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NOTES ON THE AXIOMATICS OF THE PROPOSITIONAL CALCULUS

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In this paper the proofs, unless otherwise stated, are Meredith's, and the bracketed notes introducing each item or commenting on it, Prior's. The proofs are all compressed by Meredith's device of writing 'Dmn' for the most general result (i.e. without any unnecessary identification of variables) of detaching the formula n, or some substitution in it, from the formula m, or some substitution in it.

1. *Łukasiewicz's Deduction Shortened.* (This is a very slight abridgement of Łukasiewicz's proof that *CCCpqrCCrpCsp* suffices for classical *C*. It seems worth including, as Łukasiewicz's own paper [5] is now out of print and not easily obtainable.)

1. *CCCpqrCCrpCsp*
2. *CCCpqpCrp* = DDD1D111n
3. *CCCpqrCqr* = DDD1D1D121n
4. *CpCCpqCrq* = D31
5. *CCCpqCrsCCCqtsCrs* = DDD1D1D1D141n
6. *CCCpqCrsCCpsCrs* = D51
7. *CCpCqrCCpsrCqr* = D64
8. *CCCCCpqrtCspCCrpCsp* = D71
9. *CCpqCpq* = D83
10. *CCCCrpCtpCCCpqrsCuCCCpqrs* = D18
11. *CCCCpqrCsqCCCqtsCpq* = DD10.10.n
12. *CCCCpqrCsqCCCqtpCsq* = D5.11
13. *CCCCpqrsCCsqCpq* = D12.6
14. *CCCpqrCCrpp* = D12.9
15. *CpCCpqq* = D3.14
16. *CCpqCCCprqq* = D6.15
*17. *CCpqCCqrCpr* = DD.13D.16.16.13
*18. *CCCpqpp* = D14.9
*19. *CpCqp* = D33

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2. *Two Axioms for C-Verum*. (Meredith's axiomatisation--the development is only given for the first of the two—of that fragment of the two-valued logic in which implication is supplemented by a constant true proposition, here symbolised as '*I*' --the same symbol is used for the axiom, but the context prevents confusion. This is the solution of a problem put to Meredith in 1957 by Lejewski, whose own work in [3] gave him a special interest in ways of completing the propositional calculus from its implicational fragment. It is clear that if you know of an axiom AX in *C* and *N*, which will yield the complete propositional calculus when subjoined to a basis known to be complete for *C*-pure, you can obtain a single axiom for *C*-*N* by replacing *I* in Meredith's *C*-*I* axiom by AX. It may be noted that a *C*-*I* single axiom must in the nature of the case be non-organic, i.e. must contain a law of the system as a part, namely the constant *I*. As with some systems considered in later sections, a shorter total axiomatisation seems possible with two organic axioms than with a single non-organic one. In the present case, the pair consisting of Łukasiewicz's *CCCpqrCCrpCsp* and the constant *I* is shorter than either of Meredith's single axioms.)

(*a*) *CCCpqCrCIsCCspCrCtp*
(*b*) *CCCpqCIrCsCCrpCtCup*

1. *CCCpqCrCIsCCspCrCtp* = (*a*)
2. *CCCtpCpqCCspCrCtp* = D11
3. *CCCpqCtpCCspCrCtp* = D12
4. *CCrCpCqpCtCCspCpCqp* = D31 = D33
5. *CCrCCspCpCqpCuCtCCspCpCqp* = D34
6. *CCrpCpCqp* = DDD53nn
7. *CCqrCqCpr* = D16
8. *CCrCqpCsCpCqp* = D36
9. *CCrCpCqpCtCsCpCqp* = D38
10. *CpCqp* = DDD96nn
11. *CpCqCrp* = D7.10
12. *CpCqCrCsp* = D7.11
13. *CCCsCpqpCrCtp* = D1.12
14. *CCpCrCpqCsCtCrCpq* = D1.13
15. *CCpCrCpqCrCpq* = DDDD14.14.n.n.n
16. *CCqrCsCqCpr* = D77
17. *CtCCqrCsCqCpr* = D10.16
18. *CCCqCprsCCqrCts* = D1.17
19. *CCCpqCIrCsCCrpCtCup* = D18.1
20. *CCCpqrCsCtCCrpCuCrp* = D18.19
21. *CCCpqrCCrpCsCtp* = DD15.20.n
22. *CCrpCCCpqrCsp* = D15.21
23. *CCCpqrCCrpp* = D15.22
24. *CCCCrppsCCCpqrCus* = D1.D11.23
25. *CCCpqrCtCCrpCsp* = D24.7
26. *CCpCpqCrCsCpq* = D1.D1.11

*27. *CCCpqrCCrpCsp* = DDD26.25.n.n

(In view of the next item, the proof of 7 and 24, even without 27, establishes sufficiency for *C*-pure. The above deductions will also go through if *I* in the axiom is replaced by *t*, the result of this replacement being therefore a single axiom for *C*-pure. To prove the constant *I* itself, prove *Cpp* and *CrCsCpp* in *C*-pure, and the constant is obtainable as **DDDD**.Ax. *CrCsCpp*.*Cpp*.n.n.)

3. *2-Axiom 2-Valued C-Pure.*

1. *CCCpqrCCrpp*
2. *CCqrCqCpr*
3. *CCCCrppCpqCpq* = **D**11
4. *CCCpqrCsCCrpp* = **D**21
5. *CpCCpqq* = **D**34
6. *CCCCCprqqpp* = **D**15
7. *CCpCCCprqqCCCprqq* = **D**16
8. *CCCCCpqrsCCrppCCrpp* = **D**74
9. *CCCCrppCCCpqrsCCCpqrs* = **D**18
10. *CCqrCsCqCpr* = **D**22
11. *CCCpqrCCrpCsp* = **D**9.10

4. *2-Axiom 2-Valued C-Pure (Others).* (About the time when Meredith was circulating the preceding item, it was noted by Ivo Thomas that the sufficiency of certain axiom-pairs followed easily from Łukasiewicz's proof in [6], given in D-form in [10], pp. 318-9, that in the Tarski-Bernays axioms *CCpqCCqrCpr*, *CCCpqpp*, *CpCqp*, the last one may be replaced by any formula of the form *CpCαβ*. For example, we have the following deductions, starring the *probanda*:-

1. *CCCpqpCrp*
*2. *CCpqCCqrCpr*
3. *CCpCpqCrCpq* = **DD**221
*4. *CrCCCpqpp* = *CpCαβ* = **D**31
*5. *CCCpqpp* = **D**4n, for any thesis n;

and the following:-

*1. *CCCpqpp*
2. *CCpqCsCCqrCpr*
*3. *CuCCCsCCqrCprtCCpqt* = *CpCαβ* = **D**22
*4. *CCpqCCqrCpr* = **DD**3nl, for any n.

When Thomas sent these results to Meredith, the latter replied, in a letter of August, 1958, that he knew the pair *CCCpqpCrp*, *CCpqCCqrCpr*, and (i) to the other pair he added *CCCpqpp* with *CCpqCCqrCsCpr*, *CCpqCCqrCpCsr*. He further noted (ii) that Łukasiewicz's *CpCαβ* result showed that Pierce and Syll, i.e. *CCCpqpp* and *CCpqCCqrCpr*, give Weak Syll, i.e. *CCqrCCpqCpr*. Putting capitalised variables for implications—e.g. *CPCqP* for *CCrpCqCrp*—Thomas comments, 'I fill in the reasoning thus: Peirce and Syll give themselves capitalised, *CCpqCαβ* (Syll), and so by the Łukasiewicz result (1) *CPCQP*. Peirce and Syll also give (2) *CCCpqCqrCCpqCpr*, hence by Syll, (1), (2) we get Weak Syll'. Finally, Meredith adds in his letter the theorem that follows below. It may be added that before Łukasiewicz's result, Wajsberg had the two Thomas pairs above, in [14], with more difficult proofs.) (iii) An allied result: Either Syll works with *CCCrCpqpp*.

1.	*CCCrCpqpp*		
2.	*CCqrCCpqCpr*		
3.	*CCsCCrCpqpCsp*	=	D21
4.	*CCCCrpqpCrp*	=	D32
*5.	*CCCpqpp*	=	D3DD232
6.	*CCsCqrCsCCpqCpr*	=	D22
7.	*CCqrCCsCpqCsCpr*	=	D62
8.	*CCrCsCCpqpCrCsp*	=	D75
9.	*CCrpCCCpqrp*	=	D82
10.	*CCrpCpp*	=	DD249
11.	*CqCpp*	=	D4.10
12.	*CqCrCpp*	=	DD7.11.11
13.	*CCCqrqCCqrr*	=	DD921
14.	*CCCqCpprr*	=	D13.12
15.	*CCpqCrCpq*	=	DD2.14.7
16.	*CCCpqpCrp*	=	D8.15
*17.	*CpCqp*	=	D4.16
18.	*CCCqrCpqCCqrCpr*	=	DD971
*19.	*CCpqCCqrCpr*	=	DD2.18.15

For 1 and 2. *CCpqCCqrCpr* I can give no better than DDDD22211 = *CCCpqpp* and thence via Łukasiewicz's result above (i.e.(ii)).

5. *C-Pure with Identity*. (All the axiom-pairs in the preceding sections have a total of 9 C's, distributed variously between the axioms. This set me wondering whether there could be a pair with the distribution 1-8; with results which I have described in [9]. When I put this problem to Meredith in 1959, he did not solve it, but he did in 1960 produce not a 9-C but an 8-C pair with the 1-C axiom *Cpp* as one member. His independence-proof and deductions are given below. The 4-valued matrix verifies Axiom 1 and *COp* and falsifies *Cpp*, showing independence, while the inner 3-valued matrix verifies both 1 and *CCppp*, falsifying *Cpp* and allowing no constant *k* such that *Ckp* for all *p*. The deductions are from Axiom 1 only, and illustrate the extreme difficulty of getting rid of its extra letter of simplification; but the set of section 3 is given by 12 and the detachment of *Cpp* from 20. The 'twiddle' or tilde signifies deductive equivalence. For other uses of the two axioms which are together equivalent to 1, see I.4.)

1. *CCCpqrCCrpCsCtp*
2. *Cpp*

1 with
3. *CCppp*
is saturated and
rejects both *Cpp* and *COp*

C	*1*	*2*	*3*	*0*
**1*	*1*	*3*	*2*	*0*
2	*1*	*3*	*1*	*3*
3	*1*	*1*	*2*	*2*
0	*1*	*1*	*1*	*1*

1∼*CCpqCCqrCpCsr*, *CCCrCpqpp*.

1. *CCCpqrCCrpCsCtp* Axiom
2. *CCCrCpqpCsCtp* = DDDD1D111nn
3. *CCCqprCpCsr* = DDDD121nn
4. *CrCuCCrpCsCtp* = D31
5. *CCCuqCtpCCCqrpCsCtp* = DDDD1D141nn
6. *CCCrqCsCtpCuCCrpCsCtp* = D51
7. *CCqCsCtpCuCCCqrpCsCtp* = DD64n
8. *CCCCCpqruCtpCCrpCsCtp* = DD71n
9. *CpCqCrCsp* = D32
10. *CCpCrCpqCsCtCrCpq* = DD69n
11. *CCpCrCpqCrCpq* = DDDD10.10nnn
12. *CCCpqrCCrpp* = D11.D11.1
13. *CrCqCCrpp* = D3.12
14. *CCpqCsCCCprqq* = DD6.13.n
15. *CCCCpqtqCsCCCprqq* = DD7.14.n
16. *CCCsCCCprqqCCpqtCCpqt* = D12.15
17. *CCCCCprqqtCsCCpqt* = D8.16
18. *CCpqCCqCprCpr* = DD17.12n
19. *CCCCqCprCprCCpqtCCpqt* = D18.18
20. *CCpqCCqrCpCst* = D19.6

6. *Variations on Tarski.* (I once raised with Meredith the question whether Łukasiewicz's result, that from *CCpqCCqrCpr*, *CCCpqpp* and any *CpCαβ* we could obtain the remaining Tarski-Bernays axiom *CpCqp*, would still hold if we replaced *CCCpqpp* with Tarski's original axiom *CCCpqrCCprr*. I could prove particular cases of it, e.g. *CpCpp* works as follows:-

1. *CCpqCCqrCpr*
2. *CCCpqrCCprr*
3. *CpCpp*
4. *CCCCqrCprsCCpqs* = D11
5. *CCpCqrCCsqCpCsr* = D44
6. *CCCppqCpq* = D13
7. *CCCCprrsCCCpqrs* = D12
8. *CCCpqpCpp* = D76
9. *CCpCpqCCpqCpq* = D48
10. *CCppCpp* = D93
11. *CCpCppCpp* = D2.10
12. *Cpp* = D11.3
13. *CCpCpqCpq* = D2.12

14. *CCCpqpCCpqq* = D4.13
15. *CCCpppp* = D14.8
16. *CCCpqpp* = D7.15

But I could obtain no *general* result either way. Meredith pointed out in July 1961 that the matrix

C	1	2	3
*1	1	3	3
2	1	3	3
3	1	1	1

verifies Syll, Tarski and *CpCCpqq* but falsifies *CpCqp*. In October of the same year he implicitly extended this result to three other formulae which he shows below to be equivalent to Tarski. Capitalised variables stand for implications, e.g. *CPP* is *CCpqCpq*.)

Subject to *CCpqCCqrCpr* the four theses (A) *CCCpqpCCprr*, (B) *CCpqCCCpqpr*, (C) *CCCpqrCCprr*, (D) *CCprCCCpqrr* are equivalent.

The strongest identity (derivable from Syll with any of these) is *CCPqCPq*; the strongest Peirce is *CCCCPqrCPqCPq*; the strongest Simp is *CCPqCRCPq*.

Refutation of *CPP* by

C	1	2	3	4	0
*1	1	1	0	0	0
2	0	0	0	0	0
3	0	4	0	0	0
4	1	1	0	0	0
0	1	1	1	1	1

(A) $CCprr = 1$ unless $p = 0$, but $CCOqO = 0$; hence $CCCpqrCCprr = 1$. $CCpqCCqrCpr = 1$ if $p = 0$ or $p = 2$ or $q = 2$; also if $p = 4$ $q = 1$, ($CC1rC4r = 1$); also if $p = 4$ $q = 1$ ($CC1rC1r = 1$); also if $p = 3$, $q \neq 2$. But $CC32C32 = 0$.

Deductions from Syll alone:-

1. *CCpqCCqrCpr*
2. *CCCCqrCprsCCpqs* = D11
3. *CCpCqrCCsqCpCsr* = D22
4. *CCpqCCCprsCCqrs* = D21
5. *CCCCCprsCCqrstCCpqt* = D14
6. *CCqsCCpqCCsrCpr* = D23
7. *CCtCpqCCqsCtCCsrCpr* = D36

Adding as second axiom (A) we have:-

8. *CCCpqpCCprr* (Ax)
9. *CCCpqpCCtrCCpCrsCts* = DD183

10. *CCCCpqCCCpqCqrCprss* = D89
11. *CCsCCpqpCCCCprrtCst* = D68
12. *CCCCCpqrrsCCpqs* = D10.11
13. *CPCCPqq* = D12.8
14. *CCpCQrCQCpr* = D3.13
15. *CCprCCCpqpr* = (B) = D14.8
16. *CCPqCPq* = D12.12
17. *CCCCPqrCPqCPq* = D15.16
18. *CCCprsCCpqCCqrs* = D14.4
19. *CCCCPqrsCCsCPqCPq* = D18.17
20. *CCPqCCrCPqCPq* = D12.19
21. *CCPqCRCPq* = DD1.20.12
22. *CCsCPqCCCsrCPqCPq* = D2.19
23. *CCCsrCPqCCsCPqCPq* = D14.22
24. *CCCpqCCprrCCprr* = D23.8
25. *CCCprCCpqrCCprr* = DD1.14.24
26. *CCCpqrCCprr* = (C) = DD1.21.25
27. *CCprCCCpqrr* = (D) = D14.26

Adding (B) to 1-7 we have:-

8. *CCprCCCpqpr* (Ax)
9. *CCsCCpqpCCprCsr* = D38
10. *CCCpqsCCprCCspr* = D29
11. *CCCrsCCpqpCCrtCCprt* = DD199
12. *CCCpqsCCtCspCCprCtr* = DD1.10.3
13. *CCCCtrqsCCtuCCCtrtu* = DD1.12.11
14. *CCXCCprCCCpqpr* = DD1.6.13
15. *CCCpqsCXCCCCsprtCCprt* = D5D7.14
16. *CCCpqpCCCCppprCXr* = DD1.15.9
17. *CCCpqpCCprCXr* = DD3.16.8
18. *CCCPCCCCsPrtCCPrtuCXu* = DD17.15.n
19. *CCCPPPqCPq* = DD18.9.n
20. *CCPqCPq* = DD1.8.19
21. *CCCCPqrCPqCPq* = D8.20
22. *CCPCPqCPq* = D2.21
23. *CCCpqpCCprr* = (A) = DD1.17.22

Adding (C) to 1-7 we have:-

8. *CCCpqrCCprr* (Ax)
9. *CCpqCCprrCCprr* = D88
10. *CCsCCpqrCCCCprrtCst* = D68
11. *CCCpqsCCCCprrtCCsrt* = D2.10
12. *CCCCCCprrCCsrtCCsrtuCCCpqsu* = D10.11
13. *CCCpqsCCpsCCsrr* = D12.5
14. *CCCpqpCCprr* = (A) = DD1.13.9

Adding (D) to 1-7 we have:-

8. *CCprCCCpqrr* (Ax)
9. *CCsCCpqrCCprCsr* = D38
10. *CCCpqrCCprr* = D98

(Meredith has noted that although Tarski cannot replace Peirce in Łukasiewicz's theorem, Tarski and Simp will yield the full C calculus when combined with either Syll, whereas with Pierce we must have *CCpqCCqrCpr*. With Tarski and the weaker Syll the initial deductions are

1. *CCqrCCpqCpr*
2. *CCCpqrCCprr*
3. *CpCqp*
4. *CCpqCpCrq* = D13
5. *CCqCpCqrCpCqr* = D23
6. *CpCCpqq* = D54

The rest follows from the results in the following section.)

7. *The System B-C-I.* (Meredith observed independently some of the relations between implicational calculus and combinatory logic developed in Curry and Feys [2], 9E. In particular, if we write B for *CCqrCCpqCpr*, then for any formula a, b, c, **DDD**Babc = **D**a**D**bc, just as in combinatory logic Babc = a(bc); if we put C for *CCpCqrCqCpr*, **DDD**Cabc = **DD**acb, just as in combinatory logic Cabc = acb; and if we put I for *Cpp*, **D**Ia = a, just as Ia = a in combinatory logic. *CpCqp* and *CCpCpqCpq* are similarly related to the combinators K and W. Following the practice in combinatory logic, Meredith will often write, say, *CCqrCCpqCpr* ~ λaλbλc**D**a**D**bc. The following is his 1956 summary of deductive equivalents of the set B, C, I.)

B = *CCqrCCpqCpr* = λaλbλc**D**a**D**bc
C = *CCpCqrCqCpr* = λaλbλc**DD**acb
I = *Cpp* = λaa
T = *CpCCpqq* = λaλb**D**ba
P = *CCpqCCqrCpr* = λaλbλc**D**b**D**ac

3 Axiom bases: T, I and either B or P
2 Axiom bases: I and either

Q_1 = *CCpCqrCCsqCsCpr* = λaλbλcλd**DD**ad**D**bc

or

Q_2 = *CCsqCCpCqrCsCpr* = λaλbλcλd**DD**bd**D**ac

or

R_1 = *CCCCpqrsCCqrCps* = λaλbλc**D**aλd**D**b**D**dc

or

R_2 = *CCqrCCCCpqrsCps* = λaλbλc**D**bλd**D**a**D**dc

1 Axiom bases: $\bar{Q}_1$ = λaλbλcλd**DD**ad**DD**bIc
$\bar{Q}_2$ = λaλbλcλd**DD**bd**DD**aIc

Putting α for **D**CC, i.e. *CqCCpCqrCpr*, we have

DD$\alpha\alpha\alpha$ = C
DDDPP**D**PPT = C; **DD**BBT = α; **D**CI = T.

$DDQ_1Q_1I = Q_2$; $DQ_2I = C$; $DCQ_2 = Q_1$; $DDQ_1\alpha DQ_1I = P$.
$DDR_1II = T$; $DDR_1R_1I = P$.

Meredith's main results follow from these; and for R_2 he gives

1.	*CCqrCCCCpqrsCps*	
2.	*Cpp*	
3.	*CCCCpqqrCpr*	= D12
*4.	*CpCCpqq*	= D32
5.	*CCCCsCCCpqqrCprtCst*	= D13
6.	*CCsCCCpqqrCsCpr*	= D35
*7.	*CCqrCCpqCpr*	= D61

The single axiom $\bar{Q}_1$ is the formula *CCpCqrCCssCtqCtCpr* mentioned in III.1, and the other is of comparable length—non-organic, and longer than the organic axiom-pairs, Q_1 etc. with I)

8. *Single Axiom for B-C-K* (see note introducing the last section).

		1.	*CCCpqrCCsCrtCqCst*
D11	=	2.	*CCuCCCsCrtCqCstvCrCuv*
D21	=	3.	*CrCCCxuqCuCCsCrtCst*
D23	=	4.	*CxCrCCxyCCsCrtCst*
D41	=	5.	*CrCC1yCCsCrtCst*
D25	=	6.	*CCxCyzCrCCsCrtCst*
D61	=	7.	*CrCCsCrtCst*
D77	=	8.	*CCpC*ˀ*qCpq*
D87	=	9.	*CCpCqrCqCpr* *
D88	=	10.	*CC*ˀ*C*ˀ*pp*
D1.10	=	11.	*CCqCprCC*ˀ*pCqr*
D1.11	=	12.	*CCsCCC*ˀ*pCqrtCCprCst*
D12.7	=	13.	*CCprCqCC*ˀ*pr*
D8.13	=	14.	*CCprCC*ˀ*pr*
D14.14	=	15.	*CC*ˀ*CprCC*ˀ*pr*
D1.15	=	16.	*CCqCCC*ˀ*prsCCprCqs*
D1.9	=	17.	*CCsCCqCprtCCqrCst*
D17.7	=	18.	*CCqrCsCqr*
D1.18	=	19.	*CCsCCpCqrtCrCst*
D19.7	=	20.	*CrCqCpr*
D8.20	=	21.	*CpCqp* *
D21.1	=	22.	Cul
D16.22	=	23.	*CCqrCuCCsCrtCqCst*
D8.23	=	24.	*CCpqCCsCqrCpCsr*
D9.24	=	25.	*CCsCqrCCprCpCsr*
D8.18	=	26.	*CCqrCqr*
D25.26	=	27.	*CCpqCpCCqrr*
DD9D27.27.9	=	28.	*CCpqCCqrCpr* *

9. *A Combinatory Base without C-Positive Analogue.* (If any formula is D-derivable from other formulae the combinator corresponding to it is

definable in terms of those corresponding to the other formulae. The converse, however, does not always hold, as is noted in [2], p. 315, n. 15. Meredith gives also this example: the combinator corresponding to *CCpqCCCspCqrCpr* suffices to define those corresponding to *CpCqp* and *CCpqCCpCqrCpr*, which are jointly sufficient for the positive or intuitionistic implicational calculus; but the latter two formulae are not deducible from the first one. Meredith did find them deducible, however, from the formula resulting from prefixing '*Ct*' to the latter. The following deduction is Ivo Thomas's:-

1.	*CtCCpqCCCspCqrCpr*	
2.	*CCpqCCCspCqrCpr*	= D1n
3.	*CCutC2vCtv*	= D21
4.	*CC2vv*	= D33
5.	*CCCspCqrCqCpr*	= D32
6.	*CpCCpqq*	= D54
7.	*CCCpqrCqr*	= D56
*8.	*CpCqp*	= D77
9.	*CCpqCCqrCpr* = DD7D227	= DD22D87
10.	*CCvuCC2vu*	= D94
11.	*CCCC2vuwCCvuw*	= D9.10
*12.	*CCpqCCpCqrCpr*	= DD9.2.11

Thomas has noted that 5 alone gives *CpCqp* and *Cpp*. *Cpp* = DD5DD55nn, and *CpCqp* = D5.*Cpp*. Also sufficient for C-positive are 8, i.e. in combinatory terms K, and the commutation of 12, Frege's axiom *CCpCqrCCpqCpr*, which Meredith calls A. This is derived below from P, i.e. Syll, with the combinators K and R; but R has no C-positive analogue. Items marked with a dash are all in this last position; those with one cross are in C-positive; those with two are even in the system B-C-I of Section 7.)

xx		x	-
P = λaλbλcDbDac		K = λaλba	R = λaDaa
1. DPP	=	(ab)DaDPb	xx
2. DPK	=	(ab)DaDKb	x
3. D1P	=	(abc)DbDDPac	xx
4. D23	=	(abc)DbDKDca	x
5. D4R	=	(ab)DaDKDbR	-
6. D5R	=	(a)DaR	-
7. DP4	=	(ab)DaD4b	x
8. D76	=	(ab)Dba = T	xx
9. D1R	=	(abc)DDabDac	-
10. DP8	=	(ab)DaDTb	xx
11. D10.9	=	(abc)DDbaDca	x
12. D1.10	=	(abc)DDacb = C	xx
13. D12.11	=	(abc)DDabDcb	x
14. DP.13	=	(ab)DaD.13.b	x
15. D14.12	=	(abc)DDacDbc = A	x

10. *Single Axiom Equivalent to CCCppqq and Syll.* (Wajsberg showed in [13] that from *CCCppqq* and *CCpqCCqrCpr* we may infer a substitution in *CCpCqrCqCpr*, namely *CCpCQrCQCpr*, from which we can obtain the other Syll *CCqrCCpqCpr*. Meredith obtained the same result independently in 1956, when working on pure strict implication, and Belnap also obtained it when working on entailment. Belnap's proof is in [1]; Meredith's was as follows:-

1.	*CCpqCCqrCpr*	
2.	*CCCppqq*	
3.	*CCCCqrCprsCCpqs*	= D11
4.	*CCpqCCCprsCCqrs*	= D31
5.	*CCCCCppqrsCCqrs*	= D42
6.	*CCCCqrstCCCCCppqrst*	= D15
7.	*CCCCCppqqrr*	= D62
8.	*CCCCCprsCCqrstCCpqt*	= D14
9.	*CCCppqCCqrr*	= D87
10.	*CPCCPqq*	= D39
11.	*CCpCQrCQCpr*	= DD3.3.10

In 1962 Belnap's collaborator A. R. Anderson asked Meredith if he could find a single axiom equivalent to Wajsberg's two premisses, certain general methods for obtaining single axioms, e.g. that suggested in Tarski's [12], p. 44, being unavailable in the absence of *CpCqp*; and Meredith obtained the result below. "In sub-systems of (B, C, I)", he commented, "there is complete agreement between theses and combinatory logic and I find the latter quicker to work with".)

1 = λaa ∼ *Cpp*, 2 = λaDa1 ∼ *CCCppqq*.

For brevity I omit the λ-prefix, which is understood as sufficient of λaλb .... to cover the variables. This is possible in the absence of K.

I. Ax 3 = DDDa2cDbd ∼ *CCCCCppqqCrCst* - *CCusCrCut*
4 = D33 = DDD32bDac = DDacDbd
5 = D34 = DDD42bDac = DDDac1Dbd
6 = D35 = DDD52bDac = DDDDac11Dbd
7 = D63 = DDDD3b11Dac = DDDb21Dac
*8 = D57 = DDD7b1Dac = DbDac ∼ *CCpqCCqrCpr*
9 = DD737 = DDD721D3a = DD3a1 = DDDa2bc
10 = D98 = DDD82ab = DaDb1
*11 = DD7(10)7 = DDD721D(10)a = DD(10)a1 = Da1 = 2
(11 = DDD7978).

II. The commutation also works but is much heavier, though I got it first. Very briefly:-

Ax 3 = DDDb2cDad
4 = D33 = DDDa2bD3c
5 = DDDD43333 = DbDDa1c

6 = D4D45 = DDDac1Dbd
*7 = D46 = DbDac (= D6D65)
8 = D65 = DDb1Dac
9 = DD858 = DDa1b
10 = D93 = DDDa2bc
*11 = DD(10)89 = DDD829a = Da1 = 2

III. Proof of Axiom 1 from

2 = Da1
3 = DbDac
Note: DD231 = D1D2a = D2a = 2
4 = D33 = DaD3b
5 = D44 = DDacDbd
6 = D43 = D3D3a
7 = D46 = D3D3D3a
8 = DD722 = DD3D3D322
= D2DD3D32a
= DDD3D32a1
= DaDD321
= Da2
*9 = DD385 = D5D8a = DDD8acDbd
= DDDa2cDbd

(There follows a later communications supplementing and improving these results).

With my former conventions:

(i) 1 = λaa, (ii) 2 = λaDa1, (iii) omission of the λ-prefix.

I give three more single axioms for the system (CCCppqq, CCpqCCqrCpr): the first contains encysted 2 and is obviously best possible of this kind; the others are longer but contain only encysted 1.

Concerning proofs in λ-logic versus prop. logic: I find λ easier for breakdown of a complex single axiom, but the converse process I find easier propositionally.

(A1) 3 = DDb2Dac ~ CCpqCCCCCssttCqrCpr
Note: DD3aD3b = λcDbDac
4 = D33 = DDa2D3b
5 = D43 = DD32D3a = DaD2b = DaDb1
6 = D45 = DD52D3a = D2DD3a1 = DD3a11 = D2Da1 = DDa11
*7 = D63 = D2a = Da1
8 = D4D34 = DD33D34 = D4D3a = DDD3a2D3b = DaD3b
*9 = D88 = DD3aD3b = DbDac

For the next two I use J = λaλbDab = λaDDBa1 = λaDDB1a. DDJab = Dab; if X begins with λ, DJX = X.

CCpCQrCQCpr ~ DDacDJb

(A2) 3 = DDDad1DDBbc ~ CCsCCuuCCprtCCqrCCpqCst
4 = D33 = DDDcd1DDBab
5 = D34 = DDDaDbd1DJc
6 = D53 = DDDDacd1DJb
7 = D64 = DDa1DDBbc
8 = D57 = DDDac1DJb
9 = D77 = DaDbDcd
*10 = D89 = DbDac
11 = D88 = DDDbDJa11
*12 = DD11.10.3 = Da1 = 2

(A3) 3 = DDDad1DDBcb
4 = D33 = DDDcd1DDBba
5 = D34 = DDDbdad1DJc
6 = DD543 = DDa1DDBcb
7 = DD534 = DDDac1D3b
8 = D66 = DcDbDad
9 = D74 = DDDaDJb1DJc
10 = D93 = DDDac1DJb
11 = D10.8 = DaDbc
12 = D10.10 = DDDbDJa11
*13 = DD12.11.3 = Da1 = 2
*14 = D6.13 = DbDac

11. *Two-valued C-O.* (In 1952 Meredith obtained two single axioms for the full propositional calculus in *C* and *O*. The development of one of them is given in [8]; that of the other, below.)

1.	*CCCpqCCOrsCCspCtCup*	
2.	*CCCtCupCpqCrCsCpq*	= D11
3.	*CCCrCpqCtCupCuCsCtCup*	= D12
4.	*CCCsCtCupCrCpqCwCvCrCpq*	= D13
5.	*CCCpqrCqCsr*	= DDD41nn
6.	*CCCvCrCpqCsCtCupCxCwCsCtCup*	= D14
7.	*CCCpqpCrCsp*	= DDD61nn
8.	*CCCspCpqCtCrCpq*	= D17
9.	*CCCrCpqCspCuCtCsp*	= D18
10.	*CCCtCspCrCpqCvCuCrCpq*	= D19
11.	*CCqrCqCpr*	= DDD10.1.n.n
12.	*CCCOrsCqCCspCtCup*	= D51
13.	*CsCrCqCCspCtCup*	= D5.12
14.	*CCCqCCCsrpCtCupsCwCvs*	= D1.13
15.	*CCCqCCCsrpCtCupsCxCwCvs*	= D11.14
16.	*CCCwCvsCqCCCsrpCtCupCzCyCqCCCsrpCtCup*	= D1.15
17.	*CCsCupCCCsrpCtCup*	= DDD16.1.n.n
18.	*CsCCspCtCup*	= DDD8.12.n.n
19.	*CCCsrCtCupCqCCspCtCup*	= D17.18
20.	*CpCqCrCsp*	= D55

21.	*CCpCqCprCsCtCqCpr*	= DD19.20.n
22.	*CCpCqCprCqCpr*	= DDD21.21.n.n.n
23.	*CpCCpqq*	= D22.11
24.	*CCpqCCCprCsqCsq*	= DD19.23.n
25.	*CCCCpqtCuCCCprCsqCsqCuCCCprCsqCsq*	= D24.24
26.	*CCCCCprCsqCsqCCpqtCuCvCCpqt*	= D1.25
27.	*CCCpqCCOrsCvCCspCtCup*	= D11.1
28.	*CCpqCCqCprCtCsCpr*	= DDD26.27.n.n
29.	*CCqCprCCpqCsCpr*	= D22.28
30.	*CCqCprCtCCpqCsCpr*	= D11.29
31.	*CCtCqCprCvCtCCpqCsCpr*	= D29.30
32.	*CCqrCCpqCsCpr*	= DD31.11.n
*33.	*CCpqCCqrCpr*	= D32.22
*34.	*CCCpqpp*	= DD22.7.n
*35.	*CpCqp*	= D33.7

(This proves sufficiency of the axiom for *C*-pure; to prove it for the full calculus Meredith should also prove *COp*. But given *C*-pure, and so *Cpp* and *CqCpp*, *COp* is deducible as DDDD1.*CqCpp*.*Cpp*.n.n. And *COp* with *CCCpqrCCrpCsp* would be a shorter axiomatisation, although 1 is organic).

12. *Full Propositional Calculus in N and K.* (1-3 are Rosser's axioms in [11] for this version of full p.c. Meredith deduces an alternative to Rosser's second axiom, the deduction of Rosser's axiom from Meredith's going through analogously. The other alternatives Meredith mentions look longer than Rosser's axioms but are not when all defined terms in the latter are duly expanded.)

Dpq = *NKpq* *Cpq* = *DpNq*

		1.	*CpKpp*	Rosser
		2.	*CKpqp*	Rosser
		3.	*CCpqCDqrDrp*	Rosser
D31	=	4.	*CDKppqDqp*	
D42	=	5.	*DNpp*	
D32	=	6.	*CDprDrKpq*	
D36	=	7.	*CDDrKpqsDsDpr*	
D75	=	8.	*DKrKpqDpr*	
D48	=	9.	*DDpKpqKpq*	
D79	=	10.	*CKpqKpp*	
D3.10	=	11.	*CDKpprDrKpq*	
D11.2	=	12.	*DNpKpq*	
D6.12	=	13.	*DKpqKNpr*	
D4.13	=	14.	*DKNpqp*	
D3.14	=	15.	*Cpp*	
D3.15	=	16.	*CDpqDqp*	
D3.5	=	17.	*CDpqCqNp*	
D17.16	=	18.	*CNDqpNDpq*	
D3.18	=	19.	*CDNDpqrCrDqp*	

D3.16	=	20. *CDDqprDrDpq*
D20.19	=	21. *DNCrDqpCrDpq*
D16.21	=	22. *CCrDpqCrDqp*
D3.3	=	23. *CDCDqrDrpsDsDpq*
D23.22	=	24. *DNCDqrDprCpq*
D16.24	=	25. *CCpqCDqrDpr*
(25	=	DD.22.23.22)
D20.5	=	26. *CKqpKpq*
DD25.26.2	=	27. *CKpqq*

With 27 instead of 2 we have alterations only in 6, 7, 8, 9, 10, 11, 12, 13, 14, 27.

14 and *DKqNpp* are easier than either.

13. *Two-valued E-pure*. (Łukasiewicz having shown in [4], reproduced in Polish in [7], that any one of the formulae *EEpqEErqEpr*, *EEpqEEprErq*, *EEpqEErpEqr* would suffice as a single axiom, with substitution and *E*-detachment, for that part of the propositional calculus which has no constant but material equivalence. He has also been credited with showing that no other axiom of equal length would do, but this is not so, Meredith having shown in August 1951 that either of the formulae *EEEpqrEqErp* or *ErEEqErpEpq* would do, and later that the same property is possessed by *EpEEqErpEqr*, *EEpEqrErEpq* (the easiest, he claims, in development), *EEpqErEEqrp*, *EEpqErEErqq*, *EEEpEqrrEqp* and *EEEpEqrqErp*. We give below not Meredith's original 1951 deduction from *EEEpqrEqErp* but his later improvement on it.)

1.	*EEEpqrEqErp*	
2.	*ErEEqErpEpq*	= D11 (DD222 = 1)
3.	*EEsE*(1)*tEts*	= D21
4.	*EEErpEEpqrq*	= D32
5.	*EEEpqqp*	= D42
6.	*EqEpEpq*	= D15
7.	*EErEsEsr*(6)	= DD666
8.	*EEsEsrE*(6)*r*	= D17
9.	*EEsrEE*(6)*rs*	= D18
10.	*EE*(6)*EE*(6)*rsEsr*	= D99
11.	*EEsEtEtsEEErpEEpqrq*	= DD646
12.	*EqEEEpEprqr*	= D10.11
13.	*EEsE*(12)*tEts*	= D2.12
14.	*EErqEEpEprq*	= D13.2
15.	*EEsEsEEpqrEqErp*	= D14.1
16.	*EEsEEpqrEqErp*	= D1.15
17.	*EEEprEqpErq*	= D16.14
*18.	*EEpqEErpEqr*	= D1.17

(Meredith has noted that 1 will suffice for *E*-pure not only with ordinary detachment but also with reverse detachment, i.e. the rule to infer α from $E\alpha\beta$ and β, as its primitive rule. One way of confirming this is to show that with this formula and reverse detachment, ordinary detachment is obtainable as a derived rule, thus (writing '**R**mn' for the reverse detachment of n from m):-

1.	*EEEpqrEqErp*	
2.	$E\alpha\beta$	
3.	α	
4.	*EEErpEEpqrq*	= **R**11
5.	*EErpEEpαr*	= **R**43
6.	*EEpEpqEqα*	= **R**15
7.	*EEαEpEpqq*	= **R**16
8.	*EαEpEpα*	= **R**73
9.	*EEEpααp*	= **R**18
10.	*Eαα*	= **RR**933
11.	*EEqEαpEpq*	= **R**51
12.	β	= **RR**(11)2(10).)

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