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SATISFIABILITY IN A LARGER DOMAIN

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The essential idea in the proof of the familiar result that a sentence which is satisfiable in some domain D is satisfiable in a larger domain D^+ $D \subset D^+$, is to define a predicate $\mathcal{P}^+$ over D^+ corresponding to a predicate $\mathcal{P}$ over D so that, for some fixed element $a \in D$,

$$\mathcal{P}^+(x_1, x_2, \ldots, x_n) = \mathcal{P}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$$

where $\bar{x}_i = x_i$, if $x_i \in D$, and $\bar{x}_i = a$, if $x_i \notin D$, $1 \leqslant i \leqslant n$.

It seems to me, however, that the application of this idea to achieve the proof is rather more difficult than the published accounts, for instance those in [1], [2] and my own [3], lead one to suppose. To complete the proof it is necessary to show that, *for any* $\mathcal{P}$, *and all sets of quantifiers* $Q_1, \ldots, Q_n$, *the sentences without free variables* $Q_n x_n Q_{n-1} x_{n-1} \ldots Q_1 x_1 \mathcal{P}^+$, $Q_n x_n Q_{n-1} x_{n-1} \ldots Q_1 x_1 \mathcal{P}$ *have the same truth value,* where each Q_i is an existential or universal quantifier and the quantifiers on $\mathcal{P}$ relate to the domain D, those on $\mathcal{P}^+$ to the domain D^+. Let us call this result (*).

We consider first the case of a single quantifier. If $(\forall x)\mathcal{P}(x)$ is true, then $\mathcal{P}(x)$ is true for any $x \in D$, and so $\mathcal{P}^+(x)$ is true for any $x \in D^+$ whence $(\forall x)\mathcal{P}^+(x)$ is true. If $(\exists x)\mathcal{P}(x)$ is true, there is an element $c \in D$ such that $\mathcal{P}(c)$ is true, and so $\mathcal{P}^+(c)$ is true, whence $(\exists x)\mathcal{P}^+(x)$ is true. If $(\forall x)\mathcal{P}(x)$ is false then $\mathcal{P}(c)$ is false for some $c \in D$, and so $\mathcal{P}^+(c)$ is false whence $(\forall x)\mathcal{P}^+(x)$ is false, and, finally, if $(\exists x)\mathcal{P}(x)$ is false then $\neg\mathcal{P}(b)$ is true for any $b \in D$, and so $\neg\mathcal{P}^+(x)$ is true for any $x \in D^+$, whence $(\exists x)\mathcal{P}^+(x)$ is false. Thus (*) holds in the case $n = 1$. Suppose then that (*) holds for any $\mathcal{P}(x_1, \ldots, x_n)$ and any set of n quantifiers; then if

$$(\forall y) Q_n x_n \ldots Q_1 x_1 \mathcal{P}(y, x_1, \ldots, x_n)$$

is true, we have $Q_n x_n \ldots Q_1 x_1 \mathcal{P}(b, x_1, \ldots, x_n)$ is true for any $b \in D$ and so by the inductive hypnothesis

$$Q_n x_n \ldots Q_1 x_1 \mathcal{P}^+(b, x_1, \ldots, x_n)$$

is true for any $b \in D$ and so for $b \in D^+$ and therefore

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$$(\forall y)Q_n x_n \ldots Q_1 x_1 \mathscr{P}^+(y, x_1, \ldots, x_n)$$

is true. If on the other hand $(\forall y)Q_n x_n \ldots Q_1 x_1 \mathscr{P}(y, x_1, \ldots, x_n)$ is false, and if $\overline{Q}_i$ denotes $\forall$ or $\exists$ according as Q_i denotes $\exists$ or $\forall$, then there is a $c \in D$ such that

$$\overline{Q}_n x_n \ldots \overline{Q}_1 x_1 \neg \mathscr{P}(c, x_1, \ldots, x_n)$$

is true, and so, by the inductive hypothesis

$$\overline{Q}_n x_n \ldots \overline{Q}_1 x_1 \neg \mathscr{P}^+(c, x_1, \ldots, x_n)$$

is true and therefore

$$(\forall y)Q_n x_n \ldots Q_1 x_1 \mathscr{P}^+(y, x_1, \ldots, x_n)$$

is false. In a similar way we may show that

$$(\exists y)Q_n x_n \ldots Q_1 x_1 \mathscr{P}(y, x_1, \ldots, x_n)$$

has the same truth value as

$$(\exists y)Q_n x_n \ldots Q_1 x_1 \mathscr{P}^+(y, x_1, \ldots, x_n).$$

Thus we have shown that if (*) holds for some n, it holds for $n + 1$, and so (*) holds for all $n \geqslant 1$.

Let $\mathscr{A}(P_1, P_2, \ldots, P_k)$, $\mathscr{B}(P_1, P_2, \ldots, P_k)$ be any sentences containing, at most, the predicate variables indicated, and no free individual variables, such that

$$\mathscr{A}(\mathscr{P}_1, \ldots, \mathscr{P}_k),\ \mathscr{A}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+)$$

have the same truth value, and

$$\mathscr{B}(\mathscr{P}_1, \ldots, \mathscr{P}_k),\ \mathscr{B}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+)$$

have the same truth value; then truth table considerations show that

$$\neg\mathscr{A}(\mathscr{P}_1, \ldots, \mathscr{P}_k),\ \neg\mathscr{A}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+)$$

have the same truth value, as do the disjunctions

$$\mathscr{A}(\mathscr{P}_1, \ldots, \mathscr{P}_k) \vee \mathscr{B}(\mathscr{P}_1, \ldots, \mathscr{P}_k)$$

and

$$\mathscr{A}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+) \vee \mathscr{B}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+).$$

Since every sentence without free individual variables may be expressed as a truth function of sentences of the form

$$Q_1 x_1 \ldots Q_n x_n P(x_1, \ldots, x_n)$$

it follows that for any sentence $\mathscr{S}(P_1, \ldots, P_k)$ without free individual variables, $\mathscr{S}(\mathscr{P}_1, \ldots, \mathscr{P}_k)$ and $\mathscr{S}(\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+)$ have the same truth value.

Consequently, if a sentence $\mathscr{S}(P_1, \ldots, P_k, y_1, \ldots, y_p)$ with predicate variables $P_1, \ldots, P_k$ and free individual variables $y_1, \ldots, y_p$, is satisfiable in D, by predicates $\mathscr{P}_1, \ldots, \mathscr{P}_k$ for the predicate variables and individuals $\mathbf{c}_1, \ldots, \mathbf{c}_p$ for the individual variables then it is satisfiable in D^+ by the predicates $\mathscr{P}_1^+, \ldots, \mathscr{P}_k^+$ for the predicate variables and the same individuals $\mathbf{c}_1, \ldots, \mathbf{c}_p$ for the individual variables.

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