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A NOTE ABOUT CONNECTION OF THE FIRST-ORDER FUNCTIONAL CALCULUS WITH MANY VALUED PROPOSITIONAL CALCULI

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By virtue of a generalization of the satisfiability definition, see [2], we described in [3] an approximation of the first-order functional calculus by Boolean many valued propositional calculi in which the quantifier Π had a finite meaning.

In this paper we shall describe another approximation of the calculus by many valued Boolean propositional calculi based in [4]; the proof of the approximation is analogical to [3] and it is given in [5].

We consider here a Boolean algebra with operations $\urcorner$/complemention/, $\dot{+}$ /addition/ and with elements which are n-tuples $(w_1,,\ldots\ldots, w_n)$ of numbers 0 and 1.

We use notations of [3] and especially the following:

1. variables of the calculus:
   (1') free: $x_1, \ldots$ /simply x/ ,
   (2') apparent: $a_1, \ldots$ /simply a/ .
2. relations signs: $f_1, \ldots, f_c$; $\bar{c}$ - maximum of arguments of ones.
3. $w(E)$ - the number of different free /$p(E)$ - apparent/variables occurring in E.
4. $i(E)$ - maximum of indices of those and only those variables which occur in E.
5. $n(E) = i(E) + p(E)$.
6. $E(u/z)$ - substitution of u for each occurrence of z in E /with knowing conditions/.
7. $C\{E\}$ - the set of all significant parts of E:
   $H \epsilon C\{E\} . \equiv . H = E$ or there exist $E_1 \epsilon C\{E\}$, F, G, H_1 such that:
   $(H = F) \wedge (E_1 = F') \vee \{(H = F) \vee (H = G)\} \wedge (E_1 = F + C) \vee (\exists i)\{H = H_1(x_i/a)\} \wedge (E_1 = \Pi a H_1)$.
8. $S(k)$ - the set of all atomic formulas R such that indices of free variables occurring in R are $\leqslant k$.
9. Q-function on $S(k)$ with values n-tuples $(w_1, \ldots, w_n)$ of numbers 0 and 1.

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D.1. $g(k, j, t, Q, m) . \equiv . (k \leq m) \wedge (R)\{(R \in S(k)) \wedge \{Q(R) = (w_1, \ldots, w_n)$ for some $w_1, \ldots, w_n\} \rightarrow (w_j = w_t)\}\}$

By means of the function Q we give an inductive definition of the functional V which is defined for an arbitrary formula E such that $i(E) \leq k$ and $k + p(E) \leq m$:

(1d) $V\{k, Q, m, R\} = Q(R)$, if $R \in S(m)$,

(2d) $V\{k, Q, m, F'\} = V^{\neg}\{k, Q, m, F\}$,

(3d) $V\{k, Q, m, F + G\} = V\{k, Q, m, F\} \dotplus V\{k, Q, m, G\}$,

(4d) $V\{k, Q, m, \Pi a F\} = (w_1, \ldots, w_n)$, for some $w_1, \ldots, w_n$ $. \equiv .$ $(j)\{(j \leq n) \rightarrow (w_j = 1 . \equiv .$ $(r)\{(r \leq k) \wedge (V\{k, Q, m, F(x_r/a)\} = (w_1^r, \ldots, w_n^r)$ for some $w_1^r, \ldots, w_n^r)$ $\rightarrow (w_j^r = 1)\} \wedge (t)\{(t \leq n) \wedge \bar{g}(k, j, t, Q, m) \wedge \{V\{k+1, Q, m, F(x_{k+1}/a)\} =$ $(v_1, \ldots, v_n)$ for some $v_1, \ldots, v_n\} \rightarrow (v_t = 1)\}\})$.

D.2. $J(Q, m, G) . \equiv . (k)\{(i(G) \leq k) \wedge (k + p(G) < m) \rightarrow (V\{k+1, Q, m, G\} \subset V\{k, Q, m, G\})\}$.

D.3. $F \in P(Q, m, E) . \equiv . (\exists G)\{(G \in C\{E\}) \wedge (J(Q, m, G) \rightarrow V\{i(F), Q, m, F\} = (1, \ldots, 1))\}$.

D.4. $F \in P[m, E] . \equiv .$[1] $(Q_n)\{(1 \leq n \leq 2^{cm^{\bar{c}}}) \rightarrow (F \in P(Q_n, m, E))\}$.

D.5. $F \in P\,|E| . \equiv . (\exists m)\{(m \geq n(F)) \wedge (F \in P[m, E])\}$.

D.6. $E \in P . \equiv . E \in P\,|E|$.

The meaning of the above definitions is analogical to the given in [3] and is explained in [5].

T.1. If E is a thesis, then $E \in P$.

The proof of T.1. is inductive on the length of the formal proof of E, see [3], and is given in [5].

If we replace **D.3.** by:

D.3'. $F \in P(Q, m, E) . \equiv . J(Q, m, E) \rightarrow V\{i(F), Q, m, F\} = (1, \ldots, 1)$,

then using Herbrand's proof rules, see [1], we may analogously to [3] prove:

T.2. If E is an alternative of normal forms, then E is a thesis if and only if $E \in P$, see [5].

By an extension of the calculus we mean a first-order functional calculus in which apart of the described signs there are also relations signs $f_1^1, f_2^1, \ldots$ of one argument; in this case the number c of all relations may be infinite.

Of course, all notations and theorems remain true for the extended calculus; in one we may prove:

T.3. A formula E is a thesis if and only if $E \in P$.

1. Because Q depends on n, therefore we write here $Q = Q_n$.

We note that in **T.5.** the number c which occur in D.4. may be infinite, see [5][2]; analogical remarks relevant to [3].

T.2-3. prove a new possibility of approximation of the first-order functional calculus by many valued Boolean propositional calculi; in the approximation the quantifier Π is interpreted in T.2. as a finite operator, see (4d).

Some problems connected with T.2-3. we develop in [6]; examples in another paper.

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2. We explain assuming [3]:
To prove the converse theorem to T.1. we prove an analogical theorem to T.2. from [3] in which we assume:

$$R(M) \,.\equiv .\, (i)\,(j)\,\{(M/i/ = M/j/) \rightarrow (i = j)\}$$

Then, the theorem holds for all formulas.
But to construct M with the property $R(M)$ we use a new sequence of relations $f^1_1, f^1_2, \ldots$, see [5].