SOME DEFINITIONS OF SUBJUNCTIVE IMPLICATION, OF COUNTERFACTUAL IMPLICATION, AND OF RELATED CONCEPTS

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In recent years, a difficult problem has come to the attention of analytic philosophy; this is the problem of subjunctive conditionals, i.e. the problem of reconstructing such statements of ordinary language as 'if triangles were squares, then triangles would have four sides' in a way which makes explicit whatever is being implicitly asserted in the statements. Since a statement like the one just mentioned obviously excludes the assertion of what is in some sense opposed to it, e.g. 'if triangles were squares, then triangles would not have four sides' with respect to the statement mentioned above, the required reconstruction must take this into account and so at least cannot easily be given in the form of statements which are not about statements.[1]

In this discussion,[2] we devote ourselves to definitions. Our aim is to define several closely related metamathematical relations of subjunctive and counterfactual implication and to then use these relations for the definition of some very closely related concepts. The basic idea from which all of these definitions spring is that subjunctive implication can be defined with respect to axioms and inference rules which are in a sense transportable. What is meant by this will be made more clear below.

I. PRELIMINARIES

On the assumption that the reader is acquainted with letters, numerals, subscripts, superscripts, arithmetic, and the rudiments of set theory, we begin with some preliminary conventions and definitions.[3]

Convention. The letters 'a' through 'z' range over sets.

Definition 1. If m is a positive integer, then x is an m-term sequence just in case x is a function whose domain is the set of all positive integers not greater than m.

Received July 12, 1960,

Definition 2. x is a finite sequence just in case there is a positive integer m such that x is an m-term sequence.

Definition 3. If r and s are finite sequences, then $r \frown s$ is the finite sequence t satisfying the following conditions:

(1) the domain of t is the set of all positive integers not greater than the sum $m+n$ of the greatest number m and n in the domains of r and s respectively.

(2) for any member i of the domain of t, either the difference i- the greatest number m in the domain of r is not greater than 0 and $t(i) = r(i)$ or the difference i- the greatest number m in the domain of r is greater than 0 and $t(i) = s(i\text{-}m)$.

Definition 4. If r is a finite sequence, then s is an inner sequence of r just in case s is in every set k satisfying the following conditions:

(1) for any finite sequence t, if t is in the range of r, then t is in k

(2) for any t in k and any u, if u is a finite sequence and u is in the range of t, then u is in k.

Definition 5. If s is a finite sequence, then x occurs in s just in case x is in every set k satisfying the following conditions:

(1) every member of the range of s is in k

(2) every member of the range of any inner sequence of s is in k.

Definition 6.

(1) $<x>$ is the 1-term sequence t such that $t(1) = x$

(2) $<xy>$ is the 2-term sequence t such that $t(1) = x$ and $t(2) = y$

(3) $<xyz>$ is the 3-term sequence t such that $t(1) = x$, $t(2) = y$, and $t(3) = z$.

Definition 7. x is a logical constant just in case x is either '$\sim$' or '$\rightarrow$' or '$\wedge$' or '$=$'.

Definition 8. If m is a positive integer, then the m^{th} variable is the letter 'v' with the m^{th} numeral as subscript.

Definition 9. x is a variable just in case there is a positive integer m such that x is the m^{th} variable.

Definition 10. If m and n are positive integers, then the m^{th} n-place predicate is the letter 'P' with the m^{th} numeral as subscript and the n^{th} numeral as superscript.

Definition 11. If m is a positive integer, then x is an m-place predicate just in case there is a positive integer n such that x is the n^{th} m-place predicate.

Definition 12. x is a predicate just in case there is a positive integer m such that x is an m-place predicate.

Definition 13. x is a symbol just in case x is a logical constant or a variable or a predicate.

Definition 14. x is an atomic formula just in case x satisfies one of the following conditions:

(1) there are variables v and w such that $x = <v\ \text{'='}\ w>$

(2) there is a positive integer m, an m-place predicate p, and an m-term sequence v such that the range of v is included in the set of variables and $x = \langle p \rangle ^\frown v$.

Definition 15. x is a formula just in case x is in every set k satisfying the following conditions:

(1) every atomic formula is in k

(2) for any y in k, $\langle$'$\sim$' $y\rangle$ is in k

(3) for any y and z in k, $\langle y$ '$\to$' $z\rangle$ is in k

(4) for any y in k and variable v, $\langle$'$\wedge$' $vy\rangle$ is in k.

Definition 16. x is an inference rule just in case x satisfies the following conditions:

(1) x is a function

(2) there is a positive integer m such that the domain of x is included in the set of all m-term sequences whose range is included in the set of formulas

(3) for any y in the domain of x, $x(y)$ is a set of formulas.

Definition 17. *MP* is the inference rule x satisfying the following conditions:

(1) the domain of x is the set of all 2-term sequences s such that, for some formulas f and g, $s(1) = \langle f$ '$\to$' $g\rangle$ and $s(2) = f$

(2) for any s in the domain of x, $x(s)$ = the set whose only member is $(s(1))(3)$.

Definition 18. *UG* is the inference rule x satisfying the following conditions:

(1) the domain of x is the set of all 1-term sequences s whose range is included in the class of formulas

(2) for any s in the domain of x, $x(s)$ = the set of all f such that, for some variable v, $f = \langle$'$\wedge$' $vs(1)\rangle$.

Definition 19. If x is a variable, then

(1) for any variables v and w, x is free in $\langle v$ '=' $w\rangle$ just in case $x = v$ or $x = w$

(2) for any positive integer m, m-place predicate p, and m-term sequence v whose range is included in the set of variables, x is free in $\langle p \rangle ^\frown v$ just in case x is in the range of v

(3) for any formula f, x is free in $\langle$'$\sim$' $f\rangle$ just in case x is free in f

(4) for any formula f and g, x is free in $\langle f$ '$\to$' $g\rangle$ just in case x is free in f or in g

(5) for any formula f and variable v, x is free in $\langle$'$\wedge$' $vf\rangle$ just in case x is free in f and $x \neq v$.

Definition 20. If x and y are variables, then

(1) for any variables v and w, the result of properly substituting x for y in $\langle v$ '=' $w\rangle$ is the 3-term sequence s such that, for any i in the domain of s, either $\langle v$ '=' $w\rangle$ $(i) = y$ and $s(i) = x$ or $\langle v$ '=' $w\rangle$ $(i) \neq y$ and $s(i) = \langle v$ '=' $w\rangle$ (i)

(*2*) for any positive integer *m*, *m*-place predicate *p*, and *m*-term sequence *v* whose range is included in the set of variables, the result of properly substituting *x* for *y* in <*p*>⌢*v* is the *m+1* -term sequence *s* such that, for any *i* in the domain of *s*, either (<*p*>⌢*v*) (*i*) = *y* and *s*(*i*) = *x* or (<*p*>⌢*v*) (*i*) ≠ *y* and *s*(*i*) = (<*p*>⌢*v*) (*i*)

(*3*) for any formula *f*, the result of properly substituting *x* for *y* in <'∼' *f*> is <'∼' the result of properly substituting *x* for *y* in *f*>

(*4*) for any formulas *f* and *g*, the result of properly substituting *x* for *y* in <*f* '→' *g*> is <the result of properly substituting *x* for *y* in *f* '→' the result of properly substituting *x* for *y* in *g*>

(*5*) for any formula *f* and variable *v*, the result of properly substituting *x* for *y* in <'∧' *vf*> is the *3*-term sequence *s* satisfying one of the following conditions:

(a) *y* is free in <'∧' *vf*> and either *x* ≠ *v* and *s* is <'∧' *v* the result of properly substituting *x* for *y* in *f*> or *x* = *v* and, for some positive integer *m*, *x* is the *m*th variable and *s* is <'∧' the *m+1*th variable the result of properly substituting *x* for *y* in the result of properly substituting the *m+1*th variable for *v* in *f*>

(b) *y* is not free in <'∧' *vf*> and *s* is <'∧' *vf*>.

Definition 21. *UI* = the set of all *f* such that, for some formula *g* and variables *v* and *w*, *f* = <<'∧' *vg*> '→' the result of properly substituting *w* for *v* in *g*>.

Definition 22. *VC* = the set of all *f* such that, for some formulas *g* and *h* and variable *v*, *v* is not free in *g* and *f* = <<'∧' *v* <*g* '→' *h*>> '→' <*g* '→' <'∧' *vh*>>>.

Definition 23. *x* is a standard sentential axiom just in case *x* satisfies one of the following conditions:

(*1*) for some formulas *f* and *g*, *x* = <*g* '→' <*f* '→' *g*>>

(*2*) for some formulas *f*, *g* and *h*, *x* = <<*f* '→' <*g* '→' *h*>> '→' <<*f* '→' *g*> '→' <*f* '→' *h*>>>

(*3*) for some formulas *f* and *g*, *x* = <<<'∼' *f*> '→' <'∼' *g*>> '→' <*g* '→' *f*>>.

Definition 24. *x* is a standard identity axiom just in case *x* satisfies one of the following conditions:

(*1*) for some variable *v*, *x* = <*v* '=' *v*>

(*2*) for some variables *v* and *w* and formula *f*, *v* is free in *f* and *x* = <<*v* '=' *w*> '→' <*f* '→' the result of properly substituting *w* for *v* in *f*>> or *x* = <<*v* '=' *w*> '→' <the result of properly substituting *w* for *v* in *f* '→' *f*>>.

Definition 25. If *r* is an inference rule and *1* is a set of formulas, then *1* is closed under *r* just in case, for any *s* in the domain of *r*, the range of *s* is included in *1* only if *r*(*s*) is included in *1*.

Definition 26. If *k* is a set of inference rules and *1* is a set of formulas, then *1* is closed under *k* just in case, for any *r* in *k*, *1* is closed under *r*.

Definition 27. *x* is a deductive pair just in case *x* is a *2*-term sequence satisfying the following conditions:

(*1*) x (*1*) is a set of formulas

(*2*) x (*2*) is a set of inference rules.

Definition 28. The standard pair is the deductive pair p satisfying the following conditions:

(*1*) for any f, f is in $p(1)$ just in case f is a standard sentential axiom or in *UI* or in *VC* or a standard identity axiom

(*2*) for any r, r is in $p(2)$ just in case r is *MP* or r is *UG*.

Definition 29. If k is a set of formulas, p is a deductive pair, and f is a formula, then k implies f by p just in case f is in every class *1* satisfying the following conditions:

(*1*) k is included in *1*

(*2*) $p(1)$ is included in *1*

(*3*) *1* is closed under $p(2)$.

Definition 30. The standard set of provable formulas is the set of all formulas f such that the empty set implies f by the standard pair.

II. SUBJUNCTIVE IMPLICATION

On the basis of the preceding definitions, we define a relation of subjunctive implication.

Definition 31. If k and *1* are sets of formulas, p is a deductive pair, and f is a formula, then k subjunctively implies f by p on the basis of *1* just in case the union of k and *1* implies f by p.

What is asserted in this definition can best be seen by means of examples. Suppose that, instead of ‘P^1_1’, ‘P^1_2’, and ‘P^1_3’, our first three *1*-place predicates were ‘is a triangle’, ‘is a square’, and ‘is four-sided’ in that order. Let $x = \langle$‘$\wedge$’ ‘v_1’ $\langle\langle$‘is a triangle’ ‘v_1’$\rangle$ ‘$\rightarrow$’ $\langle$‘is a square’ ‘v_1’$\rangle\rangle\rangle$ and $y = \langle$‘$\wedge$’ ‘v_1’ $\langle\langle$‘is a square’ ‘v_1’$\rangle$ ‘$\rightarrow$’ $\langle$‘is four-sided’ ‘v_1’$\rangle\rangle\rangle$ and $f = \langle$‘$\wedge$’ ‘v_1’ $\langle\langle$‘is a triangle’ ‘v_1’$\rangle$ ‘$\rightarrow$’ $\langle$‘is four-sided’ ‘v_1’$\rangle\rangle\rangle$. Also, let p be the deductive pair such that $p(1)$ = the standard set of provable formulas and $p(2)$ = the set whose only member is *MP*. Finally, let k be the set whose only member is y and let *1* be the set whose only member is x. We can then easily show both that k subjunctively implies f by p on the basis of *1* and that k does not subjunctively imply $\langle$‘$\sim$’$f\rangle$ by p on the basis of *1*. In other words, given what amounts to the usual rules of identity logic, we can show that the statement that all squares are four-sided and the statement that that all triangles are squares together imply the statement that all triangles are four-sided and not its negation.

Suppose, on the other hand, that we let $x = \langle$‘$\sim$’$\langle$‘v_1’ ‘=’ ‘v_1’$\rangle\rangle$ and $f = \langle\langle$‘v_2’ ‘=’ ‘v_1’$\rangle$ ‘$\rightarrow$’ $\langle$‘$\sim$’ $\langle$‘v_2’ ‘=’ ‘v_1’$\rangle\rangle\rangle$. Also, suppose we let k = the empty set. Then, if *1* is the set whose only member is x, and if we want to show that k subjunctively implies f by p on the basis of *1* and also that k does not subjunctively imply $\langle$‘$\sim$’$f\rangle$ by p on the basis of *1*, we cannot take p to be the same as it was in our first example; we must, so to speak, replace the deductive pair of our first example with one that leads to the desired results. For the sake of simplicity, suppose that p differs from the deductive pair of our first example only in that the only members of $p(1)$ are a relevant instance of the theorem that anything different from a given thing is different from anything identical with that thing and UI; that is, let $y = <`\wedge' `v_2' <<`\sim' <`v_2' `=' `v_1'>> `\rightarrow' <`\wedge' `v_3' <<`v_3' `=' `v_1'> `\rightarrow' <`\sim' <`v_2' `=' `v_3'>>>>>>$, let $p(1)$ be the set whose only members are $y$ and the members of $UI$, and let $p(2)$ be the set whose only member is $MP$. We can then easily show that $k$ subjunctively implies $f$ and not $<`\sim' f>$ by p on the basis of 1.

III. SEMANTIC CONSIDERATIONS

The reader may have noted that the definition of subjunctive implication given above does not in any way restrict our choice of which formulas to put into k, 1, and $p(1)$ and of which inference rules to put into $p(2)$; therefore, what we decide on depends only on what we are trying to say. However, we could always arbitrarily pick out a set m of formulas that we wish to accept and a set n of formulas which we do not wish to accept and then define a narrower conception of subjunctive implication for which we require that k is a subset of m and that 1 is a subset of n. Similar restrictions can be made with respect to $p(1)$ and $p(2)$. Yet, this is probably too arbitrary for many philosophers.

One way to narrow down the field from which we can select k, 1, $p(1)$, and $p(2)$ is to restrict our choice of these semantically, e.g., by restricting k to true formulas, 1 to untrue formulas, and so on. In order to do this precisely, we introduce some more preliminary definitions; this time, we also assume that the reader is acquainted with physical objects.

Definition 32. x is a domain just in case x is not empty.

Definition 33. If d is a domain, then x is an interpreter for d just in case x is a function satisfying the following conditions:

(1) the domain of x is the union of the set of predicates and the set whose only member is '='

(2) for any positive integer m and m-place predicate p, $x(p)$ is included in the set of all m-term sequences s whose range is included in d

(3) $x(`=')$ is the set of all 2-terms sequences s such that, for some member y of d, y is the only member of the range of s.

Definition 34. x is an interpreter just in case there is a domain d such that x is an interpreter for d.

Definition 35. If d is a domain, then x is an assigner for d just in case x is a function whose domain is the set of variables and whose range is included in d.

Definition 36. If d is a domain, a is an assigner for d, m is in d, and

v is a variable, then $a\binom{v}{m}$ is the assigner x such that, for any y in the domain of x, either $y = w$ and $x(y) = m$ or $y \neq v$ and $x(y) = a(y)$.

Definition 37. If d is a domain, i is an interpreter for d and a is an assigner for d, then

(*1*) for any variables v and w, $\langle v\,\text{'='}\,w\rangle$ is satisfied in d with respect to i and a just in case $\langle a(v)\,a(w)\rangle$ is a member of $i(\text{'='})$

(*2*) for any positive integer m, m-place predicate p, and m-term sequence v whose range is included in the set of variables, $\langle p\rangle\frown v$ is satisfied in d with respect to i and a just in case the m-term sequence s such that, for any x in the domain of s, $s(x) = a(v(x))$ is a member of $i(p)$

(*3*) for any formula f, $\langle\text{'}\sim\text{'}\,f\rangle$ is satisfied in d with respect to i and a just in case f is not satisfied in d with respect to i and a

(*4*) for any formulas f and g, $\langle f\,\text{'}\rightarrow\text{'}\,g\rangle$ is satisfied in d with respect to i and a just in case f is satisfied in d with respect to i and a only if g is satisfied in d with respect to i and a

(*5*) for any formula f and variable v, $\langle\text{'}\wedge\text{'}\,vf\rangle$ is satisfied in d with respect to i and a just in case, for any member m of d, f is satisfied in d with respect to i and $a\binom{v}{m}$.

Definition 38. If f is a formula, d a domain, and i an interpreter for d, then f is true in d with respect to i just in case, for any assigner a for d, f is satisfied in d with respect to i and a.

Definition 39. If f is a formula and k is a set of interpreters, then f is analytic with respect to k just in case, for any i in k and any domain d such that i is an interpreter for d, f is true in d with respect to i.

Definition 40. If r is an inference rule, d is a domain, and i is an interpreter for d, then r preserves truth in d with respect to i just in case, for any member s of the domain of r, every member of the range of s is true in d with respect to i only if every member of $r(s)$ is true in d with respect to i.

Definition 41. If r is an inference rule and k is a set of interpreters, then r preserves truth with respect to k just in case, for any i in k and any domain d such that i is an interpreter for d, r preserves truth in d with respect to i.

Definition 42. The world is the set of all physical objects.

Definition 43. If f is a formula and i is an interpreter, then, for any x, f is empirically significant with respect to i and x just in case, for any symbol s in the domain of i which occurs in f, the range of every t in $i(s)$ is included in the union of the world and x.

We could also define various two-place relation of empirical significance in addition to the one defined above;[4] for instance, we would call a formula f empirically significant in the first sense with respect to an interpreter i just in case f is empirically significant with respect to i and the set of real numbers. Also, we could define relations of significance analogous to all of these with respect to other domains than the world. However, since such definitions are not necessary for our purposes and since we want to leave open the question of which entities other than physical objects and their properties can be discussed in empirically significant statements, we restrict ourselves to definition 43.

Definition 44. If f is a formula, i is an interpreter, and f is empirically significant with respect to i and x, then f is empirically true with respect to i and x just in case, for any interpreter j for the union of the world and x, if, for any symbol s in the domain of i which occurs in f, $j(s) = i(s)$, then f is true in the union of the world and x with respect to j.

Definition 45. If r is an inference rule and i is an interpreter, then, for any x, r preserves empirical truth with respect to i and x just in case, for any s in the domain of r, if every member of the range of s is empirically significant and true with respect to i and x, then every member of $r(s)$ is empirically significant and true with respect to i and x.

On the basis of the preceding definitions, we define several semantic relations of subjunctive implication.

Definition 46. If k and l are sets of formulas, f is a formula, p is a deductive pair, d is a domain, and i is an interpreter for d, then k subjunctively implies f by p on the basis of l in d with respect to i in the first sense just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p
(*2*) every member of k is true in d with respect to i
(*3*) no member of l is true in d with respect to i.

Definition 47. If k and l are sets of formulas, f is a formula, p is a deductive pair, d is a domain, and i is an interpreter for d, then k subjunctively implies f by p on the basis of l in d with respect to i in the second sense just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p
(*2*) every member of k is true in d with respect to i
(*3*) no member of l is true in d with respect to i
(*4*) every member of $p(1)$ is true in d with respect to i
(*5*) every member of $p(2)$ preserves truth in d with respect to i.

Definition 48. If k and l are sets of formulas, f is a formula, p is a deductive pair, m is a set of interpreters, d is a domain, i is a member of m, and i is an interpreter for d, then k subjunctively implies f by p and m on the basis of l in d with respect to i just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p
(*2*) every member of k is true in d with respect to i
(*3*) no member of l is true in d with respect to i
(*4*) every member of $p(1)$ is analytic with respect to m
(*5*) every member of $p(2)$ preserves truth with respect to m.

These three definitions of subjunctive implication seem to be more adequate than definition 31 is in the sense of making explicit more of what is implicitly present in the subjunctive conditionals of ordinary language than definition 31 does.

IV. COUNTERFACTUAL IMPLICATION

On the basis of the definitions given above, we define several semantic relations of counterfactual implication.

Definition 49. If i is an interpreter, k and l are sets of formulas empirically significant with respect to i and x, f is a formula empirically significant with respect to i and x, and p is a deductive pair, then k counterfactually implies f by p on the basis of l with respect to i and x in the first sense just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p

(*2*) every member of k is empirically true with respect to i and x

(*3*) no member of l is empirically true with respect to i and x.

Definition 50. If i is an interpreter, k and l are sets of formulas empirically significant with respect to i and x, and p is a deductive pair, then k counterfactually implies f by p on the basis of l with respect to i and x in the second sense just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p

(*2*) every member of k is empirically true with respect to i and x

(*3*) no member of l is empirically true with respect to i and x

(*4*) every member of $p(1)$ is empirically significant and true with respect to i and x

(*5*) every member of $p(2)$ preserves empirical truth with respect to i and x.

Definition 51. If m is a set of interpreters, i is a member of m, k and l are sets of formulas empirically significant with respect to i and x, and p is a deductive pair, then k counterfactually implies f by p and m on the basis of l with respect to i and x just in case the following conditions are satisfied:

(*1*) the union of k and l implies f by p

(*2*) every member of k is empirically true with respect to i and x

(*3*) no member of l is empirically true with respect to i and x

(*4*) every member of $p(1)$ is analytic with respect to m

(*5*) every member of $p(2)$ preserves truth with respect to m.

V. THEOREMS

Although this discussion is devoted to the statement of definitions rather than to the proof of theorems, we shall at this point state a few elementary theorems which describe explicitly some of the relations between the various kinds of subjunctive and counterfactual implication.

Theorem 1. On the assumptions of definition *46*, k subjunctively implies f by p on the basis of l in d with respect to i in the first sense just in case the following conditions are satisfied:

(*1*) k subjunctively implies f by p on the basis of l

(*2*) every member of k is true in d with respect to i

(*3*) no member of l is true in d with respect to i.

This theorem follows immediately from definitions *31* and *46*.

Theorem 2. On the assumptions of definition *46*, k subjunctively implies f by p on the basis of l in d with respect to i in the second sense just in case the following conditions are satisfied:

(*1*) k subjunctively implies f by p on the basis of l in d with respect to i in the first sense

(*2*) every member of $p(1)$ is true in d with respect to i

(*3*) every member of $p(2)$ preserves truth in d with respect to i.

This theorem follows immediately from definitions *46* and *47*.

Theorem 3. On the assumptions of definition *48*, k subjunctively implies f by p and m on the basis of l in d with respect to i just in case the following conditions are satisfied:

(*1*) k subjunctively implies f by p on the basis of l in d with respect to i in either the first or the second sense.

(*2*) every member of $p(1)$ is analytic with respect to m

(*3*) every member of $p(2)$ preserves truth with respect to m.

This theorem follows immediately from definition *46*, definition *48*, and theorem *2*.

Theorem 4. On the assumptions of definition *49*, k counterfactually implies f by p on the basis of l with respect to i and x in the first sense just in case the following conditions are satisfied:

(*1*) k subjunctively implies f by p on the basis of l

(*2*) every member of k is empirically true with respect to i and x

(*3*) no member of l is empirically true with respect to i and x.

This theorem follows immediately from definition *31* and *49*.

Theorem 5. On the assumptions of definition *49*, k counterfactually implies f by p on the basis of l with respect to i and x in the second sense just in case the following conditions are satisfied:

(*1*) k counterfactually implies f by p on the basis of l with respect to i and x in the first sense

(*2*) every member of $p(1)$ is empirically significant and true with respect to i and x

(*3*) every member of $p(2)$ preserves empirical truth with respect to i and x.

This theorem follows immediately from definitions *49* and *50*.

Theorem 6. On the assumptions of definition *51*, k counterfactually implies f by p and m on the basis of l with respect to i and x just in case the following conditions are satisfied:

(*1*) k counterfactually implies f by p on the basis of *1* with respect to i and x in either the first or the second sense

(*2*) every member of $p(1)$ is analytic with respect to m

(*3*) every member of $p(2)$ preserves truth with respect to m.

This theorem follows immediately from definition *49*, definition *51*, and theorem *5*.

VI. SUBJUNCTIVE AND COUNTERFACTUAL CONDITIONALS

Now that we have defined various relations of subjunctive and counterfactual implication, we can give truth criteria for subjunctive conditionals and counterfactual conditionals in terms of any of the relations of subjunctive implication and in terms of any of the relations of counterfactual implication respectively. Since the relations of subjunctive and counterfactual implication defined in definition *48* and definition *51* respectively are the strongest ones, we shall deal only with truth criteria given in terms of them. Also, now that we have defined a relation of empirical significance and a relation of empirical truth, we can give truth criteria for expressions which begin with the phrases 'it is empirically significant that' and 'it is a fact that' in terms of the first and the second of these relations respectively. However, before we can give truth criteria for such expressions, we must have a language system in which the expressions appear.

Definition 52. x is new logical constant just in case $x =$ '**N**' or $x =$ '**S**' or $x =$ '**C**' or $x =$ '**E**' or $x =$ '**F**'.

The constant '**N**' is to be understood as the modal constant of necessity. We place it among the new logical constants for good measure.

Definition 53. If i is an interpreter, then, for any x, y is a formula with respect to i and x in the first sense just in case y is in every set k satisfying the following conditions:

(*1*) every atomic formula is in k

(*2*) for any z in k, $<$'$\sim$' $z>$ is in k

(*3*) for any formula f, $<$'**N**' $f>$ and $<$'**E**' $f>$ are in k

(*4*) for any formula f empirically significant with respect to i and x, $<$'**F**' $f>$ is in k

(*5*) for any z and w in k, $<z$'$\rightarrow$' $w>$ is in k

(*6*) for any formulas f and g, $<f$'**S**' $g>$ is in k

(*7*) for any formulas f and g empirically significant with respect to i and x, $<f$'**C**' $g>$ is in k

(*8*) for any z in k and variable v, $<$'$\wedge$' $vz>$ is in k.

Definition 54. If i is an interpreter and f is a formula with respect to i and x in the first sense, then f is a subjunctive conditional just in case $f(2) =$ '**S**'.

Definition 55. If i is an interpreter and f is a formula with respect to

i and x in the first sense, then f is a counterfactual conditional just in case $f(2) =$ '**C**'.

Now that we have language systems in which the new symbols '**N**', '**S**', '**C**', '**E**', and '**F**' appear, we define a relation of satisfaction and then a relation of truth for them which insures that '**N**', '**S**', '**C**', '**E**', and '**F**' mean what we intend them to mean.

Definition 56. If p is a deductive pair, m is a set of interpreters, d is a domain, i is a member of m, i is an interpreter for d, and a is an assigner for d, then, for any x,

(*1*) for any atomic formula f, f is satisfied in d with respect to i, a, and x by p and m just in case f is satisfied in d with respect to i and a.

(*2*) for any formula with respect to i and x in the first sense f, $<$'$\sim$'$f>$ is satisfied in d with respect to i, a, and x by p and m just in case f is not satisfied in d with respect to i, a, and x by p and m

(*3*) for any formula f, $<$'**N**'$f>$ is satisfied in d with respect to i, a, and x by p and m just in case f is analytic with respect to m

(*4*) for any formula f, $<$'**E**'$f>$ is satisfied in d with respect to i, a, and x by p and m just in case f is empirically significant with respect to i and x

(*5*) for any formula f, if f is empirically significant with respect to i and x, then $<$'**F**'$f>$ is satisfied in d with respect to i, a, and x by p and m just in case f is empirically true with respect to i and x

(*6*) for any formulas with respect to i and x in the first sense f and g, $<f$'$\rightarrow$'$g>$ is satisfied in d with respect to i, a, and x by p and m just in case f is satisfied in d with respect to i, a, and x by p and m only if g is satisfied in d with respect to i, a, and x by p and m

(*7*) for any formulas f and g, $<f$'**S**'$g>$ is satisfied in d with respect to i, a, and x by p and m just in case the empty set subjunctively implies g by p and m on the basis of the set whose only member is f in d with respect to i

(*8*) for any formulas f and g, if f and g are empirically significant with respect to i and x, then $<f$'**C**'$g>$ is satisfied in d with respect to i, a, and x by p and m just in case the empty set counterfactually implies g by p and m on the basis of the set whose only member is f with respect to i and x

(*9*) for any variable v and formula with respect to i and x in the first sense f, $<$'$\wedge$'$vf>$ is satisfied in d with respect to i, a, and x by p and m just in case, for any y in a, f is satisfied in d with respect to i, $a(\substack{v \\ y})$, and x by p and m.

Definition 57. If p is a deductive pair, m is a set of interpreters, d is a domain, i is a member of m, i is an interpreter for d, and f is a formula with respect to i and x in the first sense, then f is true in d with respect to i and x by p and m just in case, for any assigner a for d, f is satisfied in d with respect to i, a, and x by p and m.

The truth relation just defined is rather odd since it is relativized not only to a domain and an interpreter, but also to a deductive pair, a set of interpreters, and a set x.[5] Thus, truth in this sense depends on which axioms and inference rules we choose for our deductive pair, which set x with respect to which we wish to consider formulas empirically significant, and which set of interpreters with respect to which we wish to consider formulas analytic as well as on which domain and interpreter we have in mind.

Notice that, for any formulas f and g which are not empirically significant with respect to an interpreter i and a set x, such sequences as $<\text{‘F’}f>$ and $<f\text{‘C’}g>$ are not formulas with respect to i and x in the first sense. These omissions were made so that we do not have to say that such sequences are not true in the sense of definition 57 and yet, for any formula f with respect to i and x in the first sense, can also find out what we mean when we say either that f is true or that f is not true in the sense of definition 57 by means of that definition. Notice also that, for any interpreter i, any x, and any f and g which are formulas with respect to i and x in the first sense, but not formulas, the sequences $<\text{‘N’}f>$, $<\text{‘E’}f>$, $<\text{‘F’}f>$, $<f\text{‘S’}g>$, and $<f\text{‘C’}g>$ are not formulas with respect to i and x in the first sense. These omissions made our relatively simple definition of satisfaction for formulas with respect to i and x in the first sense possible. Nevertheless, a definition of satisfaction analogous to the one of definition 56 which is also applicable to most sequences of the just-mentioned kind can be defined. Before we state this definition, we define a relation of being a formula more inclusive than the one defined in definition 53.

Definition 58. If i is an interpreter, then, for any x, y is a formula with respect to i and x in the second sense just in case y is in every set k satisfying the following conditions:

(*1*) every atomic formula is in k

(*2*) for any z in k, $<\text{‘}\sim\text{’}z>$, $<\text{‘N’}z>$, and $<\text{‘E’}z>$ are in k

(*3*) for any z in k, if, for any symbol s in the domain of i which occurs in z, the range of every t in $i(s)$ is included in the union of the world and k, then $<\text{‘F’}z>$ is in k

(*4*) for any z and w in k, $<z\text{‘}\rightarrow\text{’}w>$ is in k

(*5*) for any z and w in k, $<z\text{‘S’}w>$ is in k

(*6*) for any z and w in k, if, for any symbol s in the domain of i which occurs in z or w, the range of every t in $i(s)$ is included in the union of the world and x, then $<z\text{‘C’}w>$ is in k

(7) for any z in k and variable v, $<\text{‘}\wedge\text{’}v\,z>$ is in k.

For formulas in the sense of definition 58, we should now define inference rule, closure of a set of formulas under an inference rule, closure of a set of formulas under a set of inference rules, deductive pair, implication of a formula from a set of formulas by a deductive pair, empirical significance of a formula, being a subjunctive conditional, and being a counterfactual conditional. However, since the required definitions would be completely analogous to correlates for them which have already been stated, we simply imagine that we have replaced these correlates with them. On this basis, we can, for any x and interpreter i, define both a relation of satisfaction and a relation of truth for formulas with respect to i and x in the second sense.

Definition 59. If p is a deductive pair, m is a set of interpreters, i is a member of m, i is an interpreter for d, and a is an assigner for d, then, for any x,

(*1*) for any atomic formula f, f is satisfied in d with respect to i, a, and x by p and m just in case f is satisfied in d with respect to i and a

(*2*) for any formula with respect to i and x in the second sense f, <'∼' f> is satisfied in d with respect to i, a, and x by p and m just in case f is not satisfied in d with respect to i, a, and x by p and m

(*3*) for any formula with respect to i and x in the second sense f, <'**N**' f> is satisfied in d with respect to i, a, and x by p and m just in case, for any j in m, any domain e such that j is an interpreter for e, and any assigner b for e, f is satisfied in e with respect to j, b, and x by p and m

(*4*) for any formula with respect to i and x in the second sense f, <'**E**' f> is satisfied in d with respect to i, a, and x by p and m just in case f is empirically significant with respect to i and x

(*5*) for any formula with respect to i and x in the second sense f, if f is empirically significant with respect to i and x, then <'**F**' f> is satisfied in d with respect to i, a, and x by p and m just in case, for any interpreter j for the union of the world and x, if, for any symbol s in the domain of i which occurs in f, $j(s) = i(s)$, then, for any assigner b for the union of the world and x, f is satisfied in the union of the world and x with respect to j, b, and x by p and m

(*6*) for any formulas with respect to i and x in the second sense f and g, <f'→'g> is satisfied in d with respect to i, a, and x by p and m just in case f is satisfied in d with respect to i, a, and x by p and m only if g is satisfied in d with respect to i, a, and x by p and m

(*7*) for any formulas with respect to i and x in the second sense f and g, <f'**S**'g> is satisfied in d with respect to i, a, and x by p and m just in case the following conditions are satisfied:

(a) the set whose only member is f implies g by p

(b) for some assigner b for d, f is not satisfied in d with respect to i, b, and x by p and m

(c) for any h in $p(1)$, any j in m, any domain e such that j is an interpreter for e, and any assigner b for e, h is satisfied in e with respect to j, b, and x by p and m

(d) for any r in $p(2)$, any j in m, any domain e such that j is an interpreter for e, and any s in the domain of r, if, for any t in the range of

s and any assigner b for e, t is satisfied in e with respect to j, b, and x by p and m, then, for any u in $r(s)$ and any assigner b for e, u is satisfied in e with respect to j, b, and x by p and m

(8) for any formulas with respect to i and x in the second sense f and g, if f and g are empirically significant with respect to i and x, then $\langle f\,\text{'}\mathbf{C}\text{'}\,g\rangle$ is satisfied in d with respect to i, a, and x by p and m just in case the following conditions are satisfied:

(a) the set whose only member is f implies g by p

(b) there is an interpreter j for the union of the world and x such that, for any symbol s in the domain of i which occurs in f, $j(s) = i(s)$, but, for some assigner b for the union of the world and x, f is not satisfied in the union of the world and x with respect to j, b, and x by p and m

(c) for any h in $p(1)$, any j in m, any domain e such that j is an interpreter for e, and any assigner b for e, h is satisfied in e with respect to j, b, and x by p and m

(d) for any r in $p(2)$, any j in m, any domain e such that j is an interpreter for e, and any s in the domain of r, if, for any t in the range of s and any assigner b for e, t is satisfied in e with respect to j, b, and x by p and m, then, for any u in $r(s)$ and any assigner b for e, u is satisfied in e with respect to j, b, and x by p and m

(9) for any variable v and formula with respect to i and x in the second sense f, $\langle \text{'}\wedge\text{'}\,v f\rangle$ is satisfied in d with respect to i, a, and x by p and m just in case, for any y in d, f is satisfied in d with respect to i, $a\binom{v}{y}$, and x by p and m.

Definition 60. If p is a deductive pair, m is a set of interpreters, d is a domain, i is a member of m, i is an interpreter for d, and f is a formula with respect to i and x in the second sense, then f is true in d with respect to i and x by p and m just in case, for any assigner a for d, f is satisfied in d with respect to i, a, and x by p and m.

NOTES

1. For some of the difficulties involved in reconstructing subjunctive or counterfactual conditionals as statements which are not about statements, see, for instance, R. Chisolm's paper, "The Contrary-to-Fact Conditional" (*Mind*, Vol. 55, 1946), and N. Goodman's paper, "The Problem of Counterfactual Conditionals" (*Journal of Philosophy*, Vol. 44, 1947).

2. The author wishes to express his appreciation to Richard Montague for some valuable criticisms and suggestions.

3. These definitions and those which follow are fairly standard ones when they are not partially or entirely due to the author. Experts in logic and

set theory may skip many of the initial ones and then a few of the others here and there.

4. This definition of empirical significance is less strict than most other ones in not requiring some kind of observability for empirical significance. Note, however, that any formula each of whose symbols in the domain of an interpreter i is interpreted by i to be some set of all sequences of members of the union of the world and x among which a particular attribute is observable will be empirically significant with respect to i and x.

5. The only reason that this truth relation has not also been relativized to a set of formulas true in the sense of definition 38 is that we can always place such formulas among the axioms of the deductive pair we have in mind by selecting an appropriate set of interpreters.

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