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# Weak König's Lemma Implies Brouwer's Fan Theorem: A Direct Proof

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**Abstract** Classically, weak König's lemma and Brouwer's fan theorem for detachable bars are equivalent. We give a direct constructive proof that the former implies the latter.

It is well known that weak König's lemma is the classical contrapositive of Brouwer's fan theorem for detachable bars in some formulation, and hence they are classically equivalent (see Troelstra [3], 2.3 and Troelstra and van Dalen [4], 4.7.2). In this note, we directly prove, in constructive mathematics formalized in **EL** (see Kreisel and Troelstra [2] and [4], 3.6), that weak König's lemma implies Brouwer's fan theorem, which was proved indirectly in Ishihara [1], Corollary.

Note that, although the implication is straightforward in **EL** with Markov's principle ([4], 4.5), here we give a proof without using Markov's principle. Also note that, in the presence of the axiom of countable choice $\mathrm{AC}_{00}$ ([4], 4.2), weak König's lemma, or (bounded) König's lemma for a tree with a bounding function on the number of branchings, and König's lemma for a finitely branching tree without a bounding function are equivalent over **EL**.

In **EL**, each natural number codes a finite sequence of natural numbers with the length function $|\cdot|$ and the decoding function $\pi$ such that $\pi(a, i)$ is the $i$th component of the sequence coded by $a$, if $i < |a|$, 0 otherwise (we usually write $(a)_i$ for $\pi(a, i)$). We can define a bounding function $\sigma$ for a canonical coding $\langle x_0, \ldots, x_{n-1}\rangle$ of the finite sequence $x_0, \ldots, x_{n-1}$ such that $\langle x_0, \ldots, x_{n-1}\rangle < \sigma(m, n)$ whenever $\forall i < n(x_i < 2^m)$. Also the concatenation function $*$, (the characteristic function of) the relation $\preceq$ such that $a \preceq b := |a| \leq |b| \wedge \forall i < |a|((a)_i = (b)_i)$, and the finite initial segment $\overline{\alpha}n := \langle \alpha 0, \ldots, \alpha(n-1)\rangle$ of a function $\alpha$ with length $n$ are definable in **EL**. The concatenation can be extended to concatenation of a finite sequence $a$ with an infinite sequence $\alpha$ such that $(a * \alpha)(i) = (a)_i$ if $i < |a|$, and

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$(a * \alpha)(i) = \alpha(i - |a|)$ if $i \geq |a|$. In the rest of this note, lowercase letters are used as variables ranging over the natural numbers $\mathbf{N}$, and uppercase letters and Greek lowercase letters are used as variables ranging over $\mathbf{N} \to \mathbf{N}$.

Let $\{0,1\}^{\mathbf{N}}$ denote the set of infinite binary sequences, and let $\{0,1\}^*$ stand for the set of finite binary sequences. These sets can be formulated in **EL** as

$$\begin{aligned} a \in \{0,1\}^* &:= \forall i < |a|((a)_i = 0 \vee (a)_i = 1), \\ \alpha \in \{0,1\}^{\mathbf{N}} &:= \forall i(\alpha(i) = 0 \vee \alpha(i) = 1). \end{aligned}$$

A *detachable subset* $S$ of $\{0,1\}^*$ can also be formulated as

$$\begin{aligned} S \subset_\Delta \{0,1\}^* := \forall ab(a \preceq b \wedge b \preceq a \wedge S(a) \neq 0 \to S(b) \neq 0) \wedge \\ \forall a(S(a) \neq 0 \to a \in \{0,1\}^*). \end{aligned}$$

We write $a \in S$ for $S(a) \neq 0$. Clearly, intersection, union, and complement of detachable subsets are detachable.

Let $S$ be a detachable subset of $\{0,1\}^*$. Then the characteristic functions of $\exists a \in \{0,1\}^*(|a| = n \wedge a \in S)$ and $\forall a \in \{0,1\}^*(|a| = n \to a \in S)$ are definable in **EL** (for example, consider the characteristic function of $\exists a < \sigma(1,n)(a \in \{0,1\}^* \wedge |a| = n \wedge a \in S)$ for $\exists a \in \{0,1\}^*(|a| = n \wedge a \in S)$). Similarly, letting $a \in S' := a \in \{0,1\}^* \wedge \exists b \preceq a(b \in S)$, we see that $S'$ is a detachable subset of $\{0,1\}^*$.

A (binary) *tree* is a detachable subset $T$ of $\{0,1\}^*$ such that $\langle\,\rangle \in T$ and it is closed under restriction; that is, if $a \preceq b$ and $b \in T$, then $a \in T$, or more formally,

$$T \in \mathrm{Tree} := T \subset_\Delta \{0,1\}^* \wedge \langle\rangle \in T \wedge \forall ab(a \preceq b \wedge b \in T \to a \in T).$$

A tree $T$ is *infinite* if for each $n$ there exists $a \in \{0,1\}^*$ such that $|a| = n$ and $a \in T$, and $\alpha \in \{0,1\}^{\mathbf{N}}$ is an *infinite path* in $T$ if $\forall n(\overline{\alpha}n \in T)$. Weak König's lemma is stated as

Every infinite tree has an infinite path,

or more formally,

$$\begin{aligned} \forall T \in \mathrm{Tree}[\forall n \exists a \in \{0,1\}^*(|a| = n \wedge a \in T) \to \\ \exists \alpha \in \{0,1\}^{\mathbf{N}} \forall n(\overline{\alpha}n \in T)]. \quad \text{(WKL)} \end{aligned}$$

Note that WKL is a slightly variant form of KL* in [3].

A detachable subset $B$ of $\{0,1\}^*$ is called a *detachable bar* if for each $\alpha \in \{0,1\}^{\mathbf{N}}$ there exists $n \in \mathbf{N}$ such that $\overline{\alpha}n \in B$. A detachable bar $B$ is *uniform* if there exists $n$ such that $\exists k \leq n(\overline{\alpha}k \in B)$ for all $\alpha \in \{0,1\}^{\mathbf{N}}$. Brouwer's fan theorem for detachable bars is stated as

Every detachable bar is uniform,

or more formally,

$$\begin{aligned} \forall B \subset_\Delta \{0,1\}^*[\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n(\overline{\alpha}n \in B) \to \\ \exists n \forall \alpha \in \{0,1\}^{\mathbf{N}} \exists k \leq n(\overline{\alpha}k \in B)]. \quad (\mathrm{FAN}_\Delta) \end{aligned}$$

We show that $\mathrm{FAN}_\Delta$ is equivalent to the following form of the fan theorem:

$$\begin{aligned} \forall B \in \mathrm{Upset}[\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n(\overline{\alpha}n \in B) \to \\ \exists n \forall a \in \{0,1\}^*(|a| = n \to a \in B)], \quad (\mathrm{FAN}'_\Delta) \end{aligned}$$

where $B \in \text{Upset} := B \subset_{\Delta} \{0,1\}^* \wedge \langle\rangle \notin B \wedge \forall ab(a \preceq b \wedge a \in B \to b \in B)$. Note that $\text{FAN}'_{\Delta}$ is a slight variant of KL in [3].

**Lemma 1** $\mathbf{EL} \vdash \text{FAN}_{\Delta} \leftrightarrow \text{FAN}'_{\Delta}$.

**Proof** Suppose that $\text{FAN}_{\Delta}$ holds. Then for each detachable bar $B \in \text{Upset}$, there exists $n$ such that $\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists k \le n(\overline{\alpha}k \in B)$. Hence for each $a \in \{0,1\}^*$ with $|a| = n$, letting $\alpha := a * (\lambda x.0)$, there exists $k$ with $k \le n$ such that $\overline{\alpha}k \in B$, and therefore, since $\overline{\alpha}k \preceq a$, we have $a \in B$.

Conversely, suppose that $\text{FAN}'_{\Delta}$ holds. Then for each detachable bar $B$, either $\langle\rangle \in B$ or $\langle\rangle \notin B$. In the former case, trivially, $B$ is uniform. In the latter case, letting $a \in B' := a \in \{0,1\}^* \wedge \exists c \preceq a(c \in B)$, we have $B' \in \text{Upset}$, and hence there exists $n$ such that $\forall a \in \{0,1\}^*(|a| = n \to a \in B')$. Therefore, for each $\alpha \in \{0,1\}^{\mathbf{N}}$, we have $\overline{\alpha}n \in B'$, and so there exists $c \preceq \overline{\alpha}n$ such that $c \in B$; that is, $|c| \le n \wedge \overline{\alpha}|c| \in B$. □

Let $T$ and $B$ be detachable subsets of $\{0,1\}^*$ such that each is the complement of the other. Then $T \in \text{Tree} \leftrightarrow B \in \text{Upset}$, and hence WKL can be regarded as the classical contraposition of $\text{FAN}'_{\Delta}$. Therefore WKL and $\text{FAN}'_{\Delta}$ (and $\text{FAN}_{\Delta}$) are clas-sicaly equivalent. Furthermore, if WKL holds and $\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n(\overline{\alpha}n \in B)$, then $\neg\forall n \exists a \in \{0,1\}^*(|a| = n \wedge a \in T)$ or $\neg\neg\exists n \forall a \in \{0,1\}^*(|a| = n \to a \in B)$, and hence, in the presence of Markov's principle of the form

$$\forall \alpha[\neg\neg\exists n(\alpha n \neq 0) \to \exists n(\alpha n \neq 0)], \tag{MP}$$

we have $\exists n \forall a \in \{0,1\}^*(|a| = n \to a \in B)$. Thus $\text{WKL} \to \text{FAN}_{\Delta}$ is provable in $\mathbf{EL} + \text{MP}$.

In the following, we shall prove that $\text{WKL} \to \text{FAN}_{\Delta}$ in $\mathbf{EL}$ without invoking MP. A *longest path* $\alpha$ in a tree $T$ is an infinite binary sequence such that $\forall a \in \{0,1\}^*(a \in T \to \overline{\alpha}|a| \in T)$. The longest path lemma is stated as

Every tree has a longest path,

or more formally,

$$\forall T \in \text{Tree} \exists \alpha \in \{0,1\}^{\mathbf{N}} \forall a \in \{0,1\}^*(a \in T \to \overline{\alpha}|a| \in T). \tag{LPL}$$

**Proposition 2** $\mathbf{EL} \vdash \text{LPL} \leftrightarrow \text{WKL}$.

**Proof** $\mathbf{EL} \vdash \text{LPL} \to \text{WKL}$ is trivial. To show $\mathbf{EL} \vdash \text{WKL} \to \text{LPL}$, let $T$ be a tree, and define detachable subsets $S$ and $T'$ of $\{0,1\}^*$ by

$$\begin{aligned} b \in S &:= b \in T \wedge \neg\exists c \in \{0,1\}^*(|c| = |b| + 1 \wedge c \in T), \\ a \in T' &:= a \in T \vee \exists b_0 \preceq a(b_0 \in S). \end{aligned}$$

We show that $T'$ is an infinite tree. Suppose that $a \in T'$ and $b \preceq a$. Then either $b \in T$ or $b \notin T$. In the former case, we have $b \in T'$. In the latter case, since $a \notin T$, there exists $b_0 \in S$ with $b_0 \preceq a$. Noting that $b_0 \preceq a$, $b \preceq a$, $b_0 \in T$, and $b \notin T$, we have $b_0 \preceq b$, and therefore $b \in T'$. Hence $T'$ is a tree. For given $n$, either $\exists a \in \{0,1\}^*(|a| = n \wedge a \in T)$ or $\forall a \in \{0,1\}^*(|a| = n \to a \notin T)$. In the former case, we have $\exists a \in \{0,1\}^*(|a| = n \wedge a \in T')$. In the latter case, there exists $b_0 \in T$ such that $|b_0| = \max\{|b| \mid |b| < n \wedge b \in T\}$, and hence, letting $a := b_0 * \langle 0, \dots, 0\rangle$ with $|a| = n$, we have $a \in T'$. Thus $T'$ is infinite. By WKL, there exists $\alpha_0 \in \{0,1\}^{\mathbf{N}}$ such that $\forall n(\overline{\alpha_0}n \in T')$. For any $a \in \{0,1\}^*$, suppose that $a \in T$ and $\overline{\alpha_0}|a| \notin T$. Then there exists $b_0 \in S$ with $b_0 \preceq \overline{\alpha_0}|a|$. If $|b_0| < |a|$, then there exists $c$ with

$c \preceq a$ such that $|c| = |b_0| + 1$ and $c \in T$, a contradiction. Therefore $|b_0| = |a|$, and so $b_0 = \overline{\alpha_0}|a| \in T$, a contradiction. Thus $a \in T \to \overline{\alpha_0}|a| \in T$. □

**Theorem 3** $\mathbf{EL} \vdash \mathrm{WKL} \to \mathrm{FAN}_\Delta$.

**Proof** We show that $\mathbf{EL} \vdash \mathrm{WKL} \to \mathrm{FAN}'_\Delta$. Let $B \in$ Upset be a detachable bar, and define a tree $T$ by $a \in T := a \notin B$. Then, by the previous proposition, there exists $\alpha_0 \in \{0, 1\}^{\mathbf{N}}$ such that

$$\forall a \in \{0, 1\}^*(a \in T \to \overline{\alpha_0}|a| \in T).$$

Since $B$ is a bar, there exists $n$ such that $\overline{\alpha_0}n \in B$. Let $a \in \{0, 1\}^*$ with $|a| = n$, and suppose that $a \in T$. Then $\overline{\alpha_0}n = \overline{\alpha_0}|a| \in T$, and hence $\overline{\alpha_0}n \notin B$, a contradiction. Therefore $a \notin T$, and so $a \in B$. □

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