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A STANDARDIZATION THEOREM FOR STRONG REDUCTION

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In a previous paper [2], the writer introduced a modified definition of strong reduction in combinatory logic [1a]. This paper shows that with such a strong reduction there is associated a standard reduction [3a]. All reductions are considered according to the modified definition. This allows some essential simplifications in proofs, especially of Theorem 1.

Definition of a Standard Reduction.

A strong reduction is called standard if it satisfies the following conditions:

i) The Type III steps are made last and are performed from right to left.
ii) Among the Type I and II reductions the redexes are to be contracted in the order (from left to right) of the combinators appearing at their heads. However, the redex contracted need not be of maximal extent.

The condition that Type III steps be contracted from right to left is automatically satisfied in the case that they overlap; and in case they do not, the order is irrelevant.

In the context of standard reductions, steps of Type IIc, IId, and IIf have the effect of freezing certain combinators, since they introduce expressions which can only be reduced further by Type III steps. We shall refer to combinators as being frozen without resorting to the mechanism of these steps. A combinator not frozen is called *free*.

Since the transformation of a modified strong reduction into a strong reduction in the original sense involves introduction of Type III steps ahead of Type I and II steps, a standard reduction in the present sense need not be standard in the sense of the original definition.

Two Lemmas. Two lemmas were introduced in [2] and will be used here. They are:

Lemma 1. *If* $X = [x]\mathfrak{X}$, *then* $\lambda x.Xx \succ\!\!- \lambda x.\mathfrak{X}$ *by Type I steps only. In*

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other words the contractum of a Type III step may be reversed to the original redex by a single Type II step followed by Type I steps.

Lemma 2. *The contraction of a Type II redex P may be reversed provided there are no intervening steps interior to the contractum of P.*

Standardization of a Special Case.

In this section we will prove that any reduction involving only Type I and II steps can be standardized.

Theorem 1. *If M reduces to N using only Type I and II steps, then there is a standard reduction from M to N using only Type I and II steps.*

The proof is an induction on the number of steps in the reduction from M to N. It proceeds by construction of the standard reduction from the original induction. The induction step is provided by Lemma 3. This may alter the number of steps following the point at which it is applied so the induction is applied by working backwards from N to M. In this way the induction index is not affected.

Lemma 3. *If there is a single step reduction using a Type I or II step from M to N_o and a standard reduction using only steps of Type I and II from N_o to N_n, then there is a standard reduction using only Type I and II steps from M to N_n.*

Proof of the theorem. Let $M \equiv M_0, M_1, M_2, \ldots, M_k \equiv N$ be the stages of the reduction. Since a single step reduction can always be considered standard by appropriate freezing, the reduction from M_{k-1} to M_k is standard. This is the basic step of the induction. Now if the reduction has been standardized from M_j to M_k, then Lemma 3 shows that we can standardize from M_{j-1} to M_k. We have now decreased the index and a descending induction on j gives the desired result.

Proof of the lemma: Note that if $n = 0$ the situation is trivial, since then we have only a single step reduction.

Designate the redex used to reduce M to N_o by P and the initial combinator of P by p. Designate the redex used to reduce N_{i-1} to N_i by r_i.

Look at the leftmost free combinator in M. If it is not p, then it appears in N_o. If this combinator is frozen in the reduction from N_o to N_n, then freeze it in M. Proceed to the right in M until we reach a combinator which is not frozen. If this leftmost free combinator in M is p, we have a standard reduction from M to N_n and there is nothing more to prove. If this combinator is not p, it is r_1. In the following we assume r_1 is to the left of p.

We now construct a sequence of L_i's such that we have a standard reduction from M to L_i and that a left to right reduction of all residuals of P in L_i continues this given reduction to N_{k-1}. This certainly holds for $k = 1$, if we take $i = 0$ and L_o to be M. In the induction step we construct a sequence from L_i to L_j such that a reduction of all residuals of P in L_j

gives a standard reduction from M via L_i and then to L_j and ending at M_k. We carry out the inductive step in two parts. In the first we show, assuming simply that the residuals of P in L_j are of the same type as P, and whose contraction in standard order leads to N_{k-1}; that this leads to a standard reduction from L_i via L_j to N_k. In the second part we shall show, assuming the rest of the inductive hypothesis, that we obtain a standard reduction from M to N_k by carrying out first the standard reduction from M to L_i, and then the new reduction from N_i to N_k.

The reduction from L_i to N_{k-1} is by means of the reduction of the residuals of P. Thus each of these will have a recognizable trace in N_{k-1}. Suppose that r_k lies to the left of all those traces of P in N_{k-1}. Then there will be an instance r_k' of r_k in L_i such that the reductions from L_i to N_{k-1} consist of replacement of components lying to the right of r_k'. Accordingly, there will be a component R_k' in L_k, headed by r_k' such that these replacements will be either replacements inside R_k' (or one of its traces), or entirely to the right of R_k'. Thus R_k' is a redex of the same type as R_k. Since the replacements are all steps of Type I or II, which do not allow a lambda to be eliminated, R_k' is a redex of the same type as R_k. Let L_j ($\equiv L_{i+1}$) be obtained by contracting R_k' in L_i. The effect of this is merely to change the order and multiplicity of the residuals of P. Hence if all these residuals, in standard order, are replaced by their contracta, we are led from L_i to L_j and thence to N_k by a reduction which is standard.

Suppose now that r_k lies to the right of, or inside, the trace of some residual of P. Let L_{j-1} be obtained by contracting, in standard order, exactly those residuals of P in L_i whose traces in N_{k-1} contain or lie to the left of r_k. Then since the residuals of P are non-over-lapping, those which appear in L_{j-1} will be replicas of certain ones in L_i; moreover, the contraction of all residuals of P in L_i will pass through L_{j-1} on its way to N_{k-1}. Thus L_{j-1} can take the place of L_i in the argument of the preceding paragraph; and the reduction from L_i to L_j and hence to N_k will be standard.

This completes the first part of the proof. Before starting on the second part we note the position of r_{k+1} in N_k. Since the reduction from N to N_o is standard, r_{k+1} lies in or to the right of the contractum of R_k. Accordingly no ancestor of r_{k+1} can be frozen in the reduction from M to N_k. Likewise the traces of residuals of P which still exist in L_j all lie within or to the right of the contractum of R_k. Let us suppose, as part of the inductive hypothesis, that the analogous statements hold with respect to r_k and N_k, i.e., with k *-1* in place of k. They evidently hold when $k = 0$ since then we have only a single step reduction from M to N_o, there is no R_o, $i = 0$, and $j = 1$. Thus no ancestor of r_k nor any ancestor of the head of any residual of P contracted in going from L_i to L_j can be frozen in the reduction from M to L_i. These ancestors, therefore, are in the reduction from M to L_i. Since they are not frozen in the reduction from L_{j-1} to N_k, the reduction from M to N_k is standard.

This completes the proof of the lemma and the theorem.

An Auxiliary Theorem. The following theorem is used in the proof of the standardization theorem and in other contexts.

Theorem 2. *I. If there is a strong reduction from M_o to M_n and if there is a single step reduction of Type II or Type III from M_o to N, then $N \succ M_n$.*

II. If there is a standard reduction from M_o to M_n and if there is a single step reduction using a step of Type I or Type III from M_o to N, then there is an L such that there is a standard reduction from N to L and $M_n \succ L$.

Proof: Part I. By Lemma 1 or 2 the step from M_o to N may be reversed giving $N \succ M_o \succ M_n$.

Part II. If the step from M_0 to N is a Type III step it can be reversed by Lemma 1. This introduces only Type I and II steps from N to M_o. If the reduction following this is standard, the entire reduction from N to M_n can be standardized by Theorem 1, since all Type III steps are already at the end of the reduction.

Now let the original reduction be given by

(1) $$M_o \succ M_1 \succ \ldots \succ M_n$$

and the reduction from M_0 to N be by the contraction of a Type I redex P. The proof is in the form on an induction on the number of steps in the reduction (1). We assume that the reduction from M_o to N is standard. Also that the redex P contracted in the reduction from M_o to N has initial combinator p. The reduction from M_{i-1} to M_i is by the contraction x_i of R_i with initial combinator r_i (in case R_i is Type I or II).

If $n = 0$, then N is the required L. This is the basic step of the induction.

For the induction step we assume that by contracting all residuals of P in M_{i-1}, as well as certain redexes congruent to residuals of P (reconstructed in the reduction from M_{i-1} to N_j and which we will call semiresiduals) corresponding to residuals of P destroyed by the contraction of a competing redex in (1) prior to reaching M_{i-1}, we obtain an N_j such that $M_{i-1} \succ N_j$ and $N \equiv N_o \succ N_j$. From this we show that we can contract R_i to get M_i and then by a similar sequence of steps arrive at N_k such that $M_i \succ N_k$ and $N_j \succ N_k$.

We may need Type III steps to reconstruct certain semiresiduals in these reductions. The details of definition of semiresiduals are left till appropriate points in the proof. Because P is a Type I, each residual (and semiresidual) of P must be headed by an instance of p.

It will be helpful in the following steps of the proof to recall that if a redex has more than one residual, these cannot overlap. We now consider the possible cases for the induction.

Case 1. R_i is disjoint from all residuals and semiresiduals of P. This case is the first case in the proof of Theorem L with R_i serving as P. Standardization was not a necessary hypothesis except to obtain a standard reduction.

Case 2. R_i is a residual of P. The contraction of R_i is one step of the reduction from M_{i-1} to N_j. Hence a reduction from M_{i-1} to M_i and then to

$N_k \equiv N_j$ involves at most a rearrangement of the order of contraction of residuals and semiresiduals of P.

Case 3. R_i is a part of a residual of P.

Case 3a. The combinator $\mathbf{r}_i$ is contained within an argument of P in some residual of P. Since the reduction (1) was standard, this means that the particular residual of P which involves $\mathbf{r}_i$ was frozen at some point in the reduction (1) before the stage M_i was reached. There will be zero or more residuals of R_i in N_j; but each of these residuals will have exactly the same form as R_i itself. Reducing each of these residuals of R_i occurring in N_j gives the same result as if R_i were reduced first and then the residuals and semiresiduals of P were reduced. Thus we get $M_i \succ N_k$ and $N_j \succ N_k$.

Case 3b. $\mathbf{r}_i$ is the head of a residual of P, but R_i is a subcomponent of this residual. R_i is necessarily a Type II redex. If we apply a Type III step to the contractum of R_i in M_i before any other steps are performed, we have an expression identical to M_{i-1}. In M_i the residual of P headed by $\mathbf{r}_i$ has no residual, but after the Type III step a redex is reconstructed which is identical to the destroying residual of P. This redex is what we call a semiresidual of P. In the reduction from M_i to N_j we treat this redex as if it were a residual of P and likewise in all subsequent stages of the induction process. Reducing all the residuals and semiresiduals of P after the Type III step is essentially the same as reducing the residuals and semiresiduals of P beginning with M_{i-1} and hence we get $N_k \equiv N_j$. Any Type III step other than the one introduced in this step will be handled as in the reduction from M_{i-1} to N_j.

Case 3c. R_i does not overlap a residual of P, but a residual of R_i is contained in a semiresidual of P constructed in the reduction from M_{i-1} to N_j. This can happen in one of two ways.

The residual of R_i may be part of an argument of a semiresidual which was not included in the redex which served as the R_i in the particular application of Case 3b which originally gave rise to the semiresidual of P involved. In this situation the present R_i may be handled by Case 3a after the Type III step constructing the semiresidual has been performed.

The R_i of the present case may be a redex contained within or formed from the contractum of an R_i of Case 3b. Since the R_i of Case 3b is a Type II redex its contractum is a Type III redex and hence is identifiable throughout this part of the reduction, until Type III steps are applied. Since the present R_i is a Type I or II redex, a contraction of its residual in N_j as in Case 3a will give N_k.

Case 4. Certain residuals of P are parts of R_i. One or more residuals then occur within arguments of $\mathbf{r}_i$. The most that can happen here is a change of multiplicity of one of the residuals of P. Reducing these residuals and the semiresiduals of P beginning with M_i gives us N_k. Clearly this is the same as if the residuals of R_i in N_j were reduced, so that we have $N_j \succ N_k$ as required.

Case 5. R_i is a Type III redex. This is part I of the theorem.

Two Lemmas. We now prove two simple lemmas which allow certain modifications in a reduction. These are used in the proof of the standardization theorem.

Lemma 4. *If $x \succ\!\!- M$ in k steps of Types I and II and if x contains no lambda expressions, then there is a reduction with not more than $k{\cdot}2^k$ Type I and II steps such that each Type III redex in M is the unique residual of the contractum of a Type II step.*

Proof: All Type III redexes must be introduced by Type III steps. Since the only step which actually increases the multiplicity of a component of a stage of a reduction is a step of Type Ib, this means that we can interchange steps of Type Ib and steps of Type II.

Suppose we have a single Type III redex P introduced by a Type II contraction. This redex may be part of certain other redexes which are subsequently contracted. There will be no increase in the multiplicity of residuals of P as long as they are not included in a step of Type Ib and we have no difficulty. (It cannot be a part of a Type III contraction since a Type III step may have no lambda expressions in its interior.) There may also be certain reductions interior to the Type III redex. We place no restrictions on these since they cannot affect the multiplicity of the redex in question.

Suppose that there are i steps of the reduction interior to P before the step of Type Ib is reached. Suppose further that the Type II step introducing P is the first step since previous steps are irrelevant to this part of the analysis. Let M_0 be contracted to M_1 by a Type II step replacing a component L by a component $\lambda x.L \equiv P$. Let the first step of Type Ib which overlaps this component in such a way as to contain P as an argument be the kth step. Then M_{k-1} reduces to M_k by the replacement of a component of the form $\mathsf{S}N_1N_2N_3$ by a component $N_1N_3(N_2N_3)$. If the residual of P occurs in either N_1 or N_2 there will be no change in the multiplicity of the residuals of P and we make no changes in the reduction.

If, however, the residual of P in M_{k-1} is in the component N_3, there are now two residuals of P and we modify the reduction as follows. In each stage from M_1 to M_k replace P or the unique residual of P by L. Each step from M_1 to M_k will either be a step of the same type as the original, or the two successive stages will be identical. Identical stages occur at M_0 and M_1 and whenever there were reductions interior to P in the original reduction. Delete repetitions in the reduction.

Now begin with the new M_k and replace both instances of L with P and repeat in each of these residuals exactly the same sequence of steps which occurred interior to the residual of P in the reduction from M_1 to M_{k-1} in the original reduction. Thus for each step deleted above M_k two are added below. If a reduction had k steps and introduced only one Type III redex there will be less than $2k$ steps in the modified reduction.

If there are other Type III redexes at stage M_{k+1}, each can be handled by this same procedure. Since each application no more than doubles the number of steps the total number of steps is less than $k \cdot 2^k$.

Lemma 5. *If there is a reduction from $M_{j-1} \equiv LN$ beginning with a Type II step yielding $M_j \equiv (\lambda x.L')N$ and continuing to $M_k \equiv (\lambda x.L'')N'$ where $\lambda x.L' \succ\!\!- \lambda x.L''$ and $N \succ\!\!- N'$ then there is a reduction from $M_{j-1} \equiv LM$ to $M_k' \equiv [L'/x]N''$ (where the notation $[L'/x]N''$ means the substitution of L' for x in N'') in exactly the same number of steps.*

Proof: In the reduction from M_{j-1} to M_k the reductions in N are disjoint from the reductions in L. Hence these can be interchanged without affecting the number of steps. In particular we can require all reductions in N to be performed before any reduction in L. This gives us $M_{j'-1} \equiv LN'$.

Now substitute N' for x in each stage from $M_{j'}$ to M_k. (Delete the λx prefix when the substitution is performed.) There is no change in the number of steps. Note that the result of making this substitution in M_k is M_k'. We now show that the transition from $M_{j'-1}$ to $M_{j'}$ is a valid single step reduction and the lemma will be proved. We list in tabular form the six possible kinds of Type II steps which can occur.

$M_{j'-1}$	Original Contractum	Orig. Type	Modified Contractum	Modified Type
$\mathsf{K}N_1N_2$	$(\lambda x.N)N_2$	IIa	N_1	Ia
$\mathsf{S}N_1N_2N_3$	$(\lambda x.N_1x(N_1))N_3$	IIb	$N_1N_3(N_2N_3)$	Ib
$\mathsf{I}N_1$	$(\lambda x.x)N_1$	IIc	N_1	Ic
$\mathsf{K}N_1$	$(\lambda xy.x)N_1$	IId	$\lambda y.N_1$	IIa
$\mathsf{S}N_1N_2$	$(\lambda xy.N_1y(xy))N_2$	IIe	$\lambda y.N_1y(N_2y)$	IIb
$\mathsf{S}N_1$	$(\lambda xyz.xz(yz))N_1$	IIf	$\lambda yz.N_1z(yz)$	IIe

The result of the modification in each case is another strong reduction of a different type. Hence the lemma is proved.

The Standardization Theorem

Theorem 3. *If there is a strong reduction from X to Y where neither X nor Y contain lambda expressions, then there is a Z such that there is a standard reduction from X to Z and Y reduces to Z.*

Proof: The proof of this theorem is an induction, similar to the proof of Theorem 1, beginning from the bottom of the reduction and moving upwards. Theorem 1 shows that any reduction involving only Type I and II steps can be standardized. Thus we need to show that any Type III step can be moved to a point following all Type I or II steps in the reduction, or eliminated entirely.

We assume that the reduction has been modified as in Lemma 4 so that each Type III redex is the unique residual of the contractum of a Type II step.

Let the reduction be from $M_o \equiv X$ to $M_n \equiv Y$. Assume as inductive hypothesis that the reduction from M_{k+1} to M_n is standard. This hypothesis is verified when $k = n-2$ since a one step reduction is always standard. Assuming the hypothesis we show that there is a standard reduction from M_k to some Z such that $Y >\!- Z$. Then it will follow that for every $k \leq n-1$ there is a Z_k such that there is a standard reduction from M_k to Z_k, and $Z_k >\!- Z_{k-1}$. Then Z_o will be the Z of the theorem. We further assume that the step from M_k to M_{k+1} is a Type III step which overlaps subsequent Type I or II steps. If it is Type I or II, the result follows from Theorem 1. If there are no Type I or II steps overlapping the result is trivial.

Let $R_k \equiv \lambda x.\mathfrak{U}$ be the redex contracted in going from M_k to M_{k+1}, and let the contractum of R_k be $U \equiv [x]\mathfrak{U}$. R_k is the unique residual of a Type III redex introduced in some M_j, $0 < j < k$. We treat first the case where R_k is at the head of M_k; so that we have

$$M_k \equiv (\lambda x.\mathfrak{U})N_1N_2 \ldots N_m;$$

$$M_{k+1} \equiv UN_1N_2 \ldots N_m.$$

We divide the proof into cases according to the outermost (algorithmic) step in the algorithm for obtaining U from $\lambda x.\mathfrak{U}$. It is important to notice that the argument does not depend on the value of k, but only upon the fact that a reduction from M_{k+1} on is standard.

Case 1. The step from M_k to M_{k+1} uses clause (a) of the algorithm. Then there is a V, not containing X, such that $\mathfrak{U} \equiv V$, and $U \equiv \mathsf{K}V$. The N's may be absent. Any reduction involving the N's will be disjoint from the Type III step and causes no difficulty. In this case we have $M_{k+1} \equiv \mathsf{K}VN_1 \ldots N_m$.

Case 1a. If the K in M_{k+1} is frozen in the subsequent reduction, we note that all subsequent reductions overlapping the Type III step occur entirely within V. But V passes from M_k to M_{k+1} unchanged. Hence the reduction interior to V could have started in M_k before the Type III step was applied. In other words we can move the Type III step to the end of the reduction.

If Type II steps are used in the subsequent reduction of V, the lambdas introduced by them must be removed by Type III steps (which will occur after all Type I and II steps have been made since this part of the reduction is standard by hypothesis) before the Type III step being moved to the end is contracted. It is to guarantee that this will always be possible that we require Y and Z to contain no lambda expressions. Similar situations will arise in subsequent cases, but we will not comment on them further.

Case 1b. The K introduced by the Type III step from M_k to M_{k+1} is not frozen in the subsequent reduction and the reduction from M_{k+1} to M_{k+2} is by a Type I step. There must be at least one N present since it is involved in the subsequent standard reduction as an argument of a contracted redex. We can thus apply the transformation of Lemma 5 to the reduction from M_{j-1} to M_k. The X of Lemma 5 is the present V. It is to guarantee that this transformation will always be possible that we require X in this theorem to be an H-ob. If we perform this modification we get

$M_k^! \equiv VN_2 \ldots N_m \equiv M_{k+2}$. In other words by this modification we can eliminate two steps and have a standard reduction from $M_k^!$.

Case 1c. The **K** introduced by the Type III step is not frozen and the reduction from M_{k+1} to M_{k+2} is by a Type II step. We have either $M_{k+2} \equiv (\lambda x.V)N_1 \ldots N_m$ or else $M_{k+2} \equiv (\lambda xy.x)VN_1 \ldots N_m$. In the first alternative we have $M_k \equiv M_{k+2}$, and by eliminating two steps from the reduction, we have a standard reduction from M_k to the end of the reduction. The second alternative is Case 1a.

Case 2. M_{k+1} arises from M_k by algorithm clause (b). Here we have $M_k \equiv (\lambda x.x)N_1 \ldots N_m$, and $M_{k+1} \equiv \mathbf{I}N_1 \ldots N_m$. If the **I** is frozen, then all subsequent reductions are in the N's and are disjoint from the Type III step so that we have no problem. If the **I** is not frozen, then either $M_{k+2} \equiv N_1 \ldots N_m$ or else $M_{k+2} \equiv (\lambda x.x)N_1 \ldots N_m \equiv M_k$. In the second alternative eliminating two steps from the reduction gives us a standard reduction from M_k to the end.

In the first alternative N_1 is used as an argument of a redex contracted in the subsequent standard reduction. We can thus apply the transformation of Lemma 5 to get $M_k^! \equiv N_1 \ldots N_m \equiv M_{k+2}$. Again elimination of two steps gives a standard reduction from $M_k^!$ to the end.

Case 3. M_{k+1} arises from M_k by an application of clause (c) of the algorithm. Then M_k is $(\lambda x.Ux)N_1 \ldots N_m$, and M_{k+1} is $UN_1 \ldots N_m$.

Case 3a. If the subsequent reduction is such that the result is of the form $U'N'_1 \ldots N'_m$ where $U \succ\!\!- U'$, $N_1 \succ\!\!- N'_1, \ldots, N_m \succ\!\!- N'_m$, then we observe that all the reductions in U could have been performed before the Type III step was applied.

Case 3b. The subsequent reduction is not of the special form required by the previous subcase. As in previous cases we can modify the reduction from M_{j-1} to M_k as in Lemma 5 and obtain $M_k^! \equiv UN_1 \ldots N_m \equiv M_{k+1}$. By eliminating one step we have a standard reduction from $M_k^!$ to the end of the reduction.

Case 4. M_{k+1} arises from M_k by a step beginning with an application of clause (f) of the algorithm. Then M_k is of the form $(\lambda x.\mathfrak{U}\mathfrak{B})N_1 \ldots N_m \equiv \mathbf{S}UVN_1 \ldots N_m$, and M_{k+1} is $\mathbf{S}([x]\mathfrak{U})([x]\mathfrak{B})N_1 \ldots N_m \equiv \mathbf{S}UVN_1 \ldots N_m$. This is the most difficult case. Here for the first time we have the possibility of having succeeding substeps involving additional applications of the algorithm. These do not cause difficulty since the expressions $[x]\mathfrak{U} \equiv U$ and $[x]\mathfrak{B} \equiv V$ are treated as whole units at the stage M_{k+1} of the subsequent reduction. The structure of U and V can enter in only at later stages of the reduction which is already assumed to be standard. Thus we consider only what happens to the particular instance of **S** introduced by the initial application of clause (f) of the algorithm.

Case 4a. The **S** is frozen in the reduction following stage M_{k+1}. Since any reduction involving the N's will be disjoint from the Type III step we will assume for convenience in this subcase that they are not present. Thus the end of the reduction is of the form $\mathbf{S}U'V'$ where $U \succ\!\!- U'$ and $V \succ\!\!- V'$. We will show that there is a standard sequence of steps beginning with M_k and ending with an ob to which $\mathbf{S}U'V'$ will reduce.

Lemma 1 shows that $\lambda x.SUVx$ reduces to $\lambda x.\mathfrak{U}\mathfrak{B}$ ($\equiv M_k$) using one or more Type I steps only. Since we can take $\lambda x.Ux(Vx)$ as the result of the first of these steps, we have also that $\lambda x.Ux(Vx)$ reduces to $\lambda x.\mathfrak{U}\mathfrak{B}$ using only Type I steps. We now perform a secondary induction on the number h of steps in the reduction from $\lambda x.Ux(Vx)$ to $x.\mathfrak{U}\mathfrak{B}$. By this means we construct the desired standard reduction from M_k to the ob $\lambda x.\mathfrak{Z}$ where $SU'V' \equiv M_n \succ\!\!- \lambda x.\mathfrak{Z} \succ\!\!- Z$ and Z is the ob called for by the theorem. If the number of steps in the secondary induction is zero, the auxiliary reduction from $\lambda x.Ux(Vx)$ to $\lambda x.U'x(V'x)$ is the desired reduction. This is the basic step of the induction.

The induction step is provided by Part II of Theorem 2. This says that if $M_o \succ\!\!- M_n$ in a standard reduction and $M_o \succ\!\!- N$ by a single step of Type I, then there is an L such that $M_n \succ\!\!- L$ and $N \succ\!\!- L$ is a standard reduction. In application to the present situation M_o is the i-th stage of the reduction from $\lambda x.Ux(Vx)$ to $\lambda x.\mathfrak{U}\mathfrak{B}$. N is the following stage of the same reduction. M_n is a certain expression to which $\lambda x.U'x(V'x)$ reduces, the exact form depending upon i. When $i = 0$, M_n is $\lambda x.U'x(V'x)$; and when $i = h$, M_n is $\mathfrak{Z}$.

Since the reduction $\lambda x.Ux(Vx) \succ\!\!- \lambda x.\mathfrak{U}\mathfrak{B}$ contains only Type I steps, the result of the secondary induction shows that the reduction from $\lambda x.\mathfrak{U}\mathfrak{B}$ ($\equiv M_k^!$) to $\lambda x.\mathfrak{Z}$ can be standardized. Further Theorem 2 also guarantees that $\lambda x.U'x(V'x)$ reduces to $\lambda x.\mathfrak{Z}$. Since $SU'V'$ ($\equiv M_n$ of the present theorem) reduces to $\lambda x.U'x(V'x)$ by a single Type II step, we have $M_n \succ\!\!- z.\mathfrak{Z}$. At the end of the induction we can apply Type III steps to get Z.

Case 4b. The S introduced in the initial application of clause (f) of the algorithm is not frozen and the reduction from M_{k+1} to M_{k+2} is by a Type I step. There must be at least one N. We modify the reduction as in Lemma 6. In this way we arrive at $M_k^! \equiv ([N_1/x]\mathfrak{U}\mathfrak{B})N_2 \dots N_m$. By Lemma 1 $M_{k+1} \equiv SUVN_1 \dots N_m$ reduces to $M_k^!$ using only Type I steps. By a secondary induction on the number of steps in this reduction we show that there is an ob Z to which both $M_k^!$ and M_n reduce and that the reduction from $M_k^!$ to Z is standard. The details are similar to the previous case.

Case 4c. The S is not frozen and the step from M_{k+1} to M_{k+2} is a Type II step. M_{k+2} is either $\lambda x.Ux(Vx)N_1 \dots N_m$ or else it is $(\lambda yx.Ux(yx))VN_1 \dots N_m$. The third alternative for a Type II step is the frozen case already considered. We cover the first alternative in this subcase.

We can show that $M_{k+2} \equiv \lambda x.Ux(Vx)N_1 \dots N_m$ reduces to M_k using only Type I steps by applying Lemma 1 to $Ux \equiv ([x]\mathfrak{U})x$. If we perform a secondary induction on the number of steps in this reduction from M_{k+2} to M_k, we get the desired results. Part II of Theorem 2 is again used as the induction step in the secondary induction. The details are as in the previous subcases.

Case 4d. If S is not frozen, we still need to consider the alternative $M_{k+2} \equiv (\lambda yx.Ux(yx))VN_1 \dots N_m$. Here the procedure is more complicated. We may assume for convenience that the N's are all absent. Any reduction involving them will be disjoint from the Type III step from M_k to M_{k+1} so this will cause no difficulty.

Since the reduction below M_{k+2} is standard, $M_{k+2} \equiv (\lambda yx.Ux(yx))V$

reduces to $(\lambda yx.L)V$ by Type I and II steps only and this in turn reduces to $(\lambda yx.L)V'$. There may be Type III steps at the end of the reduction from V to V'. If we then apply Type III steps to this result we can get $([yx]L)V'$. By Lemma 5 this last expression reduces to $[V'/y][x]L$ using Type I steps only. Since V' does not contain x, we have also that this is the same as $[x][V'/y]L$. Now in each stage, from M_{k+2} to the stage which looks like $(\lambda yx.L)V$, make the substitution $[V/y]$. Since V contains no lambda expressions (it is $[x]\overline{V}$, and by definition this contains no lambdas), this transforms the original reduction beginning with M_{k+2} into one which begins with $x.Ux(Vx)$ and goes as far as $\lambda x.[V/y]L$. Next reduce each of the (disjoint) instances of Y to Y'. Since the Type III steps in the separate instances of the reduction of Y to Y' are disjoint, we may leave them until all the Type I and II steps have been completed. If we now apply Type III steps to the result, we get $[x][V'/y]L$. If there are any other Type III steps than the last one specified involved in this reduction, they are either at the end of the reduction of Y to Y' or else follow all reductions of that form, since the original reduction was standard by hypothesis. This means that the reduction from $\lambda x.Ux(Vx)$ to $\lambda x.[V'/y]L$ and then to $[x][V'/y]L$ is standardizable by Theorem 1. As we have also seen M_n reduces to $[x][V'/y]L$. Thus we can now apply Case 4c to this resulting reduction to get the desired result.

This proves the theorem on the hypothesis that R_k is at the head of M_k. It remains to consider what happens if that is not the case. Let R_{k+1} be the contractum of R_k. Then it may happen that there are several replicas of R_{k+1} which are recognizable as traces of R_k with respect to I-II contractions with a head lying to the left. This introduces complications which have to be considered.

Let Δ be the given reduction, which is standard from M_{k+1} on. With Δ before us we can recognize those traces of R_k such that no contraction whose head lies in or to the right of the trace has yet occurred. Let us call these R_k traces; they are all exact replicas of R_{k+1}. Let us say further that an R_k trace is activated at a given step of Δ if the redex contracted at that step has its head in or to the right of the trace, or is of Type III. Given an R_k trace, a contraction of a I-II redex whose head is to the left of it we shall call a preparatory contraction for that trace. Then a preparatory contraction may cause multiple descendents of that trace in the next stage, or it may cancel the trace altogether.

Let M_{k+g} be the first stage in Δ such that either all R_k traces have been cancelled by preparatory contractions or the first R_k trace is to be activated in the next step. We have noted that M_{j-1} differs from M_{k+1} only in that R_{j-1} replaces R_{k+1}. If we make the same preparatory reductions starting with M_{j-1}, R_{k+1} being replaced throughout by R_{j-1}, we arrive at an M''_{j-1} which is obtained from M_{k+g} by the same replacements. If M_{k+g} contains no R_k trace, then it is identical with M''_{j-1}, and the preparatory reduction, together with the standard reduction from M_{k+g} on, will give the standard reduction from M_{j-1}. On the other hand if M_{k+g} contains only one R_k trace, then the effect from M_{k+g} is the same as if R_k were at the head of M_k, i.e., we can use the argument given above to standardize the reduction from M''_{j-1} to M_n. In fact the argument did not depend on k; and the only contractions which can contain the trace (or its descendants) as a proper part will be Type III steps at the end. Since the preparatory contractions are of Type I or II we can apply Theorem 1 to obtain a standard reduction from M_{j-1}.

If M_{k+g} contains more than one R_k trace, we continue the modification of Δ. In M''_{j-1} let us reduce to R_{k+2} only that replica of R_{j-1} which corresponds to the trace about to be activated; we take this M''_{k+g} as a new M_{j-1} and so continue. Then we can use the argument for the case where R_k is at the head of M_k to move to the end the Type III step for the R_k trace last activated; then that for the one next to the last, and so on. In the end we shall have a standard reduction starting at M_{j-1}.

This completes the proof of Theorem 3.

Remark: It is necessary that x of the theorem not contain lambda expressions as the following counterexample shows. Start with $(x.x)$ $(y.y)$. This reduces to **II** and then to **I**. If the theorem were true in this case it would say that there would be a reduction with Type III steps last such that the lambda expressions and **I** reduce to the same Z. In the context of strong reduction only Type III steps are applicable to the lambda expressions giving **II**. No further reductions can be made without using Type I or II steps. **I** does not reduce to **II**.

Corollaries to Theorem 3

Corollary A. *If there is a strong reduction from L to M then there is a Z such that there is a reduction consisting of zero or more Type III steps from L to X and a standard reduction from X to Z with* $M \succ Z$.

Proof: If X contains no lambda expressions Theorem 3 gives the result. If X contains lambda expressions, remove them by Type III steps. Now use Lemma 1 to reintroduce each indeterminate eliminated by the Type III steps. This will be done by a sequence of Type II steps and the result of this then reduces to X by Type I steps only. Thus we have a reduction from X to X' by Type III steps. X' reduces to X by Type I and II steps. Prefix this to the original reduction and apply Theorem 3 from the reduction from X' on.

If Y contains lambdas simply apply additional Type III steps to get Y' before applying the above results. This proves the corollary.

By cutting off the part of the reduction above X' in the above we get the following:

Corollary B. *If there is a strong reduction from L to M then there is an* $X \succ L$ *and a Z such that* $M \succ Z$ *and there is a standard reduction from X to Z.*

BIBLIOGRAPHY AND NOTES

[1] Curry, H. B., and Robert Feys, *Combinatory Logic Vol. 1,* (Studies in Logic and Foundations of Mathematics Series), North Holland Publishing Co., Amsterdam, 1958.

[1a] See section 6f for the original definition.

[2] Loewen, Kenneth, "Modified Strong Reduction in Combinatory Logic," *Notre Dame Journal of Formal Logic,* vol. IX, no. 1 (1968), pp. 265-270.

[3] Loewen, Kenneth, *A Study of Strong Reduction in Combinatory Logic,* A Ph.D. thesis at the Pennsylvania State University, University Park, Pennsylvania, 1962. Written under the direction of H. B. Curry.

[3a] A standardization theorem was proved for the original definition of strong reduction in this thesis. The proof of the present theorem 3 is changed very little. Proofs of the present theorems 1 and 2 are essentially simplified.

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