On representations of 1-homology classes of closed surfaces

By

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Let M^{2n} be a simply connected differentiable manifold, and let $\xi \in \pi_n (M^{2n})$ be a given homotopy class of maps: $S^n \rightarrow M^{2n}$. It is known that if $n > 2$ the class ξ can be represented by a differentiable imbedding $f: S^n \rightarrow M^{2n}$ and that if $n=2$ some classes cannot be represented.

Kervaire and Milnor have pointed out in [1] that some element of $H_2 (M^4)$ cannot be reprsented by any differentiable imbedding $f : S^2 \rightarrow M^4$, accordingly the corresponding homotopy class cannot be represented, too.

In this paper we consider an orientable closed surface M with genus m, and study whether $\xi \in H_1 (M ; Z)$ can be represented by a differentiably imbedded 1-sphere or not. (It is well known that M has a unique differentiable structure.)

§ 1. The case of $m=1$

Let T be $S^1 \times S^1$, and α, $\beta \in H_1 (T ; Z)$ be the standard generators.

THEOREM 1. *The non-zero homology class $p\alpha + q\beta$ can be represented by a differentiably imbedded 1-sphere if and only if G. C. M. $(p, q)=1$.*

Proof. $T = S^1 \times S^1$ can be represented by (x, y)-plane identifying (x, y) with $(x+i, y+j)$, $(i, j=0, \pm 1, \pm 2, \ldots)$. Putting the base point at $(0, 0)$, $p\alpha + q\beta$ is represented by a curve runs from $(0, 0)$ to (p, q).

If *G. C. M.* $(p, q)=1$, the segment which runs from $(0, 0)$ to (p, q) is an image of a differentiable mapping of a 1-sphere and has no self intersection. Thus our condition is sufficiant.

When $p=q \geqq 2$, let g be an arbitrary curve which runs from $(0, 0)$ to (p, p). By the identification mentioned in the first part of this proof, the curve g which is translated from g in parallel with the line $y=x$ by $\sqrt{2}$ represent g itself. To prove g is not an image of an imbedding we assume that g has no self intersection. Then g and g' have no intersection in the (x, y)-plane without identification.

Now we extend g and g' by translating them parallel with the line $y=x$ by $\sqrt{2}\, np$ $(n=0, \pm 1, \pm 2, \ldots)$, and denote them by G and G'. Then G and G' have no intersection, accordingly G is contained in an open component of (x, y)-plane separated by G'. Therefore the maximum lengh of oriented perpendiculars from points of G to $y=x$ is different to that related to G'. This contradicts to the fact that g' represent g.

This shows that $p\alpha+q\beta$ cannot be represented by an imbedded 1-sphere.

When $G.C.M.$ $(p, q)=m$, let $p=mp'$ and $q=mq'$. Then we can use the rectangles with faces of length p' and q', in place of the unit squares in the preceding paragraph, and similarly prove that $p\alpha+q\beta$ cannot be represented by an imbedded 1-sphere.

§ 2. The graphic representations

In this section we give some graphic representations of $p\alpha+q\beta\in H_1$ $(T\ ;\ Z)$, where α, β and T are those given in § 1.

We construct T', the connected sum of T and S^2, as follows; take away the neighborhood U of the base point in T and the interior of a disk D in S^2 and identify the boundary of U with the boundary K of D. Then T' is homeomorphic to T, accordingly [2] diffeomorphic to T.

We regard $T-U$ to be inside of K and S^2-D to be outside of K, and indicate by vectors which lie across K the differentiable curves which lie across K and have no self intersection and no mutual intersection in $T-U$.

If this graphic representation is consist of p parallel vectors and q other vectors which are orthogonal to the formers, we call it a graph of type (p, q). (Fig. 1).

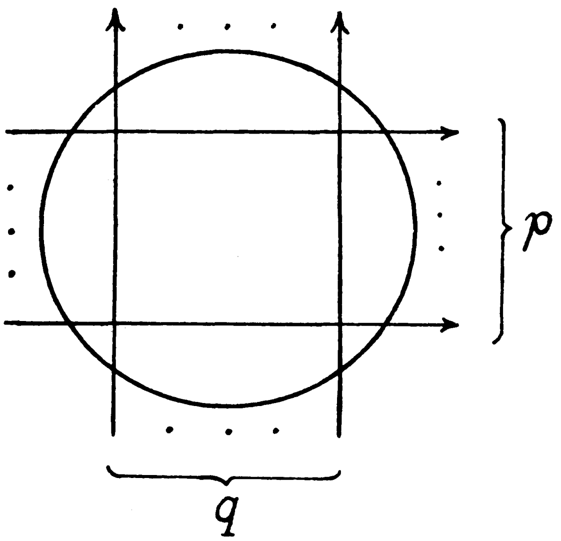

Fig. 1.

We put the base point x_0 of T' outside of K, i. e. in S^2-D, then combining every initial and end points of all vectors to x_0 we obtain a representation of some element of $H_1(T'\,;Z)=H_1(T\,;Z)$. In this case we say that the element is represented by a graph of type (p, q).

LEMMA 1. *When* $G.C.M.$ $(p, q)=r$, $p\alpha+q\beta$ *can be represented by a graph of type* (r, o).

Proof. As easily seen, $p\alpha+q\beta$ is represented by a graph of type (p, q). (Fig. 2).

At first we assume that $p>q>0$, and let $p=qq_1+p_1(0\leqq p_1<q)$. Combine the end points of q vertical vectors with the initial points of upper q horizontal vectors as indicated in Fig. 3.

Using the theorem in appendix, Fig. 3 may be transformed into Fig. 4, accordingly into Fig. 5, without changing the homology class represented by it.

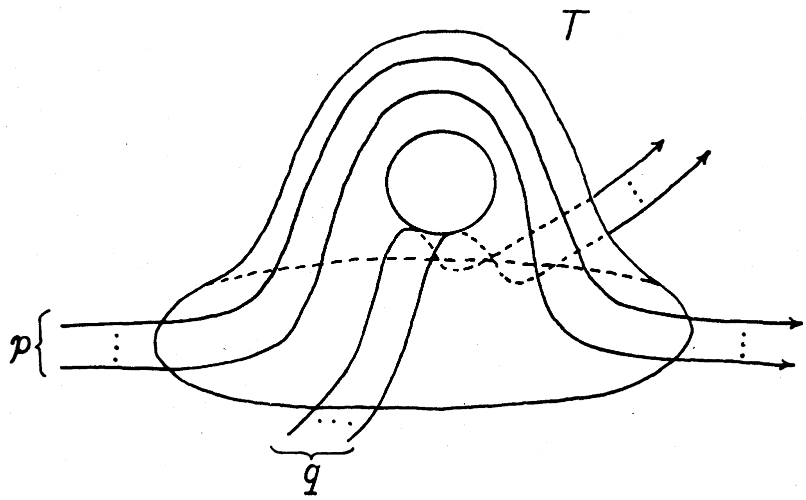

Fig. 2.

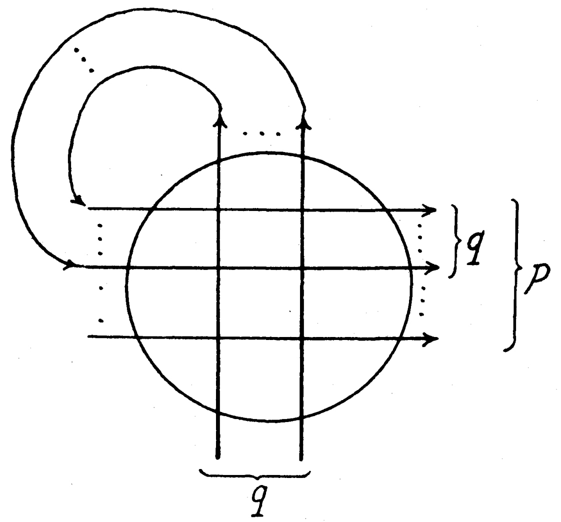

Fig. 3.

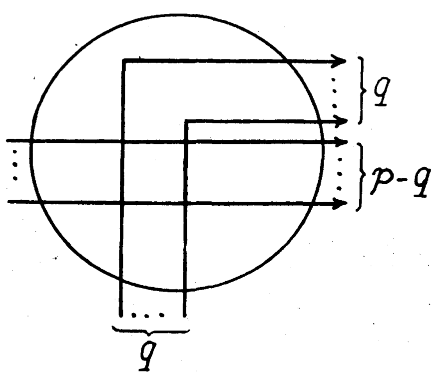

Fig. 4.

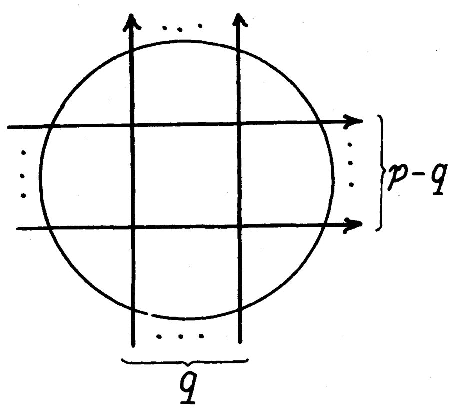

Fig. 5.

Repeating similar operations, Fig. 1 may be transformed into the graph of type (p_1, q).

Such transformation corresponds to the first step of Euclidean algorithm. Corresponding to each step of Euclidean algorithm, there is a transformation of the graph. Hence we can transform the graph of type (p, q) into the graph of type $(r, 0)$ without changing the homology class. (Note that the graph of type $(0, r)$ may be considered to be that of type $(r, 0)$).

Thus $p\alpha+q\beta$ can be represented by a graph of type $(r, 0)$.

In the cases when the above condition $p>q>0$ does not hold, the proofs are similar or trivial.

LEMMA 2. *When G. C. M.* $(p, q)=r$, $p\alpha+q\beta$ *can be represented by a graph of type* (r, r).

Proof. When $p=0$ and $q=r$ (or $p=r$ and $q=0$), $p\alpha+q\beta=r\beta$ (or $r\alpha$) is naturally represented by a graph of type $(r, 0)$ and it can be transformed into a graph of type (r, r) without changing the corresponding class as indicated in Fig. 6.

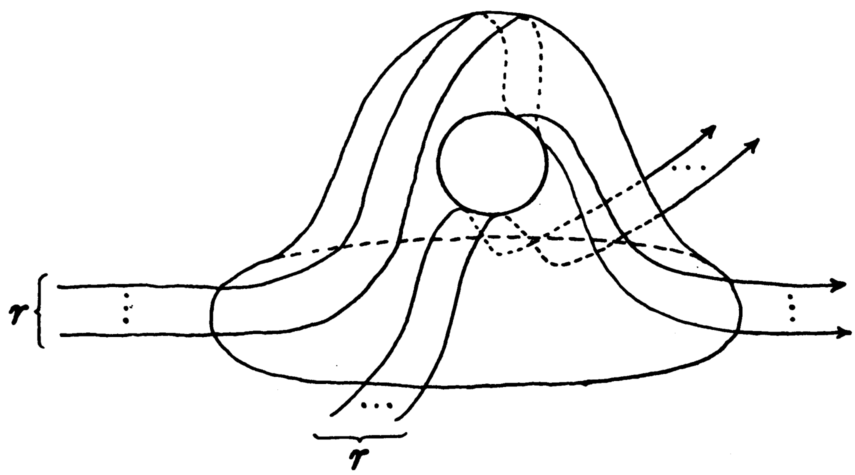

Fig. 6.

When $p\neq 0$ and $q\neq 0$, we have passed through the graph of type (r, r) in the process of the transformation of the graph which is considered in the proof of lemma 1.

LEMMA 3. *When G. C. M.* $(p, q)=1$, *for any integer* n, $p\alpha+q\beta$ *can be represented by a graph of type* $(n+1/n, 1)$, *i. e. the graph indicated in Fig.* 7.

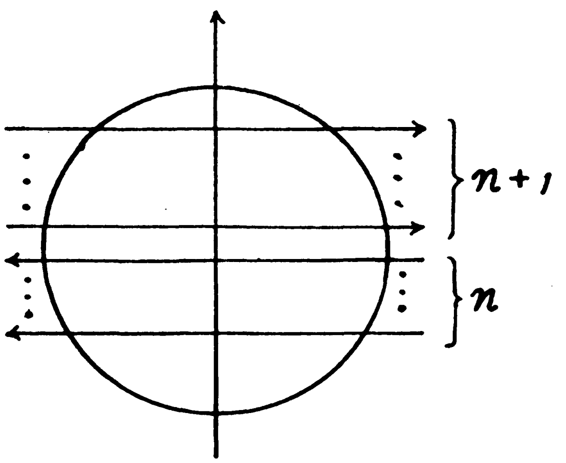

Fig. 7.

Proof. In the graph of type $(1, 1)$ obtained in lemma 2, give n curves along the curve which is represented by the horizontal vector and give other n curves along them with inverse orientation. Then the graph becomes of type $(n+1/n, 1)$ and the corresponidng homology class does not change.

§ 3. **The general case**

Let M be an orientable closed surface with genus m, and let α_i, β_i $(i=1,\dots,m)$ be the standard generators of $H_1(M; Z)$.

THEOREM 2. *The non-zero homology class* $\xi=\sum_{i=1}^{m}(p_i\alpha_i+q_i\beta_i)$ *can be represented by a differentiably imbedded* 1-*sphere if* $G. C. M. (p_i, q_i)=1$ *for some* i.

Proof. We may consider M to be as follows. Let K_i $(i=1,\dots,m)$ be circles which have mutually no intersection and stand side by side along the equator on S^2. At each K_i, we take away the interior and attach to it a holed torus T_i by the way used in § 2. Then the resulting connected sum M is an orientable closed surface with genus m, and has an unique differentiable structure.

Let $r_i=0$ if $p_i=q_i=0$ and $r_i=G. C. M. (p_i, q_i)$ otherwise, and let $\max_i r_i=n$.

Now we assume $r_1=1$. At each K $(i=2,\dots,m)$, we represent $p_i\alpha_i+q_i\beta_i$ by a graph of type $(r_i, 0)$ by lemma 1. At K_1, we represent $p_1\alpha_1+q_1\beta_1$ by a graph of type $(n/n-1, 1)$ by lemma 3.

Putting the base point x_0 of M at the north pole of S^2, we give a differentiably imbedded 1-sphere which combines x_0 and all vectors as follows. (Fig. 8).

We start from x_0 and pass through uppermost horizontal vectors of K_i $(i=1,\dots,m)$ in order, if $r_i=0$ go round the south side of K_i. After going round one time, we go towards the initial point of the second vector of K_1 and pass through all second vectors as above.

After repeating such process n times, we turn towards the initial point of the $(n+1)$-th vector of K_1 through the south hemisphere.

Then we path through the $(n-1)$ inversely oriented vectors spirally as in Fig. 8, and return to x_0 through the only vertical vector.

As easily seen, these process may be carried out such as the resulting curve has no self intersection.

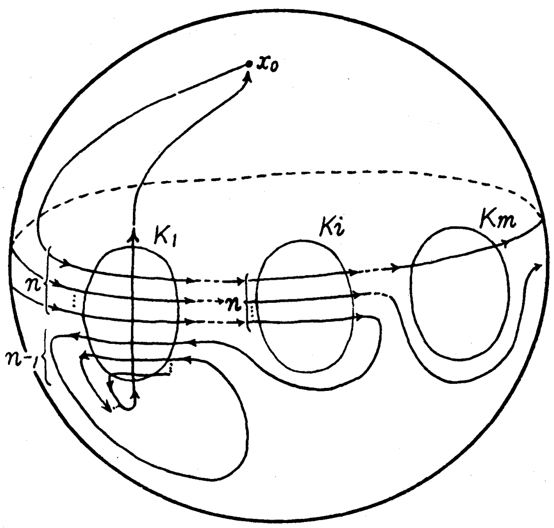

Fig. 8.

By the theorem in appendix this curve may be made differentiable. Thus we have represented ξ by a differentiably imbedded 1-sphere.

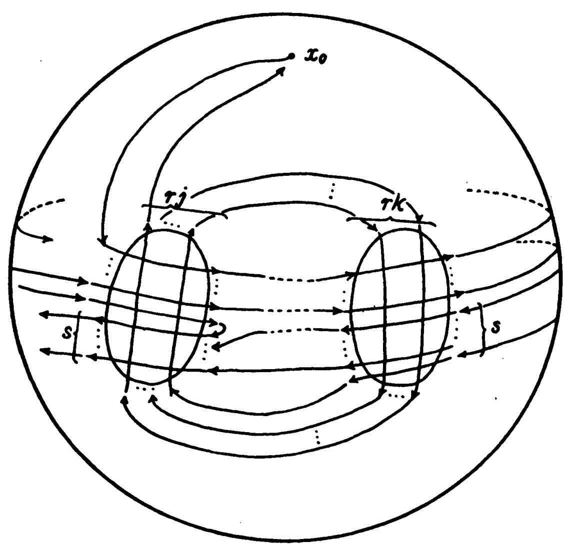

Fig. 9.

THEOREM 3. *ξ can be represented by a differentiably imbedded 1-sphere if $r_j=1+r_k$ for some j and k, where r_i $(i=1,\dots,m)$ is defined as in the proof of theorem 2.*

Proof. Let $s=max\ (0, (\max_{i\neq j,k} r_i-r_k))$. We represent $p_i\alpha_i+q_i\beta_i$ by a graph of type $(r_i, 0)$ for $i\neq j, k$.

For $i=j$ or k we represent them by the graphs of type $(r_j+s/s, r_j)$ and that of type $(r_k+s/s, 0/r_k)$ respectively. Then passing through the course indicated in Fig. 9 we obtain the required 1-sphere.

Appendix

Using the function given by Eelles, we obtain the following theorem.

THEOREM. *A polygonal line which has finite vertices and no self intersection may be approximated by a differentiable curve which has no seif intrsection.*

Proof. When two half lines $y=mx(x\leqq 0)$ and $y=m(\varepsilon-x)$ $(x\geqq\varepsilon>0)$ are given, we can differentiably connect them as follows.

Define $y=g(x)$ $(0\leqq x\leqq\varepsilon)$ by

$$g(x)=\begin{cases} m\left(1-f\left(\frac{2x}{\varepsilon}\right)\right) & \left(0\leqq x\leqq\frac{\varepsilon}{2}\right) \\ -mf\left(\frac{2x}{\varepsilon}-1\right) & \left(\frac{\varepsilon}{2}<x\leqq\varepsilon\right) \end{cases}$$

where

$$f(x)=\int_0^x e^{-\frac{1}{t(1-t)}}\,dt \Big/ \int_0^1 e^{-\frac{1}{t(1-t)}}\,dt(0\leqq x\leqq 1).$$

Let

$$h(x)=\begin{cases} mx & (x\leqq 0) \\ \int_0^x g(t)dt & (0<x<\varepsilon) \\ m(\varepsilon-x) & (x\geqq\varepsilon) \end{cases}$$

Then $y=h(x)$ $(-\infty<x<\infty)$ is differentiable as desired.

Using this fact each angle of our polygonal line may be replaced by a differentiable curve with no self intersection in arbitrary small neighborhood of the vertix.

References

[1] Kervaire, M., and J. Milnor: *On 2-spheres in 4-manifolds*, Proc. N. A. S. Vol. 47, 1961.
[2] Munkres, J. R: *Some applications of triangulation theorems (thesis)*, University of Michigan, 1955.