On extended almost analytic vectors and tensors in almost complex manifolds

By

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1. Extended contravariant almost analytic vectors.

S. Tachibana [9] generalized the notion of contravariant analytic vectors in a Kählerian manifold to an almost Kählerian manifold with structure tensor $\varphi_j{}^i$ and called v^i a contravariant almost analytic vector if it satisfies

$$\underset{v}{\mathcal{L}}\varphi_j{}^i \equiv v^r\nabla_r\varphi_j{}^i - \varphi_j{}^r\nabla_r v^i + \varphi_r{}^i\nabla_j v^r = 0 \tag{1.1}$$

where ∇ denotes the operator of covariant derivative with respect to the Riemannian connection. But this formula (1.1) is the so called concomitant and so it is independent of connection. Since, if we consider (1.1) in a Kählerian manifold, it means v^i is an analytic vector, a contravariant almost analytic vector is a generalization of a contravariant analytic vector in a Kählerian manifold. From this point of view, in an almost complex manifold, we shall call v^i an extended contravariant almost analytic vector if it satisfies

$$\underset{v}{\mathcal{L}}\varphi_j{}^i + \lambda\varphi_j{}^r N_{rl}{}^i v^l = 0 \tag{1.2}$$

where $N_{rl}{}^i$ is the Nijenhuis tensor and λ is C^∞ scalar function. This vector is also a generalization of a contravariant analytic vector in a Kählerian manifold. In fact, in a Kählerian manifold, since $N_{rl}{}^i = 0$ and $\nabla_j\varphi_i{}^h = 0$, (1.2) shows that v^i is an analytic vector [15].

Particularly, when $\lambda = -\frac{1}{2}$, this definition coincides with Sato's definition obtained from the standpoint of cross-section of a tangent bundle [4].

2. Properties of extended contravariant almost analytic vectors in K-space

By K-space (Tachibana space) we mean a Hermitian manifold M such that

$$\nabla_j\varphi_{ih} + \nabla_i\varphi_{jh} = 0. \tag{2.1}$$

In K-space, since $N_{ji}{}^{h}=4\varphi_{j}{}^{l}\nabla_{l}\varphi_{i}{}^{h}$, we have

$$v^{t}\varphi_{j}{}^{l}N_{lt}{}^{i} = 4\varphi_{j}{}^{l}\varphi_{l}{}^{s}(\nabla_{s}\varphi_{t}{}^{i})v^{t} = 4v^{t}\nabla_{t}\varphi_{j}{}^{i}.$$

Consequently, if we put $\lambda=-\frac{1}{4}$ in (1.2), we have

$$-\varphi_{j}{}^{r}\nabla_{r}v^{i}+\varphi_{r}{}^{i}\nabla_{j}v^{r} = 0$$

or transvecting this equation with $\varphi_{k}{}^{j}$, we obtain

$$\nabla_{k}v^{i}+\varphi_{r}{}^{i}\varphi_{k}{}^{l}\nabla_{l}v^{r} = 0 \tag{2.2}$$

or this is equivalent to

$$\nabla_{k}v_{i}-\varphi_{k}{}^{l}\varphi_{i}{}^{r}\nabla_{l}v_{r} = 0. \tag{2.3}$$

Generally, in an almost complex manifold, a tensor $T_{j}{}^{i}(T_{ji})$ is called pure in j, i, if it satisfies

$$T_{j}{}^{i}+\varphi_{j}{}^{a}\varphi_{b}{}^{i}T_{a}{}^{b} = 0 \qquad (T_{ji}+\varphi_{j}{}^{a}\varphi_{i}{}^{b}T_{ab}=0)$$

and $T_{j}{}^{i}(T_{ji})$ is called hybrid in j, i, if it satisfies

$$T_{j}{}^{i}-\varphi_{j}{}^{a}\varphi_{b}{}^{i}T_{a}{}^{b} = 0 \qquad (T_{ji}-\varphi_{j}{}^{a}\varphi_{i}{}^{b}T_{ab}=0).$$

For instance, our structure tensor $\varphi_{j}{}^{i}$ is pure in j, i and Riemannian metric tensor g_{ji} is hybrid in j, i.

The following proposition is easily verified.

PROPOSITION. (1) *If T_{ji} is pure (hybrid) in j, i then we have*

$$\varphi_{i}{}^{r}T_{jr} = \varphi_{j}{}^{r}T_{ri} \ (\varphi_{i}{}^{r}T_{jr}=-\varphi_{j}{}^{r}T_{ri}).$$

(2) *If $T_{j}{}^{i}$ is pure (hybrid) in j, i, then we have*

$$\varphi_{j}{}^{r}T_{r}{}^{i} = \varphi_{r}{}^{i}T_{j}{}^{r} \ (\varphi_{j}{}^{r}T_{r}{}^{i}=-\varphi_{r}{}^{i}T_{j}{}^{r}).$$

(3) *If T_{ji} is pure in j, i, and $S_{j}{}^{i}$ is pure (hybrid) in j, i, then $T_{jr}S_{i}{}^{r}$ is pure (hybrid) in j, i.*

(4) *If T_{ji} is pure in j, i and S^{ji} is hybrid in j, i, then we have $T_{ji}S^{ji} = 0$.*

Accordingly (2.2) means that $\nabla_{k}v^{i}$ is pure in k, i and (2.3) means $\nabla_{k}v_{i}$ is hybrid in k, i. If a tensor is pure with respect to all pairs of its indices, then it is called a pure tensor.

Now, it is well known that in a compact K-space with constant scalar carvature, an almost analytic vector v^{i} is decomposed in the form

$$v^{i} = p^{i}+\eta^{i}$$

where p^{i} is killing vector and η^{i} is a vector such as $\eta^{i}=\nabla^{i}r$ for a certain scalar r [13].

Hence, we have

$$\nabla_j v_i + \nabla_i v_j = \nabla_j \eta_i + \nabla_i \eta_j = 2\nabla_j \eta_i.$$

On the other hand since, by (1. 1), it is easily verified that $\nabla_j v_i + \nabla_i v_j$ is hybrid in j, i, $\nabla_j \eta_i$ is hybrid in j, i.

Thus we find that η^i in this theorem is an extended almost analytic contravariant vector. And it is well known that in an Einstein K-space with nonvanishing scalar curvature, an infinitesimal conformal transformation v^i is decomposed in the form

$$v^i = p^i + \eta^i$$

where p^i is a killing vector and η^i is a gradient defining an infinitesimal conformal transformation [1].

From this decomposition and definition of v^i, we have

$$\nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} = 2\nabla_j \eta_i$$

where ρ is a scalar function.

Thus $\nabla_j \eta_i$ is hybrid in j, i and hence η^i in this theorem is also an extended contravariant almost analytic vector.

In K-space, the following properties are known [10], [8].

$$\nabla_j \varphi_{ih} + \varphi_j{}^a \varphi_i{}^b \nabla_a \varphi_{bh} = 0, \quad \nabla_j \varphi_{ih} + \varphi_i{}^a \varphi_h{}^b \nabla_j \varphi_{ab} = 0, \quad \nabla_j \varphi_i{}^j = 0, \tag{2.4}$$

$$R_{ji} - \varphi_j{}^a \varphi_i{}^b R_{ab} = 0, \quad R^*{}_{ji} - \varphi_j{}^a \varphi_i{}^b R^*{}_{ab} = 0, \quad R_{ji} = R^*{}_{ij}, \quad (\nabla_j \varphi_{tl}) \nabla_i \varphi^{tl} = R_{ji} - R^*{}_{ji} \tag{2.5}$$

where $R_{ji} = R_{lji}{}^l$ and $R^*{}_{ji} = \frac{1}{2} \varphi^{ab} R_{abri} \varphi_j{}^r$

$$R - R^* = constant \tag{2.6}$$

where $R \equiv g^{ji} R_{ji}$ and $R^* = g^{ji} R^*{}_{ji}$,

$$\frac{1}{2} \nabla_i R = \nabla^j R_{ji}, \quad \frac{1}{2} \nabla_i R^* = \nabla^j R^*{}_{ji} \tag{2.7}$$

$$\nabla^k (R_{ik} - R^*{}_{ik}) = \frac{1}{2} \nabla_i (R - R^*) = 0. \tag{2.8}$$

Next we shall prove the following two lemmas.

Lemma 2. 1. *In a K-space, for an extended contravariant almost analytic vector v^i for $\lambda = -\frac{1}{4}$, we have*

$$\nabla^k \nabla_k \tilde{v}_i + R_{ir} \tilde{v}^r = 0, \tag{2.9}$$

$$\nabla^k \nabla_k v_i + R^*{}_{ir} v^r = 0 \tag{2.10}$$

where $\tilde{v}_i \equiv \varphi_i{}^r v_r$ and $\tilde{v}^i \equiv g^{ir} \tilde{v}_r = -\varphi_r{}^i v^r$.

Proof. By (2. 2) and (2. 4), the following identity is easily verified

(2.11) $$\nabla_j\tilde{v}_i-\varphi_j{}^a\varphi_i{}^b\nabla_a\tilde{v}_b = 2v_a\nabla_j\varphi_i{}^a.$$

Operating ∇^j to (2.11) and making use of $\nabla^j\varphi_j{}^a=0$, we obtain

(2.12) $$\nabla^j\nabla_j\tilde{v}_i-\varphi_j{}^a(\nabla^j\varphi_i{}^b)\nabla_a\tilde{v}_b-\varphi_j{}^a\varphi_i{}^b\nabla^j\nabla_a\tilde{v}_b = 2v_a\nabla^j\nabla_j\varphi_i{}^a+2(\nabla^jv_a)\nabla_j\varphi_i{}^a.$$

In this place, by Proposition and (2.5), the second term of the left hand side of (2.12) turns to

$$\varphi_j{}^a(\nabla^j\varphi_i{}^b)\nabla_a\tilde{v}_b = \varphi_j{}^a\nabla^j\varphi_i{}^b(v_s\nabla_a\varphi_b{}^s+\varphi_b{}^s\nabla_a v_s)$$
$$= (R_{ia}-R^*{}_{ia})\tilde{v}^a.$$

For the third term, by the Ricci's identities, we have

$$\varphi_j{}^a\varphi_i{}^b\nabla^j\nabla_a\tilde{v}_b=-R^*{}_{ir}\tilde{v}^r.$$

Similarly, for the first term of the right hand side, by $\nabla^j\varphi_j{}^a=0$, we have

$$v_a\nabla^j\nabla_j\varphi_i{}^a = -v_a\nabla^j\nabla_i\varphi_j{}^a = (R_{ia}-R^*{}_{ia})v^a.$$

For the last term, since ∇^jv^a is hybrid in j, a and $\nabla_j\varphi_{ia}$ is pure in j, a, by Proposition, it vanishes. Accordingly from (2.3), (2.9) follows.

It is easy to prove (2.10).

LEMMA 2.2. *In a conformally flat K-space, for an extended contravariant almost analytic vector v^i for $\lambda=-\frac{1}{4}$, we have*

(2.13) $$(R_{ji}-R^*{}_{ji})\nabla^j\tilde{v}^i = 0.$$

PROOF. Since $\nabla_j\varphi_i{}^s$ is pure in j, i and $R^{ji}-R^{*ji}$ is hybrid in j, i, we have

$$\nabla_j\varphi_i{}^s(R^{ji}-R^{*ji})v_s = 0$$

and therefore it follows

(2.14) $$(R^{ji}-R^{*ji})\nabla_j\tilde{v}_i = (R^{ji}-R^{*ji})(v_s\nabla_j\varphi_i{}^s+\varphi_i{}^s\nabla_jv_s) = (R^{ji}-R^{*ji})\varphi_i{}^s\nabla_jv_s.$$

Similarly, since $(\nabla_a\varphi_{js})\nabla^jv^s=0$, operating ∇^a to this equation, we obtain

$$(\nabla^a\nabla_a\varphi_{js})\nabla^jv^s+(\nabla_a\varphi_{js})\nabla^a\nabla^jv^s = 0.$$

Making use of the Ricci's identities and $\nabla^a\varphi_{as}=0$, the last equation becomes

(2.15) $$\varphi_s{}^r(R_{jr}-R^*{}_{jr})\nabla^jv^s+\frac{1}{2}R_{ajls}v^l\nabla^a\varphi^{js} = 0.$$

By definition of conformally flat K-space, the Riemannian curvature tensor has the following form

(2.16) $$R_{ajls}=\frac{1}{2n-2}(g_{as}R_{jl}-g_{js}R_{al}+R_{as}g_{jl}-R_{js}g_{al})-b(g_{as}g_{jl}-g_{js}g_{al}),$$

$$b=\frac{R}{(2n-2)(2n-1)}.$$

But since we have $T^{kji}R_{kjis}=0$ for anti-symmetric tensor T^{kji}, the second term of the left hand side of (2.15) vanishes and so from (2.14) and (2.15) we have (2.13).

By virture of the above two lemmas, we can prove the following

THEOREM 2.1. *In a compact conformally flat K-space of dim. $2n$ ($n>2$), if v^i is an extended contravariant almost analytic vector for $\lambda=-\frac{1}{4}$, then $\tilde{v}^i$ is a Killing vector and v_i is closed.*

PROOF. From (2.16), we have easily

$$(n-1)(R_{ji}-R^*{}_{ji})=(n-2)R_{ji}+(n-1)bg_{ji}$$

where $b>0$ [12].

Multiplying this equation by $\nabla^j\tilde{v}^i$ and making use of (2.13), we have

$$R_{ji}\nabla^j\tilde{v}^i=-\frac{n-1}{n-2}b\nabla_i\tilde{v}^i. \tag{2.17}$$

On the other hand, operating ∇^i to (2.9), we have

$$\nabla_i\nabla^r\nabla_r\tilde{v}^i+\tilde{v}^r\nabla_iR_r{}^i+R_r{}^i\nabla_i\tilde{v}^r=0. \tag{2.18}$$

But since, from (2.16) we have easily $\nabla_iR=0$, by (2.7),

$$\nabla_iR_r{}^i=\frac{1}{2}\nabla_rR=0$$

and by the Ricci's identities we have

$$\nabla_i\nabla^r\nabla_r\tilde{v}^i=\nabla_r\nabla^r\nabla_i\tilde{v}^i+R_{ri}\nabla^r\tilde{v}^i.$$

Thus (2.18) turns to

$$\nabla_r\nabla^r\nabla_i\tilde{v}^i+2R_{ri}\nabla^r\tilde{v}^i=0$$

or putting $\eta=\nabla_i\tilde{v}^i$ and substituting (2.17) in this equation, we have

$$\nabla_r\nabla^r\eta-\frac{2n-2}{n-2}b\eta=0 \quad (b>0). \tag{2.19}$$

Multiplying (2.19) by η, we have

$$\eta\nabla_r\nabla^r\eta-\frac{2n-2}{n-2}b\eta^2=0$$

or

$$\frac{1}{2}\nabla_r\nabla^r\eta^2=(\nabla^r\eta)^2+\frac{2n-2}{n-2}b\eta^2$$

where $(\nabla^r\eta)^2\equiv(\nabla^r\eta)\nabla_r\eta$.

Consequently, by virture of Green's theorem, we obtain

$$\int_M\left[(\nabla^r\eta)^2+\frac{2n-2}{n-2}b\eta^2\right]d\sigma=0$$

from which it follows $\eta=0$ i. e.

$$\nabla_i\tilde{v}^i=0. \tag{2.20}$$

But since our manifold is compact, by (2. 9) and (2. 20), $\tilde{v}^i$ is a killing vector. Thus from $\nabla_j\tilde{v}_i+\nabla_i\tilde{v}_j=0$, (2. 1) and (2. 3), it easily follows that

$$\nabla_jv_i-\nabla_iv_j=0. \qquad \text{Q. E. D.}$$

Recently one of the present authors proved the following

THEOREM 2. 2. *In a compact K-space with constant scalar curvature, if v^i is an extended contravariant almost analytic vector, i. e. if it satisfies*

$$\underset{v}{\pounds}\varphi_j{}^i+\lambda\varphi_j{}^lN_{lt}{}^iv^t=0$$

where λ is a constant satisfying $-\frac{3}{4}\leqq\lambda\leqq 0$, then v^i is decomposed in the form

$$v^i=p^i+\varphi_r{}^iq^r$$

where p^i and q^i are both Killing vectors [14].

3. Extended covariant almost analytic vectors.

Let w_i be a covariant vector in an almost complex space. If w_i satisfies the following equation

$$\varphi_h{}^l\partial_lw_j-\partial_h\tilde{w}_j+w_s\partial_j\varphi_h{}^s=0 \tag{3.1}$$

where $\tilde{w}_j\equiv\varphi_j{}^sw_s$, then w_j is called a covariant almost analytic vector. This is a generalization of a covariant analytic vector in a Kählerian manifold.

From the same point of view as §1, we can define an extended covariant almost analytic vector, that is, in an almost complex manifold, if w_i satisfies the following

$$\varphi_h{}^l\partial_lw_j-\partial_h\tilde{w}_j+w_s\partial_j\varphi_h{}^s+\lambda\varphi_h{}^lN_{lj}{}^sw_s=0 \tag{3.2}$$

where λ is C^∞ scalar function, then we shall call w_i an extended covariant almost analyltic vector.

Particularly, when $\lambda=-\frac{1}{2}$, this definition coincides with Sato's definition [5].

For instance, in an *O-space [2] which we mean a 2n-dim. almost Hermitian manifold such that

$$\nabla_j\varphi_i{}^h+\varphi_j{}^a\varphi_i{}^b\nabla_a\varphi_b{}^h=0,$$

the Nijenhuis tensor $N_{ji}{}^{h}$ becomes

$$N_{ji}{}^{h} = 2\varphi_{j}{}^{r}(\nabla_{r}\varphi_{i}{}^{h} - \nabla_{i}\varphi_{r}{}^{h}) \tag{3.3}$$

and therefore if we put $\lambda = -\frac{1}{2}$, (3.2) turns to

$$\varphi_{h}{}^{l}\nabla_{l}w_{j} - \varphi_{j}{}^{s}\nabla_{h}w_{s} = 0$$

which is equivarent to

$$\nabla_{j}w_{i} + \varphi_{j}{}^{a}\varphi_{i}{}^{b}\nabla_{a}w_{b} = 0. \tag{3.4}$$

Moreover, in an $^{*}O$-space it is easily verified that if w_{i} is an extended covariant almost analytic vector for any scalar function λ, then $\tilde{w}_{i}$ is so.

But it is well-known that a K-space is an $^{*}O$-space and an almost-Kählerian manifold which we mean a 2n-dim. almost Kählerian manifold satisfying the equation

$$\nabla_{j}\varphi_{ih} + \nabla_{i}\varphi_{hj} + \nabla_{h}\varphi_{ji} = 0, \tag{3.5}$$

is also an $^{*}O$-space.

Thus in a K-space or in an almost-Kählerian manifold, an extended covariant almost analytic vector for $\lambda = -\frac{1}{2}$ can be written as in the form (3.4).

In this place, we can prove the following

THEOREM 3.1. *In a compact $^{*}O$-space, let v^{i} be an extended contravariant almost analytic vector for $\lambda = -\frac{1}{2}$, i.e.*

$$\mathop{\pounds}_{v}\varphi_{j}{}^{i} - \frac{1}{2}\varphi_{j}{}^{l}N_{lt}{}^{i}v^{t} = 0 \tag{3.6}$$

and let w_{i} be an extended covariant almost analytic vector for $\lambda = -\frac{1}{2}$, i.e.

$$\nabla_{j}w_{i} + \varphi_{j}{}^{a}\varphi_{i}{}^{b}\nabla_{a}w_{b} = 0, \tag{3.7}$$

then the inner product $v^{i}w_{i}$ is constant.

PROOF. From (1.2) for $\lambda = -\frac{1}{2}$, it follows

$$v^{r}\nabla_{r}\varphi_{j}{}^{i} - \varphi_{j}{}^{r}\nabla_{r}v^{i} + \varphi_{r}{}^{i}\nabla_{j}v^{r} - \frac{1}{2}\varphi_{j}{}^{s}N_{sr}{}^{i}v^{r} = 0 \tag{3.8}$$

and substituting (3.3), (3.8) becomes

$$\varphi_{j}{}^{r}\nabla_{r}v^{i} - \varphi_{r}{}^{i}\nabla_{j}v^{r} - v^{r}\nabla_{j}\varphi_{r}{}^{i} = 0. \tag{3.9}$$

Multiplying (3.9) by $-\varphi_{k}{}^{j}$, we have

$$\nabla_{k}v^{i} + \varphi_{k}{}^{j}\varphi_{r}{}^{i}\nabla_{j}v^{r} + v^{r}\varphi_{k}{}^{j}\nabla_{j}\varphi_{r}{}^{i} = 0. \tag{3.10}$$

On the other hand,

(3.11)
$$\begin{aligned}&\nabla_k(v^i w_i)+\varphi_k{}^l\nabla_l(v^i w_a\varphi_i{}^a)\\&= v^i\nabla_k w_i+w_i\nabla_k v^i+\varphi_k{}^l(\nabla_l v^i)w_a\varphi_i{}^a+\varphi_k{}^l v^i(\nabla_l w_a)\varphi_i{}^a+\varphi_k{}^l v^i w_a\nabla_l\varphi_i{}^a\\&= (\nabla_k v^i+\varphi_k{}^l\varphi_s{}^i\nabla_l v^s+\varphi_k{}^l v^s\nabla_l\varphi_s{}^i)w_i+v^i\nabla_k w_i+v^i\varphi_k{}^l\varphi_i{}^a\nabla_l w_a.\end{aligned}$$

For the last term, by (3.7), we have

(3.12)
$$v^i\varphi_k{}^l\varphi_i{}^a\nabla_l w_a = v^i\varphi_k{}^l(-\varphi_l{}^s\varphi_a{}^t\nabla_s w_t)\varphi_i{}^a = -v^i\nabla_k w_i$$

and hence by (3.10) and (3.12), from (3.11) it follows

(3.13)
$$\nabla_k(v^i w_i)+\varphi_k{}^l\nabla_l(v^i w_a\varphi_i{}^a) = 0$$

or putting $f\equiv v^i w_i$ and $g=v^i\varphi_i{}^s w_s$, we have

(3.14)
$$\nabla_j f+\varphi_j{}^l\nabla_l g = 0.$$

Operating ∇^j to (3.14), and making use of $\nabla^j\varphi_j{}^r=0$, we have

(3.15)
$$\nabla^j\nabla_j f+\varphi^{jl}\nabla_j\nabla_l g = 0.$$

But since φ^{jl} is anti-symmetric in j, l and $\nabla_j\nabla_l g$ is symmetric in j, l, we have $\varphi^{jl}\nabla_j\nabla_l g=0$ and hence

$$\nabla^j\nabla_j f = 0.$$

Thus, by virture of Green's theorem, we have

$$f = \textit{constant}.$$ Q. E. D.

4. Properties of extended covariant almost analytic vectors in almost Kählerian manifold.

Let w_i be an extended covariant almost analytic vector for $\lambda=-\frac{1}{2}$ in an almost Kählerian manifold, i. e.

$$\nabla_j w_i+\varphi_j{}^a\varphi_i{}^b\nabla_a w_b = 0.$$

Operating ∇^j to this equation, we easily have

(4.1)
$$\nabla^j\nabla_j w_i-R^*{}_i{}^s w_s-\varphi_i{}^s(\nabla^j\varphi_s{}^r)\nabla_j w_r = 0$$

and operating ∇^k to (3.5), we have

$$\nabla^k\nabla_k\varphi_{ji} = \nabla^k\nabla_j\varphi_{ki}-\nabla^k\nabla_i\varphi_{kj}.$$

Making use of the Ricci's identies and $\nabla^k\varphi_{kj}=0$, the left hand side of the last equation can be written as

(4.2)
$$\nabla^k\nabla_k\varphi_{ji} = \varphi^{pq}R_{pqji}+R_j{}^r\varphi_{ri}-R_i{}^r\varphi_{rj}\ [9].$$

On the other hand, operating ∇^k to the following

$$\nabla_k \tilde{w}_i = w_s \nabla_k \varphi_i{}^s + \varphi_i{}^s \nabla_k w_s$$

and making use of the Ricci's identities and (4.2), we have

$$\nabla^k \nabla_k \tilde{w}_i = (\varphi^{pq} R_{pqil} + R_i{}^r \varphi_{rl} - R_l{}^r \varphi_{ri}) w^l$$
$$+ 2(\nabla^r \varphi_i{}^l) \nabla_r w_l + \varphi_i{}^l \nabla^r \nabla_r w_l$$

from which it follows

$$(4.3) \qquad (\nabla^r \nabla_r \tilde{w}_i - R_{ri} \tilde{w}^r) \tilde{w}^i = (\nabla^r \nabla_r w_i + R_{ri} w^r) w^i - 2R^*{}_{ri} w^r w^i - 2\nabla^r w^t (\nabla_r \varphi_{it}) \varphi_s{}^i w^s$$

where $\tilde{w}_i = \varphi_i{}^s w_s$ and $\tilde{w}^i = -\varphi_s{}^i w^s$.

And from (4.1) we have

$$(4.4) \qquad \nabla^r w^t (\nabla_r \varphi_{st}) \varphi_i{}^s = \nabla^j \nabla_j w_i - R^*{}_i{}^s w_s.$$

Then substituting (4.4) in the last term of the right hand side of (4.3), we have

$$(\nabla^r \nabla_r \tilde{w}_i - R_{ri} \tilde{w}^r) \tilde{w}^i = (\nabla^r \nabla_r w_i + R_{ri} w^r) w^i - 2R^*{}_{ri} w^r w^i - 2(\nabla^j \nabla_j w_r - R^*{}_r{}^s w_s) w^r$$

from which we have

$$(4.5) \qquad (\nabla^r \nabla_r \tilde{w}_i - R_{ri} \tilde{w}^r) \tilde{w}^i + (\nabla^r \nabla_r w_i - R_{ri} w^r) w^i = 0.$$

But the following integral formulas are well-known [15]:

$$(4.6) \qquad \int_M [(\nabla^r \nabla_r w_i - R_{ri} w^r) w^i + S(w)] d\sigma = 0,$$

$$(4.7) \qquad \int_M [(\nabla^r \nabla_r \tilde{w}_i - R_{ri} \tilde{w}^r) \tilde{w}^i + S(\tilde{w})] d\sigma = 0$$

where M denotes our almost Kählerian manifold and

$$S(w) = \frac{1}{2} (\nabla^s w^r - \nabla^r w^s)(\nabla_s w_r - \nabla_r w_s) + (\nabla^r w_r)^2,$$

$$S(\tilde{w}) = \frac{1}{2} (\nabla^s \tilde{w}^r - \nabla^r \tilde{w}^s)(\nabla_s \tilde{w}_r - \nabla_r \tilde{w}_s) + (\nabla^r \tilde{w}_r)^2.$$

Consequently, forming the sum (4.6)+(4.7) and by (4.5), we have $S(w) = 0$ and $S(\tilde{w}) = 0$.

Thus we have the following

THEOREM 4.1. *In a compact almost Kählerian manifold, if w_i is an extended covariant almost analytic vector for $\lambda = -\frac{1}{2}$, then w_i and $\tilde{w}_i$ are both harmonic vectors.*

Moreover, the following theorems are well-known.

THEOREM 4.2. (S. Tachibana [9]) *In a compact almost Kählerian manifold a necessary and sufficient condition in order that a vector w_i be a covariant almost analytic vector is that w_i and $\tilde{w}_i$ are both harmonic.*

THEOREM 4.3. *In an *O-space, a vector w_i is covariant almost analytic if and only if*

$$\nabla_j w_i + \varphi_j{}^a \varphi_i{}^b \nabla_a w_b = 0, \quad N_{ji}{}^r w_r = 0 \quad ([6], [16]).$$

By virtue of the above two theorems, we have

THEOREM 4.4. *In a compact almost Kählerian manifold, covariant almost analytic vector coincides with an extended covarient almost analytic vector for $\lambda = -\dfrac{1}{2}$.*

Would it be possible to prove the same theorem as Theorem 4.1 in a compact K-space?

5. Extended covariant almost analytic tensors.

Let $T_{j_q \dots j_1}$ be a pure tensor in an almost complex manifold. If $T_{j_q \dots j_1}$ satisfies the following equation

$$(5.1) \qquad \Phi_h T_{j_q \dots j_1} \equiv \varphi_h{}^l \partial_l T_{j_q \dots j_1} - \partial_h \tilde{T}_{j_q \dots j_1} + \sum_{r=1}^{q} (\partial_{j_r} \varphi_h{}^l) T_{j_q \dots l \dots j_1} = 0$$

where $\tilde{T}_{j_q \dots j_1} \doteq \varphi_{j_r}{}^l T_{j_q \dots l \dots j_1} (r = 1, 2, \cdots\cdots q)$, then $T_{j_q \dots j_1}$ is called an covariant almost analytic tensor. This is a generalization of covariant analytic tensor in a Kählerian manifold.

As in the case of an extended almost analytic vector, a pure tensor is called an extended covariant almost analytic tensor, if it satisfies the following

$$(5.2) \qquad \Phi_h T_{j_q \dots j_1} + \sum_{r=1}^{q} \lambda_r \varphi_h{}^l N_{l j_r}{}^t T_{j_q \dots t \dots j_1} = 0$$

where $\lambda_r (r = 1, 2, \cdots\cdots, q)$ are C^∞ scalar functions.

Now, we shall assume we are in an *O-space and consider the case when $\lambda_1 = \cdots = \lambda_{q-1} = 0$, $\lambda_q = -\dfrac{1}{2}$.

From (5.2) we have

$$(5.3) \quad \varphi_h{}^l \nabla_l T_{j_q \dots j_1} - \nabla_h (\varphi_{j_q}{}^l T_{l j_{q-1} \dots j_1}) + \sum_{r=1}^{q} (\nabla_{j_r} \varphi_h{}^l) T_{j_q \dots l \dots j_1} - \frac{1}{2} \varphi_h{}^l N_{l j_q}{}^t T_{t j_{q-1} \dots j_1} = 0$$

and substituting (3.3) in the last equation, we have

$$\varphi_h{}^l \nabla_l T_{j_q \dots j_1} - \varphi_{j_q}{}^l \nabla_h T_{l j_{q-1} \dots j_1} + \sum_{r=1}^{q-1} (\nabla_{j_r} \varphi_h{}^t) T_{j_q \dots t \dots j_1} = 0$$

or multiplying this equation by $\varphi_s{}^{j_q}$, we have

$$\nabla_h T_{s j_{q-1}\dots j_1} + \varphi_h{}^l \varphi_s{}^{j_q} \nabla_l T_{j_q\dots j_1} + \sum_{r=1}^{q-1} \varphi_s{}^{j_q} (\nabla_{j_r} \varphi_h{}^t) T_{j_q\dots t\dots j_1} = 0$$

or changing indices, we hav

$$\nabla_h T_{j_q\dots j_1} + \varphi_h{}^l \varphi_{j_q}{}^s \nabla_l T_{s j_{q-1}\dots j_1} + \sum_{r=1}^{q-1} \varphi_{j_q}{}^t (\nabla_{j_r} \varphi_h{}^l) T_{t j_{q-1}\dots l\dots j_1} = 0. \tag{5.4}$$

And from (5.3) it follows

$$\nabla_h \tilde{T}_{j_q\dots j_1} = \varphi_h{}^l \nabla_l T_{j_q\dots j_1} + \sum_{r=1}^{q-1} (\nabla_{j_r} \varphi_h{}^l) T_{j_q\dots l\dots j_1} + (\nabla_h \varphi_{j_q}{}^l) T_{l j_{q-1}\dots j_1}. \tag{5.5}$$

Moreover if we assume $T_{j_q\dots j_1}$ is anti-symmetric, and put

$$\nabla_{[h} T_{j_q\dots j_1]} \equiv \nabla_h T_{j_q\dots j_1} - (\nabla_{j_q} T_{h j_{q-1}\dots j_1} + \nabla_{j_{q-1}} T_{j_q h j_{q-2}\dots j_1} + \cdots\cdots + \nabla_{j_1} T_{j_q\dots j_2 h}),$$

then we find

$$\nabla_{[h} T_{j_q\dots j_1]} = \nabla_{[j_q} T_{j_{q-1}\dots j_1 h]} \quad \text{for even } q, \tag{5.6}$$

$$\nabla_{[h} T_{j_q\dots j_1]} = -\nabla_{[j_q} T_{j_{q-1}\dots j_1 h]} \quad \text{for odd } q.$$

On the other hand, from the following

$$\nabla_{j_q} \tilde{T}_{h j_{q-1}\dots j_1} = \varphi_h{}^l \nabla_{j_q} T_{l j_{q-1}\dots j_1} + (\nabla_{j_q} \varphi_h{}^l) T_{l j_{q-1}\dots j_1},$$

$$\nabla_{j_{q-1}} \tilde{T}_{j_q h j_{q-2}\dots j_1} = \varphi_h{}^l \nabla_{j_{q-1}} T_{j_q l j_{q-2}\dots j_1} + (\nabla_{j_{q-1}} \varphi_h{}^l) T_{j_q l j_{q-1}\dots j_1},$$

$$\cdots$$

$$\nabla_{j_1} \tilde{T}_{j_q\dots j_2 h} = \varphi_h{}^l \nabla_{j_1} T_{j_q\dots j_2 l} + (\nabla_{j_1} \varphi_h{}^l) T_{j_q\dots j_2 l}$$

and (5.5), we have

$$\nabla_{[h} \tilde{T}_{j_q\dots j_1]} = \varphi_h{}^l \nabla_{[l} T_{j_q\dots j_1]} + (\nabla_h \varphi_{j_q}{}^l - \nabla_{j_q} \varphi_h{}^l) T_{l j_{q-1}\dots j_1} \tag{5.7}$$

and similarly

$$\nabla_{[j_q} \tilde{T}_{j_{q-1}\dots j_1 h]} = \varphi_{j_q}{}^l \nabla_{[l} T_{j_{q-1}\dots j_1 h]} + (\nabla_{j_q} \varphi_{j_{q-1}}{}^l - \nabla_{j_{q-1}} \varphi_{j_q}{}^l) T_{l j_{q-1}\dots j_1}. \tag{5.8}$$

When q is even, if we notic that $\tilde{T}_{j_q\dots j_1}$ is also anti-symmetric, by (5.6) we have

$$\begin{aligned}(\varphi_h{}^l \nabla_{[l} T_{j_q\dots j_1]} - \varphi_{j_q}{}^l \nabla_{[l} T_{j_{q-1}\dots j_1 h]}) - (\nabla_{j_q} \varphi_{j_{q-1}}{}^l - \nabla_{j_{q-1}} \varphi_{j_q}{}^l) T_{l j_{q-2}\dots j_1 h} \\ = (\nabla_{j_q} \varphi_h{}^l - \nabla_h \varphi_{j_q}{}^l) T_{l j_{q-1}\dots j_1}\end{aligned} \tag{5.9}$$

and if we multiply the both sides of (5.9) by

$$*O_{ab}^{j_q h} \equiv \frac{1}{2} (\delta_a{}^{j_q} \delta_b{}^h + \varphi_a{}^{j_q} \varphi_b{}^h),$$

then the left hand side vanishes and hence we have

$$0 = (\nabla_a \varphi_b{}^l - \nabla_b \varphi_a{}^l) T_{l j_{q-1}\dots j_1}. \tag{5.10}$$

Similarly, when q is odd and $q \neq 1$, we have (5.10).

But in an *O-space, we have the following

THEOREM 5.1. *In an *O-space, a covariant pure tensor* $T_{j_q\dots j_1}$ *is almost analytic if and only if*

$$\nabla_h T_{j_q\dots j_1}+\varphi_h{}^l\varphi_{j_q}{}^s\nabla_l T_{sj_{q-1}\dots j_1}+\sum_{r=1}^{q-1}\varphi_{j_q}{}^t(\nabla_{j_r}\varphi_h{}^l)T_{tj_{q-1}\dots l\dots j_1}=0,$$

$$(\nabla_{j_q}\varphi_h{}^l-\nabla_h\varphi_{j_q}{}^l)T_{lj_{q-1}\dots j_1}=0 \ [7].$$

Thus we have the following

THEOREM 5.2. *In an *O-space, an anti-symmetric extended covariant almost analytic tensor* $T_{j_q\dots j_1}$ *for* $\lambda_1=\cdots\cdots=\lambda_{q-1}=0$, $\lambda_q=\frac{1}{2}$, *and* $b\geqq 2$ *coincides with an anti-symmetric covariant almost analytic tensor.*

6. Extended almost analytic tensors of mixed type.

Let $T_{j_q\dots j_1}{}^{i_p\dots i_1}$ or briefly $T_{(j)}{}^{(i)}$ be a pure tensor of type (p, q) in an almost complex manifold. If $T_{(j)}{}^{(i)}$ satisfies the following equation

$$\Phi_h T_{(j)}{}^{(i)}\equiv\varphi_h{}^l\partial_l T_{(j)}{}^{(i)}-\partial_h\widetilde{T}_{(j)}{}^{(i)}+\sum_{r=1}^{q}(\partial_{j_r}\varphi_h{}^l)T_{j_q\dots l\dots j_1}{}^{(i)}$$

$$+\sum_{r=1}^{q}(\partial_h\varphi_l{}^{i_r}-\partial_l\varphi_h{}^{i_r})T_{(j)}{}^{i_p\dots l\dots i_1}=0,$$

where $\widetilde{T}_{(j)}{}^{(i)}\equiv\varphi_{j_q}{}^l T_{lj_{q-1}\dots j_1}{}^{(i)}$, then it is called an almost analytic tensor and this is a generalization of an analytic tensor in a Kählerian manifold [3], [7], [11].

As in the preceding paragraph, for a pure contravariant tensor and pure tensor of mixed type we can also define an extended almost analytic tensor but since we have not yet obtained any remarkable results on these tensors, we conclude this last section with the treatment only for the tensor of type (1.1).

Let $T_j{}^i$ be a pure tensor of type (1.1). If $T_j{}^i$ satisfies the following

$$\Phi_h T_j{}^i+\lambda\varphi_h{}^l N_{lt}{}^i T_j{}^t+\mu\varphi_h{}^l N_{lj}{}^t T_t{}^i=0 \tag{6.1}$$

where λ and μ are C^∞ scalar functions, then it is called an extended almost analytic tensor.

Suppose that we are in an *O-space and $\lambda=\frac{1}{2}$, $\mu=-\frac{1}{2}$, then from (6.1) it follows

$$\varphi_h{}^l\nabla_l T_j{}^i-\varphi_j{}^l\nabla_h T_l{}^i=0$$

or multiplying the last equation by $\varphi_k{}^h$, we have

$$\nabla_k T_j{}^i+\varphi_k{}^h\varphi_j{}^l\nabla_h T_l{}^i=0. \tag{6.2}$$

But in an *O-space, by its definition, we have

$$\nabla_k \varphi_j{}^i + \varphi_k{}^h \varphi_j{}^l \nabla_h \varphi_l{}^i = 0.$$

Thus, as an example of an extended almost analytic tensor of type (1, 1) for $\lambda = \frac{1}{2}$, $\mu = -\frac{1}{2}$ in an *O-space, we have structure tensor $\varphi_j{}^i$.

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References

1. A. LICHNEROWICZ: *Some problems on transformations of Riemannian and Kählerian manifolds*, Mimeographed notes, Princeton (1965).
2. S. KOTÔ: *Some theorems on almost complex spaces*, J. Math. Soc., Japan, 12 (1960), 422–433.
3. ————: *On almost analytic tensors in almost complex spaces*, Tensor New series, 12 (1962), 110–132.
4. I. SATÔ: *Almost analytic vector fields in almost complex manifolds*, Tôhoku Math. J˙, 17(1965), 185–199.
5. ————: *Almost analytic tensor fields in almost complex manifolds*, Tensor New Series, 17 (1966), 105–119.
6. S. SAWAKI: *On almost-analytic vectors in almost-Hermitian spaces*, J. Fac. Sci. Niigata Univ., 3 (1960), 23–32.
7. ————: *On almost analytic tensors in *O-spaces*, Tôhoku Math. J., 13 (1961), 154–178.
8. ————: *On the Matsushima's theorem in a compact Einstein K-space*, Tôhoku Math. J., 13 (1961), 455–465.
9. S. TACHIBANA: *On almost-analytic vectors in almost-Kählerian manifolds*, Tôhokn Math. J., 11 (1959). 247–265.
10. ————: *On almost-analytic vectors in certain almost-Hermitian manifolds*, Tôhoku Math. J., 11 (1959), 351–363.
11. ————: *Analytic tensor and its generalization*, Tôhoku Math. J., 12 (1950), 208–221.
12. ————: *On automorphisms of conformally flat K-spaces*, J. Math. Soc. Japan, 13 (1961), 183–188.
13. K. TAKAMATSU: *On a decomposition of an almost-analytic vector in a K-space with constant scalar curvature*, Tôhoku Math. J., 16 (1964). 72–80.
14. ————: *On a decomposition of an extended almost analytic vector in a K-space with constant scalar curvature*, to appear.
15. K. YANO: *The theory of Lie derivative and its applications*, Amsterdam (1967).
16. K. YANO and M. AKO: *Almost analytic vectors in almost complex spaces*, Tôhoku Math. J., 13 (1961), 24–45.