|+CM(M+1)g(\|\varphi\|)t+2CM(M+1)f(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t]$$

$$\times p(L_1)+2C(2M)(2M+3)D\,p(L_1)t+2C(2M+3)Et+$$

$$+2(1+2C(M+1))2MCD\ \exp(2MCDt)\,p(L_1)t+2C(2M)^2(2M+3)D^2\exp(2MCDt)t^2\times$$

$$\times(4+g(\|\varphi\|)t)\,p(L_1),\ \text{where}\ E=r(L_1)+q(2L_1)(P(L_1)+L_2h(L_1)).$$

Now let L_1 and L_2 be so large that

$$\|\varphi\|\leq L_1,\ C(M+1)\|\varphi\|<L_1;\ \|A(0)\varphi\|\leq L_2,\ \frac{3}{2}C(M+1)\|A(0)\varphi\|<L_2.$$

Then we can find positive t satisfying the following four inequalities:

$$Mf(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}t\leq 1/2,\ g(\|\varphi\|)t\leq 1/2,$$

$$2C(M+1)\|\varphi\|(\exp(2MCDt)-1)+2C(M+1)\exp(2MCDt)p(L_1)t\leq L_1-C(M+1)\|\varphi\|$$

and

$$2\{\|I-\exp(-tA(0))\|+CM(M+1)g(\|\varphi\|)t$$

$$+2CM(M+1)f(L_1+\|\varphi\|)(p(L_1)+L_2h(L_1))t\}\{2MCD\exp(2MCDt)\|\varphi\|+p(L_1)\}$$

$$+2C(2M)^2(2M+3)D^2\exp(2MCDt)(4+g(\|\varphi\|)t)t\{\|\varphi\|+p(L_1)t\}$$

$$+2C(2M+3)Dp(L_1)t+2C(2M+3)Et$$

$$+2(1+2C(M+1))2MCD\exp(2MCDt)\,p(L_1)t\leq L_2-\frac{3}{2}C(M+1)\|A(0)\varphi\|.$$

In fact, all the terms on the left sides of the above inequalities decrease to vanish as $t\downarrow 0$. Hence, denoting the maximum value of such t by $T''=T''(\varphi)$, which as well as L_1 and L_2 depends only on φ, we obtain

$$\|u_{m+1}(t)\|\leq L_1 \text{ and } \|A(t)u_{m+1}(t)\|\leq L_2,\ t\in[0, T''].$$

On the other hand, recalling

$$\|\lambda(\lambda-A(t))^{-1}\|\leq M+1,\ \|A(t)(\lambda-A(t))^{-1}\|\leq M,\ \lambda\in\Sigma$$

and

$$\begin{cases}\|\{A(t)-A(s)\}A(s)^{-1}\|\leq g(\|\varphi\|)|t-s|,\\ \|F(t, \varphi)-F(s, \varphi)\|\leq r(\|\varphi\|)|t-s|,\ t, s\in[0, T'']\end{cases}$$

and noting that $g(\|\varphi\|)$, $r(\|\varphi\|)$ are smaller than D, E respectively, we have

$$\|u_0(t)\|\leq L_1 \text{ and } \|A(t)u_0(t)\|\leq L_2,\ t\in[0, T''].$$

Thus it has been proved by induction that

$$\|u_k(t)\|\leq L_1 \text{ and } \|A(t)u_k(t)\|\leq L_2,\ t\in[0,\ T''],\ k=0,1,\ldots\ldots .$$

Letting R be a positive number such that $1\leq R$ and $2C(M+1)<R$, and $T_0=T_0(\varphi)$ one such that $T_0(\varphi)\leq T''$ and

$$2C(M+1)(\exp(2MCDT_0)-1)\leq R-2C(M+1),$$

we have

$$\|U_k(t,\ s)\|\leq R,\ 0\leq s\leq t\leq T_0,\ k=0,\ 1,\ldots\ldots .$$

LEMMA 3. If $\|U_m(t,s)\|\leq R$; $\|u_m(t)\|\leq L_1$, $\|A(t)u_m(t)\|\leq L_2$ for $t, s\in[0, T_0]$ and $m=0, 1,\ldots\ldots$, then

$$\|u_m(t)-u_{m-1}(t)\|\leq(H+L_2)\{R(q(2L_1)+L_2f(2L_1))\}^m t^{m+1}/(m+1)!,$$
$$m=1, 2,\ldots\ldots .$$

PROOF. This is a simple consequence of the inequalities:

$$\begin{cases} \|u_0(t)-\varphi\|\leq(H+L_2)t, \\ \|u_1(t)-u_0(t)\|\leq\int_0^t R\{q(L_1+\|\varphi\|)+L_2f(L_1+\|\varphi\|)\}\|u_0(r)-\varphi\|dr, \\ \|u_m(t)-u_{m-1}(t)\|\leq\int_0^t R\{q(2L_1)+L_2f(2L_1)\}\|u_{m-1}(r)-u_{m-2}(r)\|dr, \end{cases}$$
$$m=1, 2,\ldots\ldots .$$

3. The proof of theorems.

In the section we shall establish there theorems and prove them making use of the lemmas prepared in the previous section.

THEOREM 1. If, for an arbitrary $\varphi\in D$, the assumptions $(A)_\varphi$, $(B)_\varphi$ and (C) are satisfied by $A(t,\ u)$, $A(t,\varphi)$ and $F(t,u)$, then there exist positive numbers L_1, L_2 and T_0, all of which depend only on φ, and (1) with $u(0)=\varphi$ admits a solution $u(t)$ on $[0,\ T_0]\subset[0,\ T]$ with the properties:

$$\|u(t)\|\leq L_1 \text{ and } \|A(t)u(t)\|\leq L_2 \text{ for all } t\in[0,\ T_0].$$

PROOF. From Lemma 2 and Lemma 3 we get a sequence $u_m(t)$, $t\in[0,\ T_0]$, $m=0, 1,\ldots\ldots$ satisfying

(3.1) $\|u_m(t)\|\leq L_1$,

(3.2) $\|A(t)u_m(t)\|\leq L_2$

and hence

(3.3) $$\|u_m(t)-u_m(s)\|\leq\{p(L_1)+L_2h(L_1)\}\,|t-s|,$$

(3.4) $$\|u_{m+1}(t)-u_m(t)\|\leq(H+L_2)\{R(q(2L_1)+L_2f(2L_1))\}^{m+1}t^{m+2}/(m+2)!.$$

Hence (3.4) implies that $u_m(t)$ converges uniformly on $[0, T_0]$ as $m\to\infty$. Putting $u(t)=s\text{-}\lim_{m\to\infty} u_m(t)$, from (3.1) and (3.3) we get

(3.5) $$\|u(t)\|\leq L_1,$$

(3.6) $$\|u(t)-u(s)\|\leq\{p(L_1)+L_2h(L_1)\}\,|t-s|,$$

from which we have, for $t, s\in[0, T_0]$,

(3.7) $$\begin{cases}\|\{A(t, u(t))-A(s, u(s))\}A(s)^{-1}\|\leq D|t-s|,\\ \|F(t, u(t))-F(s, u(s))\|\leq E|t-s|.\end{cases}$$

Noting

$$\|\{A(t, u)-A(t)\}(\lambda-A(t))^{-1}\|\leq f(L_1+\|\varphi\|)\{p(L_1)+L_2h(L_1)\}Mt\leq\frac{1}{2},$$

we have easily

(3.8) $$\|\lambda(\lambda-A(t, u))^{-1}\|\leq 2(M+1),\ \|A(t)(\lambda-A(t, u))^{-1}\|\leq 2M$$

for $\lambda\in\Sigma$ and $t\in[0, T_0]$.

Using (3.7) and (3.8) and recalling (2), we can construct the fundamental solution $U_u(t, s)$, $0\leq s\leq t\leq T_0$ of

$(1)_u$ $$dv/dt+A(t, u)v=F(t, u),\ 0\leq t\leq T$$

and get the solution of $(1)_u$ with $v(0)=\varphi$ in the from

$$v(t)=U_u(t, 0)\varphi+\int_0^t U_u(t, r)F(r, u)dr.$$

From (3.5)-(3.8) we get

(3.9) $$\|U_u(t, s)\|\leq R,$$

(3.10) $$\|A(t)v(t)\|\leq L_2$$

after similar calculation to that in the proof of Lemma 2.

Using (3.1), (3.2), (3.5) and (3.9), we have

$$\begin{aligned}\|u(t)-v(t)\|&\leq\|u(t)-u_{m+1}(t)\|+\|u_{m+1}(t)-v(t)\|\\ &\leq\|u(t)-u_{m+1}(t)\|+\int_0^t R\{q(2L_1)+L_2f(2L_1)\}\|u_m(r)-u(r)\|dr,\ m=0, 1,\ldots,\end{aligned}$$

which concludes $u(t)=v(t)$, that is,

$$u(t)=U_u(t, 0)\varphi+\int_0^t U_u(t, r)F(r, u)dr,\ 0\leq t\leq T_0.$$

Remark. From (3.5) and (3.10) we obtain

$$\|du(t)/dt\| \leq p(L_1)+L_2h(L_1),\ t\in[0,\ T_0].$$

The difference the solution $u(t)$ of (1) with $u(0)=\varphi$ and an approximate solution $u_m(t)$ is estimated as

$$\|u(t)-u_m(t)\| \leq \sum_{k=m+1}^{\infty}(H+L_2)[R\{q(2L_1)+L_2f(2L_1)\}]^k t^{k+1}/(k+1)!$$

for $t\in[0,\ T_0]$.

COROLLARY. Let $f(t)$ be a function on $[0,\ T]$ to X such that

$$\|f(t)-f(s)\| \leq K|t-s|\ (0<K) \text{ for } t,\ s\in[0,\ T].$$

Then, under $(\mathrm{A})_\varphi$ and $(\mathrm{B})_\varphi$, $\varphi\in D$, there exist positive numbers S; M_1, M_2 and T_1 dependent only on φ and there exists a solution $u(t)$ on $[0,\ T_1]\subset[0,\ T]$ of

$$du/dt+A(t,\ u)u=f(t),\ 0\leq t\leq T,$$

$$u(0)=\varphi.$$

And $u(t)$ satisfies $\|u(t)\|\leq M_1$ and $\|A(t)u(t)\|\leq M_2$, $t\in[0,\ T_1]$.

THEOREM 2. If $(\mathrm{A})_{\varphi i}$, $(\mathrm{B})_{\varphi i}$ and (C) are satisfied by $A(t,\ u)$, $A(t,\ \varphi_i)$ and $F(t,\ u)$ for $\varphi_i\in D(i=1,\ 2)$, then the solution $u^i(t)$, $t\in[0,\ T_0(\varphi_i)]$ of (1) with $u(0)=\varphi_i(i=1,\ 2)$, whose existence has been established in Theorem 1, satisfy:

$$\|u^1(t)-u^2(t)\| \leq \|\varphi_1-\varphi_2\|R^1\exp[tR^1\{p(L_1^1+L_1^2)+L_2^2f\varphi_2(L_1^1+L_1^2)\}]$$

for $t\in[0,\ T_0(\varphi_1)]\cap[0,\ T_0(\varphi_2)]$, where $R^1=R(\varphi_1,)$ $L_1^i=L_1(\varphi_i)(i=1,\ 2)$, $L_2^2=L_2(\varphi_2)$ and f_{φ_2} is the function which appears in $(\mathrm{I})_{\varphi_2}$.

Therefore, under the assumptions of Theorem 1, the solution $u(t)$ of (1)-(2) is uniquely determined by the initial value φ.

PROOF. Writing

$$u^1(t)-u^2(t)=U_{u^1}(t,\ 0)(\varphi_1-\varphi_2)+\int_0^t U_{u^1}(t,\ r)\{F(r,\ u^1)-F(r,\ u^2)\}dr$$
$$-\int_0^t U_{u^1}(t,\ r)\{A(r,\ u^1)-A(r,\ u^2)\}u^2(r)dr,$$

we have $\|u^1(t)-u^2(t)\| \leq R^1\|\varphi_1-\varphi_2\|+\int_0^t R^1F\|u^1(r)-u^2(r)\|dr,$

$$F=q(L_1^1+L_1^2)+f_{\varphi 2}(L_1^1+L_1^2)L_2^2 \text{ for } t\in[0,\ T_0(\varphi_1)]\cap[0,\ T_0(\varphi_2)].$$

From this inequality we get

$$\|u^1(t)-u^2(t)\| \leq R^1\|\varphi_1-\varphi_2\|\sum_{k=0}^{m}(R^1Ft)^k/k!\ +(L_1^1+L_1^2)(R^1F)^{m+1}t^{m+1}/(m+1)!,$$

for $m=0,\ 1,\ldots\ldots$, which conludes

$$\|u^1(t)-u^2(t)\| \leq R^1\|\varphi_1-\varphi_2\|e^{R^1Ft},\ t\in[0,\ T_0(\varphi_1)]\cap[0,\ T_0(\varphi)].$$

REMARK. Under the assumption of Theorem 1, there exists a solution $v(t)$,

$t\in[0, T_0(\Psi)]$ of (1) with $u(0)=\Psi$ for every $\Psi\in D$ such that $\|\Psi-\varphi\|\leq\delta$ where δ is a positive number with $f(2\|\varphi\|+\delta)\delta M\leq\frac{1}{2}$, and

$$\|u(t)-v(t)\|\leq\|\varphi-\Psi\|R\ \exp[tR\{q(L_1+L_1(\Psi))+L_2(\Psi)f_\Psi(L_1+L_1(\Psi))\}]$$

is valid for any $t\in[0, T_0)\cap[0, T_0(\Psi)]$.

In fact, noting

$$\|\{A(t, \Psi)-A(t, \varphi)\}A(t)^{-1}\|\leq f(\|\Psi\|+\|\varphi\|)\|\Psi-\varphi\|\leq f(2\|\varphi\|+\delta)\delta\leq\frac{1}{2M},$$

we have $\|\lambda(\lambda-A(t, \Psi))^{-1}\|\leq 2(M+1)$, $\lambda\in\Sigma$ and $\|A(t)A(t, \Psi)^{-1}\|\leq 2$,

which implies $A(t, u)$ and $A(t, \Psi)$ satisfy $(A)_\Psi$ and $(B)_\Psi$.

Finally, we shall compare the solution $u^\varepsilon(t)$ of the Cauchy problem containing a positive parameter $\varepsilon\leq\varepsilon_0$:

$(1)^\varepsilon$ $\qquad du/dt+\varepsilon A(t, u)u=F(t, u),\ 0\leq t\leq T,$

(2) $\qquad u(0)=\varphi$

with the solution $v(t)$ of the limit problem

$(1)^0$ $\qquad du/dt=F(t, u),\ 0\leq t\leq T,$

$$u(0)=\varphi.$$

Here we let $A(t, u)$ and $A(t, \varphi)$ fulfill $(A)_\varphi$ and $(B)_\varphi$ and let $F(t, u)$ fulfill, in addition to (C),

$(V)_\varphi$ If $u(t)$ belongs to D and $A_\varphi(t)u(t)$ is strongly continuous in $t\in[0, T]$, then so does $F(t, u(t))$ and $\|A_\varphi(t)F(t, u(t))\|\leq k(\|A_\varphi(t)u(t)\|)$, where k is such a function as f, g, q and r and is generally dependent on φ.

Then, as is easily seen, $\varepsilon A(t, u)$ and $\varepsilon A(t, \varphi)$ satisfy $(A)_\varphi$ and $(B)_\varphi$ with f, g; θ, M unchanged for every ε. Therefore, following Theorem 1, we can obtain the solution $u^\varepsilon(t)$, $t\in[0, T_0]$ of $(1)^\varepsilon$–(2) such that

$$\|U_{u^\varepsilon}(t, s)\|\leq R;\ \|u^\varepsilon(t)\|\leq L_1,\ \varepsilon\|A(t)u^\varepsilon(t)\|\leq L_2,$$

where R; L_1, L_2 and T_0 are independent of ε.

On the other hand $v(t)$ is given uniquely as a strong limit of the sequence $v_m(t)$, $m=0, 1,\ldots\ldots$ defined as follows:

$$\begin{cases} dv_0/dt=F(t, \varphi),\quad v_0(0)=\varphi,\ 0\leq t\leq T, \\ dv_{m+1}/dt=F(t, v_m),\quad v_m(0)=\varphi,\ 0\leq t\leq T,\ m=0, 1,\ldots\ldots . \end{cases}$$

Letting N_1 and N_2 be positive numbers larger than $\|\varphi\|$ and $\|A(0)\varphi\|$ respectively and $T_2(\leq T)$ be the maximum value of positive t satisfying

$$p(N_1)t\leq N_1-\|\varphi\|,\ k(N_2)t+\{\|A(0)\varphi\|+k(N_2)t\}tg(\|\varphi\|)\leq N_2-\|A(0)\varphi\|,$$

we can easily see by induction that $v_m(t)$ belong to D and $A(t)v_m(t)$ are strongly continuous in $t\in[0, T_2]$ with $\|v_m(t)\|\leq N_1$ and $\|A(t)v_m(t)\leq N_2$ and that

$$\|v_{m+1}(t)-v_m(t)\|\leq Hq(2N_1)^{m+1}t^{m+2}/(m+2)!,\ t\in[0, T_2],\ m=0,1,\ldots\ldots .$$

Noting

$$u^\varepsilon(t)-v_{m+1}(t)$$
$$=\int_0^t U_{u^\varepsilon}(t, r)\{F(r, u^\varepsilon)-F(r, v_m)\}dr-\int_0^t U_{u^\varepsilon}(t, r)\varepsilon A(r, u^\varepsilon)v_{m+1}(r)dr,\ m=0,1,\ldots\ldots,$$
$$u^\varepsilon(t)-v_0(t)=-\int_0^t U_{u^\varepsilon}(t, r)\varepsilon A(r, u^\varepsilon)v_0(r)dr+\int_0^t U_{u^\varepsilon}(t, r)\{F(r, u^\varepsilon)-F(r, \varphi)\}dr,$$

we have $\|u^\varepsilon(t)-v_m(t)\|$

$$\leq\frac{\varepsilon h(L_1)N_2}{q(L_1+N_1)}\sum_{k=1}^{m+1}\{Rq(L_1+N_1)t\}^k/k!+R^{m+1}q(L_1+N_1)^m\{p(L_1)+H\}t^{m+1}/(m+1)!,$$

$$m=0,1,\ldots\ldots$$

and hence

$$\|u^\varepsilon(t)-v(t)\|\leq\varepsilon h(L_1)N_2q(L_1+N_1)^{-1}[\exp\{Rq(L_1+N_1)t\}-1]$$

for all $t\in[0, T_0]\cap[0, T_2]$.

Thus we have estabished

THEOREM 3. Under the assumptions $(A)_\varphi$, $(B)_\varphi$ and (C)-$(V)_\varphi$, the solution $u^\varepsilon(t)$, $t\in[0, T_0]$ of $(1)^\varepsilon$ with $u(0)=\varphi$ converges to the solution $v(t)$, $t\in[0, T_2]$ of $(1)^0$ with $u(0)=\varphi$ in the following sense:

$$\|u^\varepsilon(t)-v(t)\|\leq\varepsilon h(L_1)N_2q(L_1+N_1)^{-1}[\exp\{Rq(L_1+N_1)t\}-1],$$

for all $t\in[0, T_0]\cap[0, T_2]$.

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