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ON SOME RECENT CRITICISM OF CHURCH'S THESIS*

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A function $f(x_1,\ldots,x_n)$ is said to be effectively computable if there is an effective procedure[1] which will provide the value of the function for any values of its arguments. Clearly the notion of effectively computable function is not mathematically precise; it depends upon the hazy notion of an effective procedure. Various attempts have been made to give well-defined mathematical equivalents of these vague ideas, and all of these attempts have been shown to be equivalent to the notion of general recursive function (cf. Kleene [3]). It seems clear from the definition of general recursive functions that every such function is effectively computable, and, on the other hand, every known effectively computable function has turned out to be general recursive. All this evidence has led many logicans to accept what is known as Church's Thesis: A function is effectively computable if and only if it is general recursive.[2] It is impossible to prove Church's Thesis since it involves the fuzzy notion of effectively computable function; all that one can expect (and has obtained) is a vast amount of confirming evidence.[3] However, it is quite possible to obtain firm refutations of Church's Thesis; namely, if we find a general recursive function which is not effectively computable, or, conversely, if we find an effectively computable function which can be shown not to be general recursive. Thus, it is possible to believe that the notion of general recursive function is wider or narrower than the notion of effectively computable function. Recently, Porte [5] and Péter [4] have claimed that it is too wide, while Kalmár [1] has suggested that it may

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1. Given in advance and requiring no ingenuity for its performance.

2. For partial functions $f(x_1,\ldots x_n)$ (i.e. functions which are not necessarily defined for all n-tuples) there is an extended form of Church's Thesis: A partial function is effectively computable if and only if it is partial recursive (Kleene [3]).

3. The situation here is quite analogous to what happens in mathematics when we define continuous function, curve, area, etc.

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be too narrow. We shall try to show that this criticism of Church's Thesis is unfounded.

1. Porte [5] has proved an interesting theorem which has, as a result, that there are general recursive functions $f(z)$[4] such that, for any general recursive function $g(x)$, there exist infinitely many numbers x in the range of f such that, for any argument z_0 with $f(z_0) = x$, the number of steps[5] necessary to compute $f(z_0)$ exceeds $g(x)$. Now, if g grows very fast, say $g(x) = 100^{100^{100^{100^{x}}}}$, then it will be impossible to carry out the computation of $f(z_0)$ within the life-span of a human being or probably within the life-span of the human race. From this fact Porte concludes that the general recursive function is not humanly computable, and therefore, not effectively computable. But this is not correct: human computability is not the same as effective computability. A function is considered effectively computable if its value can be computed in an effective way in a finite number of steps, but there is no bound on the number of steps required for any given computation. Thus, the fact that there are effectively computable functions which may not be humanly computable has nothing to do with Church's Thesis.

2. According to the precise mathematical definition, a function $f(x_1,\ldots,x_n)$ is general recursive if there exists a system of equations **E** for computing f, i.e. for any $x_1, \ldots, x_n$, there exists a computation from **E** of the value of $f(x_1,\ldots,x_n)$ (Kleene[3]). Both occurrences of the existential quantifier "there exists" are meant here in the non-constructive classical sense. To this, Péter ([4], p. 229) makes the following objections: (i) The existential quantifier must be interpreted constructively; otherwise, the functions defined in this way cannot be considered constructive. (ii) If the existential quantifiers are meant in the constructive sense, and if the notion of "constructive" is defined in terms of general recursive functions, then this procedure contains a vicious circle.

Both objections seem to be without foundation.[6] In the case of (i), the general recursive functions defined using the non-constructive existential quantifiers are certainly effectively computable in the sense in which this expression is used in Church's Thesis; no bound is set in advance on the number of steps required for computing the value of an effectively computable function, and it is not demanded that the computer know in advance how many steps will be needed. In addition, for a function to be computable by a system of equations it is not necessary that human beings ever know this fact, just as it is not necessary for human beings to prove a given function continuous in order that the function be continuous. Since objection (i) is thus seen to be unjustified, there is no need to assume, as is done in (ii), that the existential quantifiers are interpreted constructively. However,

4. Namely, any general recursive functions with non-recursive range.

5. Say, the number of steps in the calculation of the value of f from a system of equations for the computation of f.

6. I am assuming that Péter intends "constructive" to have the same meaning as "effectively computable."

there is another error in (ii); "constructive" (or "effectively computable") is not *defined* in terms of general recursive functions. Church's Thesis is not a definition; rather it states that the class of general recursive functions has the same extension as the class of effectively computable functions; and the latter class has its own independent intuitive meaning. Thus, there is no vicious circle implicit in Church's Thesis.

3. That the class of general recursive functions is a proper subclass of the class of effectively computable functions has been argued by Kalmár [1].[7] By a theorem of Kleene ([2], Th. XIV) there is a general recursive function $\phi(x,y)$ such that the function

$$\psi(x) = \begin{cases} \mu y\ (\phi(x,y) = 0) & \text{if } (\exists y)(\phi(x,y) = 0) \\ 0 & \text{otherwise} \end{cases}$$

is not recursive. If we accept Church's Thesis, it follows that ϕ is effectively computable. Now, for any p, if $(\exists y)(\phi(p,y) = 0)$, then we can calculate such a y, since ϕ is effectively computable; hence we can calculate $\psi(p)$. On the other hand, if we can prove by "arbitrary correct means" that not-$(\exists y)(\phi(p,y) = 0)$, then we can also calculate $\psi(p)$; namely, $\psi(p) = 0$. Thus, for any p, if $(\exists y)(\phi(p,y) = 0)$ or if we can prove by "arbitrary correct means" that not-$(\exists y)(\phi(p,y) = 0)$, then we can calculate ψ (p).

Kalmár then claims that, since ψ is not effectively computable, there must be some p for which not-$(\exists y)(\phi(p,y) = 0)$ and this fact cannot be proved by "arbitrary correct means." For, if there were no such p, then, for any p, we could start calculating $\phi(p,y)$ for $y = 0,1,\ldots$, and also, at the same time, start enumerating the proofs by "arbitrary correct means." Eventually we would obtain the value of $\psi(p)$, and thus ψ would be effectively computable, which we know cannot be true. But, a hidden assumption has been made which Kalmár does not mention, namely, (**H**): *The set of proofs by "arbitrary correct means" is effectively enumerable.* Without this assumption, I see no way of obtaining Kalmár's conclusion. To sum up, if we assume Church's Thesis together with the Hypothesis (H), then there exists some p such that not-$(\exists y)(\phi(p,y) = 0)$ and "not-$(\exists y)(\phi(p,y) = 0)$" cannot be proved by "arbitrary correct means." Thus, $(\exists y)(\phi(p,y) = 0)$ would be absolutely undecidable by "arbitrary correct means." Kalmár then states that we know this preposition $(\exists y)(\phi(p,y) = 0)$ to be absolutely undecidable by "arbitrary correct means" and we also know that this proposition is false. But this is not so. All that we know (under the assumption of Church's Thesis and Hypothesis (**H**)) is that there is some (unspecified) p such that not-$(\exists y)(\phi(p,y) = 0)$ is true and is not provable by "arbitrary correct means"; we do not know any particular p for which this holds and thus we cannot point to any particular absolutely undecidable proposition which we know to be false. (Of course, if we were able to identify such a

7. Kalmár claims only that his arguments make Church's Thesis implausible. It will be clear, however, that if his arguments were correct, then Church's Thesis would be false.

proposition, then, this would be a contradiction, and therefore, either Church's Thesis *or* hypothesis (**H**) would be false.)

Kalmár also points out that the absolutely undecidable proposition $(\exists y)(\phi(p,y) = 0)$ mentioned above cannot be proved by "arbitrary correct means" to be absolutely undecidable. For, if we could prove it to be absolutely undecidable, then, for any q, it would be impossible to prove $\phi(p,q) = 0$. But, since ϕ is effectively computable, it would follow that $\phi(p,q) \neq 0$ for every q, and so we would know that $(\exists y)(\phi(p,y) = 0)$ is false, contradicting its absolute undecidability. Kalmár then states ([1], p. 76): "The fact that some consequence of Church's Thesis cannot be proved by any correct means can be regarded, I think, as arguments against its plausibility." This is misleading on two counts: (a) It is the conjunction of Church's Thesis and Hypothesis (**H**), not Church's Thesis alone, from which all the other results have been derived; (b) What we have deduced is not that some particular proposition $(\exists y)(\phi(p,y) = 0)$ is absolutely undecidable, but only that *there is some (unspecified) p such that $(\exists y)(\phi(p,y) = 0)$ is absolutely undecidable.* We have no basis for knowing that the italicized sentence cannot be proved by "arbitrary correct means."

Since the notion of "proof by arbitrary correct means" is rather vague, Kalmár attempts to give a more rigorous argument against Church's Thesis. Let $\phi_1, \phi_2, \ldots, \phi_r$ be the function symbols occurring in a system **S** of equations defining the function ϕ. Let $\mathcal{P}$ be the first-order theory having $\phi_1, \ldots, \phi_r$ as function symbols, 0 as individual constant, $=$ as predicate letter, and arbitrary predicate variables. As non-logical axioms of $\mathcal{P}$, we take the equations of **S**. In the previous argument against Church's Thesis, replace "proof by arbitrary correct means" by "proof in some consistent extension of $\mathcal{P}$." Notice that if $\sim(\exists y)(\phi(p,y) = 0)$ is provable in some consistent extension $\mathcal{P}'$ of $\mathcal{P}$, then $\sim(\exists y)(\phi(p,y) = 0)$ is true. (For, if $\phi(p,q) = 0$ for some q, then $\phi(p,q) = 0$ is provable in $\mathcal{P}$ and hence also in $\mathcal{P}'$. So, $(\exists y)(\phi(p,y) = 0)$ would be provable in $\mathcal{P}'$, contradicting the consistency of $\mathcal{P}'$.) Just as before, if we assume Church's Thesis and the Hypothesis (**H#**): *The class of proofs in arbitrary consistent extensions of $\mathcal{P}$ is effectively enumerable*, it follows from the fact that ψ is not effectively computable that there is some p such that $\sim(\exists y)(\phi(p,y) = 0)$ is true but is not provable in any consistent extension of $\mathcal{P}$. Now, let $\mathcal{P}'$ be the extension of $\mathcal{P}$ obtained by adding $\sim(\phi(p,y) = 0)$ as an axiom to $\mathcal{P}$, where y is a free variable. Then, $\sim(\exists y)(\phi(p,y) = 0)$ is provable in $\mathcal{P}'$, and the consistency of $\mathcal{P}'$ follows from Gentzen's consistency proof for arithmetic. Thus, we have a contradiction. This means that either Church's Thesis is false or Hypothesis (**H#**) is false.[8] Therefore, Church's Thesis implies the falsity of (**H#**). Since there is no evidence at all for (**H#**), Kalmár's proof offers us no reason to question the validity of Church's Thesis.

8. Since Kalmár failed to notice the necessity of (**H#**) in his derivation, he should have concluded from his argument that Church's Thesis is false, instead of claiming, as he did, that it is just implausible.

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