PREREQUISITES

The essential prerequisites for these notes are contained in Bourbaki, "Commutative Algebra", Chapters I through IV. Results from Chapters V through VII will sometimes (but not often) be referred to. We shall denote them throughout by B.C.A., so that when we write, say, Proposition 4, B.C.A., III, 3, 2 we mean proposition 4 to be found in Bourbaki's "Commutative Algebra", Chapter III, §3, no 2.

We begin by recalling some of the elementary fundamental notions of Commutative Algebra and modern Algebraic Geometry. No attempt at proofs will be made here, most proofs being available either from the above mentioned chapters of Bourbaki, or from Grothendieck's EGA.

We consider only commutative rings A with unit element, and only ring homomorphisms such that $1 \rightsquigarrow 1$.

Unless otherwise specified, the rings considered will be noetherian. This means that the set of ideals of A satisfies the ascending chain condition, or equivalently, that every ideal of A admits a finite basis.

We call A semi-local if it has a finite number of maximal ideals. If A has a unique maximal ideal (when no danger of ambiguity exists, ideal will always mean proper ideal), A is said to be a local ring.

We call A a Jacobson ring if every prime ideal $\mathfrak{p} \subset A$ is the intersection of the maximal ideals containing it, $\mathfrak{p} = \bigcap_{\mathfrak{m} \supset \mathfrak{p}} \mathfrak{m}$.

The radical of A, rad(A), is defined as the intersection of all the maximal ideals of A, $\mathrm{rad}(A) = \bigcap_{\mathfrak{m} \subset A} \mathfrak{m}$.

The nilradical of A, $\mathfrak{n}(A)$, is the intersection of all prime ideals of A, $\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$. $\mathfrak{n}(A)$ is easily seen to consist precisely of the nilpotent elements of A. When $\mathfrak{n}(A) = (0)$ i.e. when A has no nilpotent elements, A is said to be <u>reduced</u>. If A is a Jacobson ring $\mathrm{rad}(A) = \mathfrak{n}(A)$, but already when A is a nontrivial local ring (i.e. not a field) $\mathrm{rad}(A) = \mathfrak{m} \neq \mathfrak{n}(A)$ in general, where $\mathfrak{m}$ denotes the unique maximal ideal of A.

One result which will be used often is the following

<u>Nakayama's Lemma</u>. Let A be a ring, M, N two finitely generated A-modules. Let $u: M \to N$ be an A-morphism, and let $\mathfrak{a}$ be an ideal of A with $\mathfrak{a} \subset \mathrm{rad}(A)$. If $u \otimes \mathrm{id}_{A/\mathfrak{a}} : M \otimes (A/\mathfrak{a}) \to N \otimes (A/\mathfrak{a})$ is surjective, so is u.

Let A be a ring, S a multiplicatively closed subset of A. On the set-theoretical product AxS define the following equivalence relation

$$(a, s) \sim (a', s') \iff \text{there exists } s'' \in S \text{ with } s''(as' - a's) = 0.$$

One easily checks that the following operations

$$(a, s) + (a', s') = (as' + a's, ss')$$
$$(a, s) \cdot (a', s') = (aa', ss')$$

define a ring structure on the set of equivalence classes of AxS. We denote such ring by A_S, and call it the <u>localization</u> of A at S. We denote the equivalence class of (a, s) by a/s. The homomorphism $\tau: A \to A_S$ defined by $\tau(a) = as/s$ (for any $s \in S$) turns A_S into an A-module. We caution that τ need not be injective.

Let M be an A-module. We define $M_S = A_S \otimes_A M$. It is easy to check that M_S can be obtained also by repeating verbatim the above procedure for the construction of A_S, simply substituting M for A.

In the category of rings and ring homomorphisms, A_S can be more simply defined as follows:

> the localization of A at S consists of a ring C, and a homomorphism $\rho \in \mathrm{Hom}(A, C)$ such that for all rings B the function $\mathrm{Hom}(C, B) \to \mathrm{Hom}(A, B)'$ is bijective, where $\mathrm{Hom}(A, B)'$ consists of all those morphisms $u \in \mathrm{Hom}(A, B)$ such that all elements of $u(S)$ are units in B.

In most applications to Algebraic Geometry the set S is of one of two types. In the first, S consists of the non negative powers of an element $t \in A$, and we write A_t instead of A_S. In the second type, S is the complement of a prime ideal $\mathfrak{p}$ of A. In this case we use the notation $A_{\mathfrak{p}}$ instead of $A_{A-\mathfrak{p}}$.

Let M be a finitely generated A-module. We define

$$\mathrm{Ass}(M) = \{\mathfrak{p} \text{ a prime ideal of } A \mid \mathfrak{p} \text{ is the annihilator of some } x \in M,\ x \neq 0\}$$

$$\mathrm{Supp}(M) = \{\mathfrak{p} \text{ a prime ideal of } A \mid A_{\mathfrak{p}} \otimes_A M \neq 0\}.$$

Ass(M) is a finite set when A is noetherian, and is related to Supp(M) by the following property: the minimal primes of Ass(M) coincide with the minimal primes of Supp(M). We call Ass(M) the <u>set of associated ideals of M</u>, and Supp(M) the <u>support of M</u>.

In the case that M = A the following statements are true:

1) Ass(A) = the prime ideals (isolated and imbedded) corresponding to (0).

2) $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$ = the set of zero divisors.

3) {the minimal primes of Supp(A)} = {the isolated primes of (0)} = {the minimal primes of A}.

4) $\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$.

We give some examples of the above notions, again without any attempts at proofs.

The <u>local rings</u> most commonly met in Algebraic Geometry are of the form $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime ideal of A. It is immediate to check that the complement of $\mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ consists of units, whence $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$.

An example of a <u>Jacobson ring</u> is given by $A/\mathfrak{p}$, where A is a finitely generated algebra over an algebraically closed field k, and $\mathfrak{p}$ is a prime ideal of A.

Finally, we leave as an exercise to the reader to prove that, if M is a finitely generated A-module with annihilator $\mathfrak{a}$, then Ass(M) = {the prime ideals corresponding to an irredundant primary decomposition of $\mathfrak{a}$}.

GEOMETRIC NOTIONS

Let A be a ring. We recall that <u>Spec(A)</u> is defined, as a set, to consist of all the prime ideals $\mathfrak{p}$ of A. Such set is made into a topological space by defining a subbasis of open sets