THE GROUP CONFIGURATION – after E. Hrushovski

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We present here some results of E. Hrushovski which give, in the context of stable theories, an "abstract" or geometrical (in terms of dependence relations), characterization of the presence of some group acting definably on a weight one type.

Preliminaries

We will use freely definitions and basic facts concerning local weight (i.e. p–weight, for p a given regular type), as introduced in [Hr1]; these can also be found in [Hr2] or [Po].

We just introduce the following definition:

Definition: Let p be a fixed regular type (over $\varnothing$) and let $\bar{a}$,$\bar{b}$ be such that $t(\bar{a}/A)$ and $t(\bar{b}/A)$ are p–simple. We say that $\bar{a}$ and $\bar{b}$ are <u>p–independent</u> over A (denoted $\bar{a} \perp_p \bar{b}$) if $w_p(\bar{a}\bar{b}/A = w_p(\bar{a}/A) + w_p(\bar{b}/A)$.

We need to recall briefly what is meant by the canonical basis of a non stationary type.

We begin with the following definitions and theorems which can be found in [Ls., Chapt. 3–2] or in [Pi., Chapt. 4].

Definition:

Let T be a stable theory, $p \in S(A)$. A definition of p is a map d, which takes each formula $\varphi(\bar{v},\bar{y})$ to a formula $d_\varphi(\bar{y})$ such that:

i) for all $\bar{a} \in A$, $p \vdash \varphi(\bar{v},\bar{a})$ iff $\models d_\varphi(\bar{a})$

ii) for all $B \supseteq A$, for all formulas $\varphi(\bar{v},\bar{y})$, and for all $\bar{b} \in B$, $\models d\varphi(\bar{b})$ iff all non forking extensions of p over B satisfy the formula $\varphi(\bar{v},\bar{b})$. We say that $p \in S(A)$ is <u>definable over</u> A_0 if there is a map d satisfying conditions (i) and (ii) such that, for all $\varphi(\bar{v},\bar{y})$, the formula $d\varphi(\bar{y})$ has its parameters in A_0.

Weak definability theorem:

Let T be a stable theory, $p \in S(A)$, then p is definable over A. In fact, there is $A_0 \subseteq A$, $|A_0| \leq |L| + \aleph_0$ such that p is definable over A_0.

Theorem:

Let T be a stable theory, $A \subseteq B$, $p \in S(B)$; p is definable over A if and only if p is the unique non forking extension in $S(B)$ of its restriction to A.

Notation – $\mathrm{dcl}(A)$ is the definable closure of A

– $\mathrm{acl}(A)$ is the algebraic closure of A.

Recall that we are working inside a big saturated model of T, $\mathfrak{C}$. The following theorem is the analogue of Theorem 6–10, Chapt. III–6 in [Sh.], where the canonical basis is defined for a stationary type.

Theorem: Existence of the canonical basis

Let T be a stable theory; in T^{eq}, for every type $p \in S(A)$, there is a set $C(p)$ such that

(i) $C(p) \subseteq \mathrm{dcl}(A)$; p is definable over $C(p)$

(ii) an automorphism σ of $\mathfrak{C}^{eq}$ leaves $C(p)$ pointwise fixed if and only if σ leaves the set of non forking extensions of p globally invariant

(iii) $D \subseteq A$ is such that p does not fork over D, if and only if $C(p) \subseteq \mathrm{acl}(D)$.

(We say that $C(p)$ is the canonical basis of p.)

In the case of a stationary type q, the canonical basis of q is contained in the definable closure of any Morley sequence in q. The analoque in the non stationary case is:

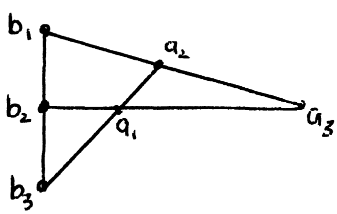
Fact: Let T be a stable theory, $p \in S(A)$ and Let I be an independent set of realizations of p over A, containing a Morley sequence in each strong type extending p. Then C(p) is contained in the definable closure of I.

Configurations

We will assume that we are working with T a complete superstable theory, but the theorem is in fact true for any stable theory, with only minor changes in the proofs.

Let p be a fixed regular type (over $\emptyset$).

Definition: The set $\{a_1,a_2,a_3,b_1,b_2,b_3\}$ is called a <u>p–configuration</u> over A if it satisfies the following:



(a) For each i, $t(b_i/A)$ is p–simple of p–weight n and $t(a_i/A)$ is p–simple of p–weight 1.

(b) All elements are pairwise p–independent over A.

(c) $w_p(b_1b_2b_3/A) = 2n$; for all $i \neq j \neq k$, $w_p(b_ia_ja_k/A) = n + 1$; $w_p(\overline{ab}/A) = 2n + 1$.

Remarks:

- With the assumptions in (b), (c) implies that b_k is in the p–closure of Ab_ib_j, and that a_k is in the p–closure of Ab_ia_j.
- If any element x of the configuration is replaced by some p–simple element y such that $cl_p(Ax) = l_p(Ay)$, then we still have a p–configuration over A.

Main Theorem:

Suppose there is a p–configuration in some model of T. Then there is an ∞ definable group G and a definable generic action of G on a regular type domination equivalent to p.

Lemma 1:

There is a p–configuration $\bar{a},\bar{b}$ over some model M such that $t(\bar{a}\bar{b}/M)$ is equivalent (i.e. domination equivalent) to some power of p (in fact p^{2n+1}).

Proof: First, replace A (which we can suppose to be finite) by an $\aleph_\varepsilon$ – saturated model $M' \supseteq A$, such that the configuration $\bar{a}\bar{b} \perp M'$, then of course we now have a configuration over M′.

Now let N be $\aleph_\varepsilon$ –prime over $M'\bar{a}\bar{b}$ and let $M, M' \leq M \leq N$ be maximal orthogonal to p over M′. Then in N, if $t(e/M')$ is p–simple of p–weight 0, $e \in M$: if not, Me contradicts the maximality of M. If $e \in N$ and $t(e/M')$ is p–simple of weight $n > o$, then $t(e/M)\ \Delta\ p^n$, more precisely, there is $\bar{\alpha}$ in N realizing p^n such that $\bar{\alpha}\ \Delta\ e$ over M′ and $e\ \Delta\ \bar{\alpha}$ over M: as $t(e/M')$ is p–simple, we know that there is $\bar{\alpha}\bar{\beta}$ in N such that $\bar{\alpha}$ realizes p^n and $t(\bar{\beta}/M')$ is hereditary orthogonal to p, and e and $\bar{\alpha}\bar{\beta}$ are domination equivalent over M′. Then $\bar{\beta}$ must be in M and the rest follows. It is now easy to check that $\bar{a}\bar{b}$ is a p–configuration over M. □

Remark: Note that now, in N, for p–simple elements, p–independence over M is equivalent to independence over M.

Lemma 2:

There is, in N, a p–configuration over M such that $a_1 \in dcl(Ma_3b_2)$ and $a_2 \in dcl(Ma_3b_1)$.

Proof: We will replace the given configuration by one where

$a_1 \in \mathrm{dcl}\,(Ma_3b_2)$, from this one, we then get exactly in the same way, another one satisfying also the second condition.

Let $C \subseteq M$, finite, be such that $t(\overline{a}\overline{b}/M)$ is based on C (i.e. $t(\overline{a}\overline{b}/M)$ does not fork over C and $t(\overline{a}\overline{b}/C)$ is stationary) and let $b'_1 \in M$ have the same type as b_1 over C. Then, as b_1 and $b_2\,a_1\,a_3$ are independent over C $b_1\,b_2\,a_1\,a_3$ and $b'_1\,b_2\,a_1\,a_3$ have the same type over C; let N be the $\aleph_\varepsilon$-prime model over $M\overline{a}\overline{b}$ and let b'_3 and a'_2 in N be such $b_1\,b_2\,b_3\,a_1\,a_2\,a_3$ and $b'_1\,b_2\,b'_3\,a_1\,a'_2\,a_3$ have the same type over C. It follows that $cl_p(Mb_2) = cl_p(Mb_2b'_3)$: by condition (c), we have that b'_3 is in the p–closure of Cb'_1b_2, so $b'_1 \in cl_p(Mb_2)$. Similarly, a_3 and $a_3\,a'_2$ have the same p–closure over M.

Now let a'_1 be the canonical basis of $t(a_1/Mb_2b'_1a_3a'_2)$ (in the sense described above in the preliminaries) or more precisely a finite subset of the basis over which it is algebraic.

Now we will see that a'_1 and a_1 have the same p–closure over M: let e be any element realizing the same type as a_1 over $Mb_2b'_3a_3a'_2$. Now,

$$w_p(a_1eb_2b'_3a_3a'_2/C) = w_p\,(a_1e/C) \;+\; w_p\,(b_2a_3/Ca_1e) \\ + w_p\,(b'_3a'_2/Ca_1eb_2a_3).$$

As both a_1 and e are in the p–closure of Cb_2a_3,

$$w_p(a_1eb_2a_3/C) = w_p(b_2a_3/C) = n+1 = w_p(a_1e/C) + w_p(b_2a_3/Ca_1e).$$

We know that $w_p(a_1e/C) \leq 2$; suppose it were equal to 2, then it would follow that $w_p(b_2a_3/Ca_1e) = n - 1$. It would follow as easily that $w_p(b'_3a'_2/Ca_1eb_2a_3)$ must be at most $n - 1$. But this contradicts the fact that, as $b'_1 \in cl_p\,(Cb_2b'_3)$, $w_p\,(a_1eb_2b'_3a_3a'_2/C)$ must be $2n+1$. Hence $w_p(a_1e/M) \leq w_p(a_1e/C) = 1$, that is, $e \in cl_p\,(Ma_1)$.

Now let I be an independent set of realizations of $t(a_1/Mb_2b'_3a_2a'_3)$ such that $a'_1 \in \mathrm{dcl}\,MI$. By the above I is in the p–closure of Ma_1, hence so is a'_1. As a_1 has p–weight 1 over M, and a'_1 must have p–weight > 0 over M it follows that $cl_p(Ma_1) = cl_p(Ma'_1)$.

So, as remarked at the beginning, if we replace b_2 by $b_2b'_3$ a_3 by a'_2a_3, and a_1 by a'_1, we still have a p–configuration over M. Now a'_1 is in the canonical basis of $t(a_1/Mb_2b'_3a_3a'_2)$, hence it is in the definable closure of $b_2b'_3a_3a'_2$ over M. □

Lemma 3:

There is a p–configuration over M such that $a_3 \in dcl\ (Ma_1b_2) \cap dcl\ (Ma_2b_1)$, and which still satisfies the conditions in lemmas 1 and 2.

Proof: Let $\overline{ab}$ be the configuration over M given by the preceeding lemma. Let D be the canonical basis of $t(a_3/Mb_1b_2a_1a_2)$. Then $a_1 \in dcl(MDb_2)$: By the properties of D mentioned in the preliminaries $t(a_3/D)$ has a unique nonforking extension over MDb_2a_1. Now let a'_1 be such that $t(a'_1/MDb_2) = t(a_1/MDb_2)$, and let a''_3 realise a nonforking extension of $t(a_3/D)$ over $MDb_2a_1a'_1$. By what we have just said $t(a''_3a'_1b_2DM), = t(a''_3a'_1b_2DM)$, $(=t(a_3a_1b_2DM))$, so as $a_1 \in dcl(Ma_3b_2)$, we see that $a_1 = a'_1$. For the same reason, $a_2 \in dcl\ (MDb_1)$ also. Now let a'_3 be a finite subset of D such that D is in the algebraic closure of a'_3 and both a_1 and a_2 are definable over Ma'_3b_2, Ma'_3b_1 respectively. Arguing similarly as we did in Lemma 2 for a'_1, we see that a_3 and a'_3 have the same p–closure over M. We can therefore replace a_3 by a'_3 without any loss and we now have a configuration above M satisfying all the preceding conditions and the added one that $a_3 \in dcl\ (Mb_1b_2a_1a_2)$.

Let $C \subseteq M$, finite be such that $t(\bar{a}\bar{b}/M)$ is based over C. Now let this time $b'_3 \in M$ have same type as b_3 over C. As in lemma 2, find b'_1 and a'_2 in N, the $\aleph_\varepsilon$–prime model over $M\bar{a}\bar{b}$, such that $b'_1b_2b'_3a_1a'_2a_3$ is a configuration isomorphic to the original one over C, find also b'_2 and a'_1 such that $b_1b'_2b'_3a'_1a_2a_3$ is also isomorphic to the original configuration over C.

We want to replace, in our original configuration, b_1 by $b_1b'_2$, b_2 by $b_2b'_1$, a_1 by $a_1a'_12$ and a_2 by $a_2a'_1$. We must check that the new elements have the same p–closures as the old ones over M. As in lemma 2, this follows directly from condition (c) in a configuration.

Let us check now that this new configuration over M satisfies all the requirements:

–$a_1 \in \mathrm{dcl}(Ma_3b_2)$ and $a'_2 \in \mathrm{dcl}(Ma_3b'_1)$, so $a_1a'_2 \in \mathrm{dcl}(Ma_3b_2b'_1)$

–$a_2 \in \mathrm{dcl}(Ma_3b_1)$ and $a'_1 \in \mathrm{dcl}(Ma_3b'_2)$, so $a_2a'_1 \in \mathrm{dcl}(Ma_3b_1b'_2)$

–by isomorphism also, as this was true for a_3 in the original configuration, we have that $a_3 \in \mathrm{dcl}(Ma_1a'_2b'_1b_2)$ and $a_3 \in \mathrm{dcl}(Ma'_1a_2b_1b'_2)$. □

By lemmas 2 and 3 and the definability relations they give, let $a_3 = f_{b_1}(a_2)$ and $a_1 = g_{b_2}(a_3)$, where f_b is an invertible Mb_1–definable function, and g_{b_2} is an Mb_2–definable invertible function.

Let us denote by q_i the type of a_i over M.

Let $h_{b_1b_2}(a_2)$ denote $g_{b_2}(f_{b_1}(a_2))$. Define the germ of $h_{b_1b_2}$ as the equivalence class of b_1b_2 modulo the (definable) relation b_1b_2 and $b'_1\, b'_2$ are the equivalent if for all a realizing q_2, independent from $b_1b_2b'_1b'_2$, $h_{b_1b_2}(a) = h_{b'_1b'_2}(a)$.

Lemma 4:

The germ of $h_{b_1b_2}$ is in the p–closure of Mb_3.

Proof: Indeed, $h_{b_1b_2}(a_2) = a_1$ is in $cl_p(Ma_2b_3)$: as a_2 is independent from $\bar{b}$ ($\bar{b} = b_1b_2$) over M, for any a realizing q_2 and independent from $\bar{b}$, we also have that $h_{b_1b_2}(a) \in cl_p(Mab_2)$. Let $(e_i)_{i\in I}$ be a Morley sequence of q_2 over $M\bar{b}$. Then the germ of $h_{b_1b_2}$ is definable over $\{(e_i)_{i\in I}, (h_{b_1b_2}(e_i))_{i\in I}\} \subseteq cl_p\{M, b_3, (e_i)_{i\in I}\}$. Now as $b_1b_2 \perp_p (e_i)_{i\in I}$, over Mb_3, so is the germ, and in fact, it must be in $cl_p(Mb_3)$. □

Define F to be the set of germs of $\{f_b; t(b/M) = t(b_1/M)\}$ and let G be the set of germs of $\{g_e; t(e/M) = t(b_2/M)\}$.

From now on it is understood that we are always working over the model M, and that we are always considering germs of functions, even if we do not mention it explicitly.

Lemma 5:

(1) If $g \in G, f \in F$, f and g are independent, then $g\circ f$ and f, and $g\circ f$ and g are independent.

(2) If f, f' are independent in F, then $h = f^{-1}\circ f'$ is independent from f and f' and is q_2-internal.

(3) If $f_0, f_1, f_2, f_3 \in F$ and are independent, then there are $f_5, f_6 \in F$, independent, such that $f_0^{-1}\circ f_1 \circ f_2^{-1}\circ f_3 = f_5^{-1}\circ f_6$, and $f_5^{-1}\circ f_6$ is independent from both $f_0^{-1}\circ f_1$ and $f_2^{-1}\circ f_3$.

Proof:

(1) g corresponds to some b_2 and f to some b_1, then by lemma 4, $g\circ f$ is in the p-closure of b_3, where b_3 and b_1 and b_3 and b_2 are independent.

(2) Let f be f_{b_1} and f' be f_{b_1} with b_1 and b_1' independent over a_3. Then let $f_{b_1}(a_2') = a_3$ and $f^{-1}(a_3) = a_2$, so $h(a_2') = a_2$. Now h is definable over a Morley sequence e_i in type a_2' over b_1b_1' together with the $h(e_i)$'s, so it is q_2-internal. Now the sequence $e_i\, h(e_i)$ is independent from b_1 and from b_1' hence so is h.

(3) Let $g \in G$ be independent from $\{f_0, f_1, f_2, f_3\}$. Note first that $(g \circ f_0)^{-1}$, $(g \circ f_1)$, $(f_0^{-1} \circ f_1)$ are pairwise independent:

g corresponds to some b_2, f_0 to some b_1 and f_1 to some b_1', such that $t(b_1 b_2) = t(b_1' b_2)$. Let b_3' complete the diagram, i.e. be such that $b_1 b_2 b_3$ and $b_1') b_2 b_3'$ have the same type. Then $(g \circ f_0)^{-1} \in cl_p(b_3)$, $(g \circ f_1) \in cl_p(b_3')$, $(f_0^{-1} \circ f_1) \in cl_p(b_1 b_1'))$. The result follows by the independence relations.

Now let us write $(f_0^{-1} \circ f_1) = (g \circ f_0)^{-1} \text{ o } (g \circ f_1)$. By independence, $(f_0^{-1} \circ f_1), (g \circ f_0)^{-1}$ and $(f_2^{-1} \circ f_3), (g \circ f_1)^{-1}$ have the same type, hence there is some h of the form $(g' \circ f')$ such that $(f_2^{-1} \circ f_3) = (g \circ f_1)^{-1} \circ h$. Now $(g \circ f_0)^{-1}, h$ have the same type as $(g \circ f_1)^{-1}$, h, therefore $(g \circ f_0)^{-1} \circ h$ is equal to some $(f_5^{-1} \circ f_6)$ and the result follows. Now, $(f_5^{-1} \circ f_6)$ and $(f_2^{-1} \circ f_3)$ are equidefinable over $(f_0^{-1} \circ f_1)$, hence have same p-weight; the independence follows.

Now let $H = \{f^{-1} \circ f'; f, f' \text{ independent in } F\}$. Then H is closed under inverse, and by the above lemma it is closed under generic composition. Now let $G = \{h \circ h'; h, h' \in H\}$. Then G is closed under inverse, and G is also closed under composition: let $h, h', h'' \in H$, we see that $h \circ h' \circ h'' \in G$: choose g in H independent from all the rest, then by the above lemma, there is g' in H, g' independent from g, such that $h' = g \circ g'$ and g' independent from h'. So $h \circ h' h'' = h \circ g \circ g' \circ h''$, now $h \circ g \in H$ because we chose g independent from h; $g \underset{h'}{\perp} h''$ hence $g' \underset{h'}{\perp} h''$, $g' \circ h'' \in H$. This finishes the proof of the main theorem. □

So we have the group G acting generically on the type q_2.

Proposition 1:

The group G is connected, with generic any element of H, that is elements of the form $f^{-1} \circ f'$ where f and $f' \in F$ are independent. G is q_2-internal.

Proof: In order to see that elements of H realize the unique generic of G, we check that for $g \in G$ and $h \in H$ independent, $g \circ h \in H$ and is independent from g. Let $g = h_1 \circ h_2$, wlog, suppose that $h \perp h_1 h_2$. Now $g \circ h = h_1 \circ (h_2 \circ h)$, $h_2 \circ h \in H$ and is independent from h_1, so this again is the product of two independent elements of H and the rest follows directly from lemma 5.

Lemma 6:

If $h \in H$, then the p-weight of h is equal to the p-weight of F.

Proof: h is of the form $f \circ f'$, with f, f' in F, independent, and $h \perp f$, $h \perp f'$. As h and f are equidefinable over f', they must have the same p-weight over f', and both h and f are independent from f'. □

Definition: We say that a p-configuration is <u>minimal</u> if for all i, if there is some $b_i' \in cl_p(b_i)$ such that we still have a p-configuration if we replace b_i by b_i', then $cl_p(b_i') = cl_p(b_i)$.

Proposition 2:

If we start from a minimal p-configuration, then the p-weight of G must be equal to the p-weight of the b_i's.

Proof: Note first that all the changes we have made in lemmas 1, 2, and 3, in order to replace the conditions of p-dependence by conditions of definability, maintain the condition of minimality. Now if we replace in our configuration b_1 by f_{b_1}, b_2 by g_{b_2} and b_3 by $g_{b_2} \circ f_{b_1}$, then we still have a p-configuration. By minimality, then the p-weights of b_1 and f_{b_1} must be the same, and the result follows by the above lemma. □

We have a "generic" transitive action of G on a type q of p-weight 1. By results in [Hr1], we then get a transitive action of a group of p-weight n on an infinitely definable set S, with q generic type of S for the action. The possibilities for such an action are known [Hr1]:

There are only 3 cases:

- $n = 1$: the group G is abelian and the action is simply transitive.
- $n = 2$: S is the affine line over a definable field K and G acts as $AGL_1(K)$.
- $n = 3$: S is the projective line over a definable field K and G acts as $PGL_1(K)$.

It is not possible to have $n \geq 4$.

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