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A Note on Monothetic BCI

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Abstract In "Variations on a theme of Curry," Humberstone conjectured that a certain logic, intermediate between BCI and BCK, is none other than *monothetic* BCI—the smallest extension of BCI in which all theorems are provably equivalent. In this note, we present a proof of this conjecture.

1 Introduction

In "Variations on a theme of Curry" [1], Humberstone described a logic, intermediate between BCI and BCK, labeled BCI*. BCI* is obtained from BCI by the addition of the single axiom schema:

$$(*) \qquad (A \to A) \to (B \to B).$$

Humberstone conjectured that BCI* is an axiomatization of *monothetic* BCI (μBCI)—defined to be the smallest extension of BCI logic in which all theorems are provably equivalent (and hence interreplaceable in any formula *salva provabilitate*). Humberstone shows ([1], Proposition 4.1) that this conjecture is equivalent to the following:

$$\vdash_{\mathrm{BCI}^*} A \;\Rightarrow\; \vdash_{\mathrm{BCI}^*} A \to (B \to B) \quad \text{for some formula } B \tag{1}$$

That is, every theorem of BCI* provably implies a *self-implication* (a formula of the form $B \to B$).[1] An obvious proof strategy for (1) is to attempt to establish it by induction on the length of the shortest proof of A. For the base case of such an induction, one would need to show that each axiom of BCI* provably implies a self-implication, while for the induction step one would need to show that the property of provably implying a self-implication is preserved by the rule modus ponens. The latter is proved as Proposition 4.6 in [1]. For the base case, it is easy to show that the axioms C, I, and (*) provably imply a self-implication, since the converse of each of these axioms is provable (being in fact just a relettered instance of the very

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same formula) and in BCI any formula with a provable converse provably implies a self-implication ([1], Proposition 4.2).

To complete the proof of (1) then, it remains to show that the axiom B provably implies a self-implication. This was left as an open question in [1]. It is the purpose of the present note to exhibit a proof of this claim. The proof was originally discovered by one author using the automated theorem prover OTTER [2].[2] The following is a "tidied up" presentation of that proof.

2 Proof of the Conjecture

We begin by proving three lemmas.

Lemma 2.1 $\vdash_{\mathrm{BCI}^*} [A \to (B \to (C \to C))] \to [A \to (B \to (D \to D))]$.

Proof We prove the following representative instance of the above schema:

$$\vdash_{\mathrm{BCI}^*} (p \to (q \to (r \to r))) \to (p \to (q \to (s \to s))).$$

(1)	$(r \to r) \to (s \to s)$	Axiom (*)
(2)	$(q \to (r \to r)) \to (q \to (s \to s))$	1, prefixing q
(3)	$(p \to (q \to (r \to r))) \to (p \to (q \to (s \to s)))$	2, prefixing p

□

Lemma 2.2 $\vdash_{\mathrm{BCI}^*} [(A \to B) \to (A \to C)] \to [(C \to B) \to (D \to D)]$.

Proof We will first show that the instance of this schema, with $A = p$, $B = q$, $C = r$, and $D = p \to q$ is provable in BCI (and hence, of course, in BCI*):

$$\vdash_{\mathrm{BCI}} [(p \to q) \to (p \to r)] \to [(r \to q) \to ((p \to q) \to (p \to q))].$$

(1)	$(r \to q) \to ((p \to r) \to (p \to q))$	Axiom B
(2)	$((p \to r) \to (p \to q)) \to ((\alpha \to (p \to r)) \to (\alpha \to (p \to q)))$	Axiom B
(3)	$(r \to q) \to ((\alpha \to (p \to r)) \to (\alpha \to (p \to q)))$	1, 2 trans. $\to$
(4)	$[\alpha \to (p \to r)] \to [(r \to q) \to (\alpha \to (p \to q))]$	3, permuting
(5)	$[(p \to q) \to (p \to r)] \to [(r \to q) \to ((p \to q) \to (p \to q))]$	4, $\alpha = p \to q$

Now setting $A = (p \to q) \to (p \to r)$, $B = r \to q$, and $C = p \to q$, we have shown that

$$\vdash_{\mathrm{BCI}^*} A \to (B \to (C \to C)).$$

So, applying Lemma 2.1 we have

$$\vdash_{\mathrm{BCI}^*} A \to (B \to (D \to D)).$$

That is, putting $D = s$,

$$\vdash_{\mathrm{BCI}^*} [(p \to q) \to (p \to r)] \to [(r \to q) \to (s \to s)],$$

which is a representative instance of the schema to be proved. □

Lemma 2.3 $\vdash_{\mathrm{BCI}^*} [A \to (B \to A)] \to [((B \to C) \to D) \to (C \to D)]$.

Proof We prove the representative instance:

$$\vdash_{\mathrm{BCI}^*} [p \to (q \to p)] \to [((q \to r) \to s) \to (r \to s)].$$

(1)	$(p \to p) \to (r \to r)$	Axiom (*)
(2)	$(q \to (p \to p)) \to (q \to (r \to r))$	1, prefixing q
(3)	$(p \to (q \to p)) \to (q \to (r \to r))$	2, permuting
(4)	$(p \to (q \to p)) \to (r \to (q \to r))$	3, permuting
(5)	$((q \to r) \to s) \to ((r \to (q \to r)) \to (r \to s))$	Axiom B
(6)	$(r \to (q \to r)) \to (((q \to r) \to s) \to (r \to s))$	5, permuting
(7)	$[\alpha \to (r \to (q \to r))] \to$ $[\alpha \to (((q \to r) \to s) \to (r \to s))]$	6, prefixing α
(8)	$[(p \to (q \to p)) \to (r \to (q \to r))] \to$ $[(p \to (q \to p)) \to (((q \to r) \to s) \to (r \to s))]$	7, $\alpha =$ $p \to (q \to p)$
(9)	$[p \to (q \to p)] \to [((q \to r) \to s) \to (r \to s)]$	4, 8, modus ponens

□

We can now use Lemmas 2.2 and 2.3 to establish our main result.

Theorem 2.4 *Every instance of the axiom B provably implies a self-implication in* BCI*.

Proof We will show that the following formula, in which the antecedent of the main conditional is a representative instance of the axiom B, is provable:

$$\vdash_{\mathrm{BCI}*} [(p \to q) \to ((r \to p) \to (r \to q))] \to (s \to s).$$

Substituting $A = (r \to p) \to q$, $B = r$, $C = p$, $D = q$ in Lemma 2.3, we have

$$[((r \to p) \to q) \to (r \to ((r \to p) \to q))] \to [((r \to p) \to q) \to (p \to q)]. \quad (2)$$

Permuting r and $r \to p$ in (2),

$$[((r \to p) \to q) \to ((r \to p) \to (r \to q))] \to [((r \to p) \to q) \to (p \to q)]. \quad (3)$$

Now we substitute $A = (r \to p) \to q$, $B = (r \to p) \to (r \to q)$, $C = p \to q$, $D = s$ in Lemma 2.2:

$$([((r \to p) \to q) \to ((r \to p) \to (r \to q))] \to [((r \to p) \to q) \to (p \to q)]) \to [(C \to B) \to (D \to D)]. \quad (4)$$

From (3) and (4), using modus ponens,

$$(C \to B) \to (D \to D). \quad (5)$$

Completing the substitution, we have the desired result:

$$[(p \to q) \to ((r \to p) \to (r \to q))] \to (s \to s). \quad (6)$$

□

3 Comments

Theorem 2.4 completes the proof that every theorem of BCI* provably implies a self-implication and thereby establishes Humberstone's conjecture that BCI* = μBCI. Several problems remain open. The proof given above that B implies a self-implication appealed to the axiom (*) twice: once in the derivation of Lemma 2.1 and then again in the derivation of Lemma 2.2. One question then is whether the appeal to the axiom (*) is necessary. Does every instance of the axiom B provably imply a self-implication in BCI? More generally, does every *theorem* of BCI provably imply a self-implication?

Notes

1. Given axiom (*), all self-implications are equivalent. So in the case of BCI*, if a formula provably implies *some* self-implication, it provably implies *all* self-implications.

2. The OTTER software and documentation are available from the Argonne National Laboratory website: http://www-unix.mcs.anl.gov/AR/otter/.

References

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[2] Kalman, J. A., *Automated Reasoning with* OTTER, Rinton Press, Incorporated, Princeton, 2001. With a foreword by Larry Wos. Zbl 1009.68145. MR 1892796. 542

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