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A THEOREM CONCERNING A RESTRICTED RULE OF SUBSTITUTION IN THE FIELD OF PROPOSITIONAL CALCULI. II

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6* It follows from definition Df.3, Remark IV and **5.4** that in $\mathfrak{D}_0$ for every m, $1 \leqslant m \leqslant y$, and for every k, $2 \leqslant k \leqslant z$, $\{\mathbf{s}_1^m\} \vdash_{\overline{\mathrm{R1}}} \mathbf{s}_k^m$, i.e., that $\mathbf{s}_k^m$ is a consequence by R1 of the first term of $\mathbf{S}_m$. We indicate by $\mathfrak{D}_1 = \{\mathbf{A}; \mathbf{V}_{1\mathbf{E}}^*; \mathbf{V}_{2\mathbf{E}}^*; \mathbf{S}_1^+; \mathbf{S}_2; \ldots; \mathbf{S}_y\}$ an augmentation of $\mathfrak{D}_0$ such that $\mathfrak{D}_1$ is a proof sequence of $\mathbf{b}$ in which $\mathbf{S}_1^+ = \mathbf{S}_1$, but in which for every k, $1 < k < z$, there are two terms σ and τ such that they precede $\mathbf{s}_1^1$, i.e., the first term of $\mathbf{S}_1^+$, and $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathbf{s}_k^1$. In the other words, in $\mathfrak{D}_1$ every term of $\mathbf{S}_1$ is a consequence by R2 of two terms belonging to $\mathfrak{D}_1$ and preceding the first term of $\mathbf{S}_1$. Obviously, if $\mathbf{S}_1 = \{\mathbf{s}_1^1\}$, then $\mathfrak{D}_1 = \mathfrak{D}_0$. But, in such a case $\mathfrak{D}_0$ can be considered as a particular instance of $\mathfrak{D}_1$ which will not be analyzed separately. In a similar way we indicate by $\mathfrak{D}_2 = \{\mathbf{A}; \mathbf{V}_{1\mathbf{E}}^*; \mathbf{V}_{2\mathbf{E}}^*; \mathbf{S}_1^{+*}; \mathbf{S}_2^+; \ldots; \mathbf{S}_y\}$ an analogous augmentation of $\mathfrak{D}_1$ and so forth.

In this section we will prove that we can replace $\mathfrak{D}_0$ by its augmentation

$$\mathfrak{D}_y = \{\mathbf{A}; \mathbf{V}_{1\mathbf{E}}^*; \mathbf{V}_{2\mathbf{E}}^*; \mathbf{S}_1^{+*}; \ldots; \mathbf{S}_{y-1}^{+*}; \mathbf{S}_y^+\}$$

such that $\mathfrak{D}_y$ is a proof sequence of $\mathbf{b}$ in which for every m, $1 \leqslant m \leqslant y - 1$, $\mathbf{A} \vdash_{\overline{\mathrm{R1}^*, \mathrm{R2}}} \mathbf{S}_m^{+*}$ and, moreover, $\mathbf{A} \vdash_{\overline{\mathrm{R1}^*, \mathrm{R2}}} \mathbf{S}_y^+$.

Since in order to prove this statement we shall use deductions entirely analogous to those that were presented in section **4**, the proof given below will be rather concise.

6.1 Let us assume that in $\mathfrak{D}_0$ $\mathbf{S}_1 \neq \{\mathbf{s}_1^1\}$ and, moreover, that $\mathbf{s}_k^1$, $2 \leqslant k \leqslant z$, is an arbitrary term of $\mathbf{S}_1$ such that $\mathbf{s}_1^1 \neq \mathbf{s}_k^1$. Then, *cf.*, **5.3** and Remark IV, in $\mathfrak{D}_0$ there are two terms σ and τ such that they precede $\mathbf{s}_1^1$, $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\} \vdash_{\overline{\mathrm{R1}}} \mathbf{s}_k^1$. Since, obviously, $\mathbf{s}_k^1$ is a substitution instance of $\mathbf{s}_1^1$, there is a

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formula μ such that $\{\tau\}\vdash_{\overline{R1}} \mu$ and $\mu \approx C\rho\, \mathbf{s}_k^1$. Therefore, since there are five generic cases of $\mathbf{S}_1$, *cf.*, **5.5.3**, we have to analyze five possible cases:

Case 1. σ is a term of $\mathbf{A}$, τ is a term of $\mathbf{V}_{2\mathbf{E}}$, $\tau \approx C\sigma\mathbf{s}_1^1$, $\{\sigma, \tau\}\vdash_{\overline{R2}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\}\vdash_{\overline{R1}} \mathbf{s}_k^1$.

Additionally, we have $\{\tau\}\vdash_{\overline{R1}} \mu$ and $\mu \approx C\rho\mathbf{s}_k^1$. If in $\mathfrak{D}_0$ μ is not a term of $\mathbf{V}_{2\mathbf{E}}$, then, since τ is a term of $\mathbf{V}_{2\mathbf{E}}$ and $\{\tau\}\vdash_{\overline{R1}} \mu$, μ possesses the same formal properties as term γ_k discussed in section **4.1**. Therefore, since σ is a term of $\mathbf{A}$, using the same reasoning as given in **4.1** we are able to replace $\mathfrak{D}_0$ by its augmentation $\mathfrak{D}_0^* = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^*;\ \mathbf{V}_{2\mathbf{E}}^*;\ \mathbf{S}_1;\ .\ .\ .;\ \mathbf{S}_y\}$ such that $\mathfrak{D}_0^*$ is a proof sequence of $\mathbf{b}$ in which $\mathbf{A}\vdash_{\overline{R1^*,R2}} \mathbf{s}_k^1$. Whence, assume that in $\mathfrak{D}_0$ μ is a term of $\mathbf{V}_{2\mathbf{E}}$. Clearly, either $\sigma \approx \rho$ or, since σ is a term of $\mathbf{A}$, $\{\sigma\}\vdash_{\overline{R1^*}}\rho$. Hence, if $\sigma \approx \rho$ or in $\mathfrak{D}_0$ ρ is a term of $\mathbf{V}_{1\mathbf{E}}$, in $\mathfrak{D}_0$ $\mathbf{A}\vdash_{\overline{R1^*,R2}} \mathbf{s}_k^1$. On the other hand, if $\{\sigma\}\vdash_{\overline{R1^*}}\rho$ and in $\mathfrak{D}_0$ ρ is not a term of $\mathbf{V}_{1\mathbf{E}}$, then we construct an augmentation $\mathbf{V}_{1\mathbf{E}}^{*1}$ of $\mathbf{V}_{1\mathbf{E}}$ by adding to $\mathbf{V}_{1\mathbf{E}}$ ρ as its last term, *cf.*, definition (a) in **4.1**. Then, we are able to replace $\mathfrak{D}_0$ by its augmentation $\mathfrak{D}_0^{*1} = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^{*1};\ \mathbf{V}_{2\mathbf{E}};\ \mathbf{S}_1;\ .\ .\ .;\ \mathbf{S}_y\}$ such that $\mathfrak{D}_0^{*1}$ is a proof sequence of $\mathbf{b}$ in which $\{\rho, \mu\}\vdash_{\overline{R2}} \mathbf{s}_k^1$. Therefore, it is obvious that if Case 1 holds for $\mathbf{s}_k^1$ then we can always replace $\mathfrak{D}_0$ by its augmentation $\mathfrak{D}_{0C1} = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^*;\ \mathbf{V}_{2\mathbf{E}}^*;\ \mathbf{S}_1;\ .\ .\ .;\ \mathbf{S}_y\}$ such that $\mathfrak{D}_{0C1}$ is a proof sequence of $\mathbf{b}$ and such that in it $\mathbf{s}_k^1$ is a consequence by R2 of two terms belonging to $\mathfrak{D}_{0C1}$ and preceding $\mathbf{s}_1^1$. Thus, Case 1 is solved.

Clearly, *cf.*, **4.1**, in an analogous way we can obtain the solutions to the remaining four cases, viz.

Case 2. σ is a term of $\mathbf{V}_{1\mathbf{E}}$, τ is a term of $\mathbf{V}_{2\mathbf{E}}$, $\tau \approx C\sigma\mathbf{s}_1^1$, $\{\sigma, \tau\}\vdash_{\overline{R2}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\}\vdash_{\overline{R1}} \mathbf{s}_k^1$.
Case 3. σ is a term of $\mathbf{V}_{2\mathbf{E}}$, τ is a term of $\mathbf{A}$, $\tau \approx C\sigma\mathbf{s}_1^1$, $\{\sigma, \tau\}\vdash_{\overline{R2}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\}\vdash_{\overline{R1}} \mathbf{s}_k^1$.
Case 4. σ is a term of $\mathbf{V}_{2\mathbf{E}}$, τ is a term of $\mathbf{V}_{1\mathbf{E}}$, $\tau \approx C\sigma\mathbf{s}_1^1$, $\{\sigma, \tau\}\vdash_{\overline{R2}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\}\vdash_{\overline{R1}} \mathbf{s}_k^1$.
Case 5. Both σ and τ are the terms of $\mathbf{V}_{1\mathbf{E}}$, $\tau \approx C\sigma\mathbf{s}_1^1$, $\{\sigma, \tau\}\vdash_{\overline{R2}} \mathbf{s}_1^1$ and $\{\mathbf{s}_1^1\}\vdash_{\overline{R1}} \mathbf{s}_k^1$.

It means, *cf.*, **4**, that if one of the cases 2-5 holds for $\mathbf{s}_k^1$, then we can always replace $\mathfrak{D}_0$ by its augmentation such that it is a proof sequence of $\mathbf{b}$ and such that in it $\mathbf{s}_k^1$ is a consequence by R2 of two terms belonging to this augmentation and preceding $\mathbf{s}_k^1$. Since cases 1-5 are mutually disjoint, we can conclude that for $\mathbf{s}_k^1$ under consideration there is an unique augmentation of $\mathfrak{D}_0$ which satisfies the required properties.

6.2 Since in the proof given in **6.1** it is assumed that $\mathbf{s}_k^1$, $2 \leqslant k \leqslant y$, is an arbitrary term of $\mathbf{S}_1$ such that $\mathbf{s}_1^1 \neq \mathbf{s}_k^1$ and since sequence $\mathbf{S}_1$ is finite, it follows from the discussion presented in **4.3** and **4.3.1**, that we are able to replace $\mathfrak{D}_0$ by its augmentation $\mathfrak{D}_1 = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^*;\ \mathbf{V}_{2\mathbf{E}}^*;\ \mathbf{S}_1^+;\ \mathbf{S}_2;\ .\ .\ .;\ \mathbf{S}_y\}$ such that $\mathfrak{D}_1$ is a proof sequence of $\mathbf{b}$ and such that in it every term of $\mathbf{S}_1^+$ is a consequence by R2 of two terms belonging to $\mathfrak{D}_1$ and preceding $\mathbf{s}_1^1$.

6.3 In the subsections of **6** given below the letters y and z will always represent the numbers y and z as defined respectively in **5.4** and **5.3**. Moreover, in order to present the deductions given in those subsections in a compact way: (1) we presuppose a familiarity with the formal properties of subsequences $\mathbf{A}$, $\mathbf{V}_{1\mathbf{E}}$, $\mathbf{V}_{2\mathbf{E}}$, $\mathbf{S}_1$, $\mathbf{S}_1^+$ and so forth, (2) we assume tacitly the applications of Formula $\mathfrak{S}$, and (3) we introduce the following two purely abbreviational definitions:

Df. 5 For any δ, δ is a term of $\mathfrak{M}$ if and only if there is X such that X is a proof sequence of $\mathbf{b}$; δ is a term of X; $\mathbf{A}$, $\mathbf{V}_{1\mathbf{E}}$, and $\mathbf{V}_{2\mathbf{E}}$ are the subsequences of X and δ is a term either of $\mathbf{A}$ or of $\mathbf{V}_{1\mathbf{E}}$ or of $\mathbf{V}_{2\mathbf{E}}$,

and

Df. 6 For any δ and m, δ is a term of $\mathfrak{N}_m$ if and only if there is X such that X is a proof sequence of $\mathbf{b}$; δ is a term of X; $\mathbf{S}_1^{+*}, \mathbf{S}_2^{+*}, \ldots, \mathbf{S}_m^{+*}$ are the subsequences of X and δ is a term either of $\mathbf{S}_1^{+*}$, or of $\mathbf{S}_2^{+*}$, or of $\ldots$, or of $\mathbf{S}_m^{+*}$.

6.4 Now, we have to prove the following lemma:

Lemma 1 *For any k, m, and n such that $2 \leqslant k \leqslant z$, $1 \leqslant m \leqslant n$ and $1 < n < y$, if*

$$\mathfrak{D}_{n+1} = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^*;\ \mathbf{V}_{2\mathbf{E}}^*;\ \mathbf{S}_1^{+*};\ \ldots;\ \mathbf{S}_{m-1}^{+*};\ \mathbf{S}_m^{+*};\ \mathbf{S}_{m+1}^{+*};\ \ldots;\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

is a proof sequence of $\mathbf{b}$, $\mathbf{s}_k^m$ *is a term of* $\mathbf{S}_m^{+*}$, $\{\mathbf{s}_k^m\}\vdash_{\mathsf{R1}} \mu$ *and* μ *is not a term of* $\mathfrak{D}_{n+1}$, *then there is a sequence*

$$\mathfrak{D}_{n+1}^m = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^*;\ \mathbf{V}_{2\mathbf{E}}^*;\ \mathbf{S}_1^{+*};\ \ldots;\ \mathbf{S}_{m-1}^{+*};\ \mathbf{S}_m^{+*1};\ \mathbf{S}_{m+1}^{+*};\ \ldots;\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^m$ *is an augmentation of* $\mathfrak{D}_{n+1}$ *and such that it is a proof sequence of* $\mathbf{b}$, μ *is its term, and in which* $\mathbf{A}\vdash_{\mathsf{R1^*,R2}} \mu$ *and* μ *is the last term of* $\mathbf{S}_m^{+*1}$.

Explanation: Lemma 1 says: Assume that $\mathfrak{D}_{n+1}$ is a proof sequence of $\mathbf{b}$ which satisfies certain conditions, *cf.*, **6**, and, moreover, that $\mathbf{s}_k^m$ is its term and that a formula μ which is not a term of $\mathfrak{D}_{n+1}$ is a substitution instance by R1 of $\mathbf{s}_k^m$. Clearly, we do not need to use μ in the proof of $\mathbf{b}$. But, there is an augmentation of $\mathfrak{D}_{n+1}$, say $\mathfrak{D}_{n+1}^m$, such that $\mathfrak{D}_{n+1}^m$ is a proof sequence of $\mathbf{b}$ such that it contains μ as its term and in which $\mathbf{A}\vdash_{\mathsf{R1^*,R2}} \mu$. But, in $\mathfrak{D}_{n+1}^m$ μ is not used in the proof of $\mathbf{b}$. We shall see that an application of Lemma 1 is essential in the proof of Theorem A.

In order to prove Lemma 1 let us assume its antecedent. Then:

6.5 Since $\mathbf{s}_k^m$ is a term of $\mathbf{S}_m^{+*}$, it follows from the definition of $\mathfrak{D}_{n+1}$, *cf.*, **6**, that in $\mathfrak{D}_{n+1}$ there are two terms σ_1 and τ_1 such that they precede $\mathbf{s}_1^m$, $\tau_1 \approx C\sigma_1\mathbf{s}_k^m$ and $\{\sigma_1, \tau_1\}\vdash_{\mathsf{R2}} \mathbf{s}_k^m$. Since $\{\mathbf{s}_k^m\}\vdash_{\mathsf{R1}} \mu$, there is a formula μ_1 such that $\{\tau_1\}\vdash_{\mathsf{R1}} \mu_1$ and $\mu_1 \approx C\rho_1\mu$. Obviously, either $\sigma_1 \approx \rho_1$ or $\{\sigma_1\}\vdash_{\mathsf{R1}} \rho_1$.

6.5.1 Assume that μ_1 is a term of $\mathfrak{D}_{n+1}$ and, moreover, that $\sigma_1 \approx \rho_1$. Since τ_1 is a term either of $\mathfrak{M}$ or of $\mathfrak{N}_{m-1}$, $\{\tau_1\}\vdash_{\mathsf{R1}} \mu_1$ and μ_1 is a term of $\mathfrak{D}_{m+1}$, it follows from the definitions of A, $\mathsf{V}_{1\mathsf{E}}$, $\mathsf{V}_{2\mathsf{E}}$, and S_n^+ that μ_1 precedes s_1^m. Hence, μ_1 is a term either of $\mathfrak{M}$ or of $\mathfrak{N}_{m-1}$, and, therefore, $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu_1$. Since $\sigma_1 \approx \rho_1$, it yields that $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu$. But, according to the antecedent of Lemma 1 μ is not a term of $\mathfrak{D}_{n+1}$. However, since $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu$, we are able to construct an augmentation S_m^{+*1} of S_m^{+*} by adding to S_m^{+*} μ as its last term and consequently, to replace $\mathfrak{D}_{n+1}$ by its augmentation

$$\mathfrak{D}_{n+1}^1 = \{\mathsf{A};\ \mathsf{V}_{1\mathsf{E}}^*;\ \mathsf{V}_{2\mathsf{E}}^*;\ \mathsf{S}_1^{+*};\ \ldots;\ \mathsf{S}_{m-1}^{+*};\ \mathsf{S}_m^{+*1};\ \mathsf{S}_{m+1}^{+*};\ \ldots;\ \mathsf{S}_n^{+*};\ \mathsf{S}_{n+1}^{+};\ \mathsf{S}_{n+2};\ \ldots;\ \mathsf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^1$ is a proof sequence of b in which μ is a term of S_m^{+*1} and and $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu$. Obviously, if such a case holds, Lemma 1 is proved.

6.5.2 Hence, assume that at least one of the formulas, ρ_1 or μ_1, is not a term of $\mathfrak{D}_{n+1}$. Since each of the terms σ_1 and τ_1 is a term either of $\mathfrak{M}$ or of $\mathfrak{N}_{m-1}$ we have to distinguish the following four cases:

Case 1. Both σ_1 and τ_1 are the terms of $\mathfrak{M}$.
Case 2. σ_1 is a term of $\mathfrak{M}$ and τ_1 is a term of $\mathfrak{N}_{m-1}$.
Case 3. σ_1 is a term of $\mathfrak{N}_{m-1}$ and τ_1 is a term of $\mathfrak{M}$.
Case 4. Both σ_1 and τ_1 are the terms of $\mathfrak{N}_{m-1}$.

6.6 Assume that Case 1 holds. Hence both σ_1 and τ_1 are the terms of $\mathfrak{M}$. Moreover, by assumptions, μ and at least one of the formulas, ρ_1 or μ_1 are not a term of $\mathfrak{D}_{n+1}$. Since, *cf.*, definition Df. 5, both σ_1 and τ_1 are the terms either of A or of $\mathsf{V}_{1\mathsf{E}}^*$ or of $\mathsf{V}_{2\mathsf{E}}^*$, we have to analyze the following subcases:

6.6.1 σ_1 is a term of A. Then we have to investigate all possible subcases created by the fact that τ_1 is a term either of A or of $\mathsf{V}_{1\mathsf{E}}^*$ or of $\mathsf{V}_{2\mathsf{E}}^*$. Viz.:

(i) Suppose that $\sigma_1 \approx \rho_1$. In such a case μ_1 is not a term of $\mathfrak{D}_{n+1}$. Whence:

(1) Assume that τ_1 is a term of A. Since $\{\tau_1\} \vdash_{\overline{R1}} \mu_1$, it yields clearly that $\mathsf{A} \vdash_{\overline{R1^*}} \mu_1$. Hence, by assumptions, $\{\sigma_1, \tau_1\} \vdash_{\overline{R2}} \mu$, i.e., $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu$. Therefore, due to the fact that $\mathsf{A} \vdash_{\overline{R1^*,R2}} \{\mu_1, \mu\}$ we are able to construct the augmentations $\mathsf{V}_{1\mathsf{E}}^{*1}$ and S_m^{+*1} of $\mathsf{V}_{1\mathsf{E}}^*$ and S_m^{+*} respectively adding μ_1 to $\mathsf{V}_{1\mathsf{E}}^*$ and μ to S^{+*} as their last terms, and, consequently to replace $\mathfrak{D}_{n+1}$ by its augmentation

$$\mathfrak{D}_{n+1}^1 = \{\mathsf{A};\ \mathsf{V}_{1\mathsf{E}}^{*1};\ \mathsf{V}_{2\mathsf{E}}^*;\ \mathsf{S}_1^{+*};\ \ldots;\ \mathsf{S}_{m-1}^{+*};\ \mathsf{S}_m^{+*1};\ \mathsf{S}_{m+1}^{+*};\ \ldots;\ \mathsf{S}_n^{+*};\ \mathsf{S}_{n+1}^{+};\ \mathsf{S}_{n+2};\ \ldots;\ \mathsf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^1$ is a proof sequence of b in which μ is a term of S_m^{+*1}. Therefore, in $\mathfrak{D}_{n+1}^1$ $\mathsf{A} \vdash_{\overline{R1^*,R2}} \mu$.

(2) Assume that τ_1 is a term of $\mathsf{V}_{1\mathsf{E}}^*$. It implies that in $\mathfrak{D}_{n+1}$ there is a term δ such that δ is a term of A and $\{\delta\} \vdash_{\overline{R1^*}} \tau_1$. Therefore, since $\{\sigma_1\} \vdash_{\overline{R1}} \mu_1$, $\{\delta\} \vdash_{\overline{R1^*}} \mu_1$, i.e., $\mathsf{A} \vdash_{\overline{R1^*}} \mu_1$. Hence, it is self-evident that the solution to this eventuality is the same exactly as given in point (1) above.

(3) Assume that τ_1 is a term of $\mathsf{V}_{2\mathsf{E}}^*$. This together with the assumption that $\{\tau_1\} \vdash_{\overline{R1}} \mu_1$ yields that, although μ_1 is not a term of $\mathfrak{D}_{n+1}$, it possesses the same formal properties as term γ_k discussed in section **4.1**. For example, if in $\mathfrak{D}_{n+1}$ subsequence E were not empty and if μ_1 were a term of $\mathfrak{D}_{n+1}$, then μ_1 would be a term of E. Hence, using exactly the same reasonings as those which were presented in section **4**, we can replace $\mathfrak{D}_{n+1}$ by its augmentation

$$\mathfrak{D}^1_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*1}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ .\ .\ .;\ \mathbf{S}^{+*}_m;\ .\ .\ .;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ .\ .\ .;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^1_{n+1}$ is a proof sequence of **b** in which μ_1 is a term of $\mathbf{V}^{*1}_{2\mathbf{E}}$. Hence, *cf.*, **4**, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu_1$ and, therefore, since $\{\sigma_1,\ \mu_1\}\vdash_{\overline{R2}}\mu$, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu$. Due to this we are able to construct an augmentation $\mathbf{S}^{+*1}_m$ of $\mathbf{S}^{+*}_m$ adding μ to $\mathbf{S}^{+*}_m$ as its last term, and, consequently, to replace $\mathfrak{D}^1_{n+1}$ by its augmentation

$$\mathfrak{D}^2_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*1}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ .\ .\ .;\ \mathbf{S}^{+*}_{m-1};\ \mathbf{S}^{+*1}_m;\ \mathbf{S}^{+*}_{m+1};\ .\ .\ .;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ .\ .\ .;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^2_{n+1}$ is a proof sequence of **b** in which μ is a term of $\mathbf{S}^{+*1}_m$. Hence in $\mathfrak{D}^2_{n+1}$ $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu$.

(ii) Suppose that $\sigma_1\vdash_{\overline{R1}}\rho_1$ and ρ_1 is a term of $\mathfrak{D}_{n+1}$. In such a case, since σ_1 is a term of **A**, clearly, ρ_1 is a term of $\mathbf{V}^{*}_{1\mathbf{E}}$ and $\mathbf{A}\vdash_{\overline{R1^*}}\rho_1$. On the other hand, since ρ_1 is a term of $\mathfrak{D}_{n+1}$, μ_1 is not a term of $\mathfrak{D}_{n+1}$. Whence:

(4) Assume that τ_1 is a term of **A**. Since $\{\tau_1\}\vdash_{\overline{R1^*}}\mu_1$, obviously, $\mathbf{A}\vdash_{\overline{R1^*}}\mu_1$. Hence, since $\{\rho_1,\ \mu_1\}\vdash_{\overline{R2}}\mu$, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu$. Therefore, we are able to solve this possibility in exactly the same way as for case (1) in (i) above.

(5) Assume that τ_1 is a term of $\mathbf{V}^{*}_{1\mathbf{E}}$. Hence, *cf.*, point (2) in (i) above, $\mathbf{A}\vdash_{\overline{R1^*}}\mu_1$. Therefore, it is self-evident that the solution to this eventuality is the same exactly as given in point (4) above.

(6) Assume that τ_1 is a term of $\mathbf{V}^{*}_{2\mathbf{E}}$. Then, *cf.*, **3.2**, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\tau_1$ and, therefore, since $\tau_1\vdash_{\overline{R1}}\mu_1$, clearly, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu$. Hence, *cf.*, point (3) in (i) above, we are able to solve this eventuality in exactly the same way as in point (3) above.

(iii) Suppose that $\{\sigma_1\}\vdash_{\overline{R1}}\rho_1$ and ρ_1 is not a term of $\mathfrak{D}_{n+1}$. Since σ_1 is a term of **A**, it yields that $\mathbf{A}\vdash_{\overline{R1^*}}\rho_1$. Therefore, due to this fact we are able to construct an augmentation $\mathbf{V}^{*1}_{1\mathbf{E}}$ of $\mathbf{V}^{*}_{1\mathbf{E}}$ adding ρ_1 to $\mathbf{V}^{*}_{1\mathbf{E}}$ as its last term. And, consequently, we are able to replace $\mathfrak{D}_{n+1}$ by its augmentation

$$\mathfrak{D}^1_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*1}_{1\mathbf{E}};\ \mathbf{V}^{*}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ .\ .\ .;\ \mathbf{S}^{+*}_m;\ .\ .\ .;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ .\ .\ .;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^1_{n+1}$ is a proof sequence of **b** in which ρ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. And, therefore, in $\mathfrak{D}^1_{n+1}$ $\mathbf{A}\vdash_{\overline{R1^*}}\rho_1$. In the discussions presented in this section below instead of $\mathfrak{D}_{n+1}$ we shall consider always $\mathfrak{D}^1_{n+1}$ as the proof sequence under investigation.

Since ρ_1 is not a term of $\mathfrak{D}_{n+1}$, it follows from the assumptions accepted in **6.6** that either μ_1 is a term of $\mathfrak{D}_{n+1}$, i.e., of $\mathfrak{D}^1_{n+1}$, or μ_1 is not a term of $\mathfrak{D}_{n+1}$, i.e., of $\mathfrak{D}^1_{n+1}$. Hence, we have to investigate two eventualities, namely:

(a) Suppose that μ_1 is a term of $\mathfrak{D}^1_{n+1}$. Then, since τ_1 is a term of $\mathfrak{M}$, we have the following subcases:

(7) Assume that τ_1 is a term of **A**. It implies, since $\{\tau_1\}\vdash_{\overline{R1}}\mu_1$, that $\mathbf{A}\vdash_{\overline{R1^*}}\mu_1$ and, therefore, since μ_1 is a term of $\mathfrak{D}^1_{n+1}$, that μ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Hence, since $\{\rho_1,\ \mu_1\}\vdash_{\overline{R2}}\mu$, $\mathbf{A}\vdash_{\overline{R1^*,R2}}\mu$. Therefore, due to this fact, we are able to construct an augmentation $\mathbf{S}^{+*1}_m$ of $\mathbf{S}^{+*}_m$ adding μ to $\mathbf{S}^{+*}_m$ as its last term. And, consequently, since ρ_1 and μ_1 are the terms of $\mathfrak{D}^1_{n+1}$, we are able to replace $\mathfrak{D}^1_{n+1}$ by its augmentation

$$\mathfrak{D}^2_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*1}_{1\mathbf{E}};\ \mathbf{V}^{*}_{2\mathbf{E}};\ \mathbf{S}^{+*}_{1};\ \ldots;\ \mathbf{S}^{+*}_{m-1};\ \mathbf{S}^{+*1}_{m};\ \mathbf{S}^{+*}_{m+1};\ \ldots;\ \mathbf{S}^{+*}_{n};\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \mathbf{S}_{y}\}$$

such that $\mathfrak{D}^2_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ is a term of $\mathbf{S}^{+*1}_{m}$. Hence, in $\mathfrak{D}^2_{n+1}$ $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$.

(8) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$ clearly, *cf.*, discussion presented in point (1) of (i), it yields that $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$, and, therefore, since μ_1 is a term of $\mathfrak{D}^1_{n+1}$, μ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Hence, it is self-evident that we can solve this eventuality in exactly the same way as in point (7) above.

(9) Assume that τ_1 is a term of $\mathbf{V}^{*}_{2\mathbf{E}}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$ and μ_1 is a term of $\mathfrak{D}^1_{n+1}$, such a case is impossible because otherwise μ_1 would be a term of $\mathbf{E}$. But, in $\mathfrak{D}^1_{n+1}$, $\mathbf{E}$ is empty.

(b) Suppose that μ_1 is not a term of $\mathfrak{D}^1_{n+1}$. Then, we have again three subcases to be investigated, namely:

(10) Assume that τ_1 is a term of $\mathbf{A}$. Then, since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, we have clearly that $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$. Since $\{\rho_1,\ \mu_1\}\vdash_{\overline{\mathrm{R2}}}\mu$, it implies that $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$. Since $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$, we are allowed to construct an augmentation $\mathbf{V}^{*2}_{1\mathbf{E}}$ of $\mathbf{V}^{*1}_{1\mathbf{E}}$ in regard to formula μ_1. But, since any augmentation used in our deductions must fulfill the condition established in Remark I above and since $\mathbf{V}^{*1}_{1\mathbf{E}}$ is already an augmentation of $\mathbf{V}^{*}_{1\mathbf{E}}$ in regard to formula ρ_1, it is self-evident that $\mathbf{V}^{*2}_{1\mathbf{E}}$ can have one of the following forms: $\mathbf{V}^{*2}_{1\mathbf{E}} = \{\mathbf{V}^{*1}_{1\mathbf{E}},\ \mu_1\}$ or $\mathbf{V}^{*2}_{1\mathbf{E}} = \{\mathbf{V}^{*}_{1\mathbf{E}},\ \mu_1,\ \rho_1\}$. Moreover, since $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$, we are also allowed to construct an augmentation $\mathbf{S}^{+*1}_{m}$ of $\mathbf{S}^{+*}_{m}$ adding μ to $\mathbf{S}^{+*}_{m}$ as its last term. Then, consequently, we are able to replace $\mathfrak{D}^1_{n+1}$ by its augmentation

$$\mathfrak{D}^2_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*2}_{1\mathbf{E}};\ \mathbf{V}^{*}_{2\mathbf{E}};\ \mathbf{S}^{+*}_{1};\ \ldots;\ \mathbf{S}^{+*}_{m-1};\ \mathbf{S}^{+*1}_{m};\ \mathbf{S}^{+*}_{m+1};\ \ldots;\ \mathbf{S}^{+*}_{n};\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_{y}\}$$

such that $\mathfrak{D}^2_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ is a term of $\mathbf{S}^{+*1}_{m}$. Hence, in $\mathfrak{D}^2_{n+1}$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$.

(11) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Then, since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, clearly, *cf.*, point (2) of (i), we have $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$. Hence, since $\{\rho_1,\ \mu_1\}\vdash_{\overline{\mathrm{R2}}}\mu$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$. Therefore, it is self-evident that, since ρ_1 is a term of $\mathfrak{D}^1_{n+1}$, we are able to solve this eventuality exactly in the same way as in point (10) above.

(12) Assume that τ_1 is a term of $\mathbf{V}^{*}_{2\mathbf{E}}$. This, together with the assumption that $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, yields that although μ_1 is not a term of $\mathfrak{D}^1_{n+1}$ it possesses the same formal properties which the formula μ_1 discussed in point (3) of (i) above has. Therefore, using, as in point (3), exactly the same reasonings as those which were presented in section **4** we can construct the suitable augmentations $\mathbf{V}^{*2}_{1\mathbf{E}}$ and $\mathbf{V}^{*1}_{2\mathbf{E}}$ of $\mathbf{V}^{*1}_{1\mathbf{E}}$ and $\mathbf{V}^{*}_{2\mathbf{E}}$ respectively, *cf.*, **4.1**, such that $\mathbf{V}^{*1}_{2\mathbf{E}}$ will contain μ_1 as its last term. And, consequently, *cf.*, **4**, we can replace $\mathfrak{D}^1_{n+1}$ by its augmentation

$$\mathfrak{D}^2_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*2}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_{1};\ \ldots;\ \mathbf{S}^{+*}_{m};\ \ldots;\ \mathbf{S}^{+*}_{n};\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_{y}\}$$

such that $\mathfrak{D}^2_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ_1 is a term of $\mathbf{V}^{*2}_{1\mathbf{E}}$ and ρ_1 is a term of $\mathbf{V}^{*1}_{2\mathbf{E}}$. Hence, in $\mathfrak{D}^2_{n+1}$ $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\{\rho_1,\ \mu_1\}$. Therefore, since $\{\rho_1,\ \mu_1\}\vdash_{\overline{\mathrm{R2}}}\mu$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$. Whence, due to this fact, we are able to construct an augmentation $\mathbf{S}_m^{+*1}$ of $\mathbf{S}_m^{+*}$ adding μ to $\mathbf{S}_m^{+*}$ as its last term and, consequently, to replace $\mathfrak{D}_{n+1}^2$ by its augmentation

$$\mathfrak{D}_{n+1}^3 = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^{*2};\ \mathbf{V}_{2\mathbf{E}}^{*1};\ \mathbf{S}_1^{+*};\ \ldots;\ \mathbf{S}_{m-1}^{+*};\ \mathbf{S}_m^{+*1};\ \mathbf{S}_{m+1}^{+*};\ \ldots;\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^3$ is a proof sequence of $\mathbf{b}$ in which μ is a term of $\mathbf{S}_m^{+*1}$. Hence, in $\mathfrak{D}_{n+1}^3$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$.

(iv) It is obvious that the eventualities discussed in points (1)-(12) above exhaust all possible subcases generated by the assumption that in $\mathfrak{D}_{n+1}$ σ_1 is a term of $\mathbf{A}$ and, moreover, that these eventualities are mutually disjoint. Therefore, the deductions presented in this section show that if Case 1 holds for its instance mentioned above, then Lemma 1 is proved.

6.6.2 σ_1 is a term of $\mathbf{V}_{1\mathbf{E}}^*$. It implies that in $\mathfrak{D}_{n+1}$ there is a term η such that η is a term of $\mathbf{A}$ and $\{\eta\}\vdash_{\overline{\mathrm{R1^*}}}\sigma_1$. Hence, since $\{\sigma_1\}\vdash_{\overline{\mathrm{R1}}}\rho_1$, $\{\eta\}\vdash_{\overline{\mathrm{R1^*}}}\rho_1$ and, therefore, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\rho_1$. Whence, we have to consider two possibilities:

(v) Formula ρ_1 is a term of $\mathfrak{D}_{n+1}$. In such a case, since τ_1 is a term of $\mathfrak{M}$, it is self-evident that we can solve this possibility exactly in the same way as in point (a) of (iii) in section **6.6.1**.

(vi) Formula ρ_1 is not a term of $\mathfrak{D}_{n+1}$. In such a case, since τ_1 is a term of $\mathfrak{M}$, we can solve this eventuality in exactly the same way as in point (b) of (iii) in section **6.6.1**.

(vii) Thus, if Case 1 holds for its instance, discussed in this section, Lemma 1 is proved.

6.6.3 σ_1 is a term of $\mathbf{V}_{2\mathbf{E}}^*$. Since $\{\sigma_1\}\vdash_{\overline{\mathrm{R1}}}\rho_1$, we have to investigate two cases:

(viii) Formula ρ_1 is a term of $\mathfrak{D}_{n+1}$. Since $\{\sigma_1\}\vdash_{\overline{\mathrm{R1}}}\rho_1$, such a case is impossible because otherwise ρ_1 would be a term of $\mathbf{E}$, *cf.*, **3.3**. But, in $\mathfrak{D}_{n+1}$, $\mathbf{E}$ is empty.

(ix) Formula ρ_1 is not a term of $\mathfrak{D}_{n+1}$. This together with the assumption that $\{\sigma_1\}\vdash_{\overline{\mathrm{R1}}}\rho_1$ yields that although ρ_1 is not a term of $\mathfrak{D}_{n+1}$ it belongs to the class of the formulas whose formal properties were discussed already in **6.1** and points (3), (6), and (12) in section **6.6.1**. Therefore, using, as in those points, exactly the same reasonings as those which were presented in section **4**, we are able to construct the suitable augmentations $\mathbf{V}_{1\mathbf{E}}^{*1}$ and $\mathbf{V}_{2\mathbf{E}}^{*1}$ of $\mathbf{V}_{1\mathbf{E}}^*$ and $\mathbf{V}_{2\mathbf{E}}^*$ respectively, *cf.*, **4.1**, such that $\mathbf{V}_{1\mathbf{E}}^{*1}$ will contain ρ_1 as its last term. And, consequently, *cf.*, **4**, we are able to replace $\mathfrak{D}_{n+1}$ by its augmentation

$$\mathfrak{D}_{n+1}^1 = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^{*1};\ \mathbf{V}_{2\mathbf{E}}^{*1};\ \mathbf{S}_1^{+*};\ \ldots;\ \mathbf{S}_m^{+*};\ \ldots;\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ \ldots\ \mathbf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^1$ is a proof sequence of $\mathbf{b}$ in which ρ_1 is a term of $\mathbf{V}_{2\mathbf{E}}^{*1}$. Hence, in $\mathfrak{D}_{n+1}^1$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\rho_1$. In the discussion presented in this section below instead of $\mathfrak{D}_{n+1}$ we shall consider always $\mathfrak{D}_{n+1}^1$ defined above as the proof sequence under investigation.

Since ρ_1 is not a term of $\mathfrak{D}_{n+1}$, we have to analyze, *cf.*, (iii) in **6.6.1**, two cases, namely:

(c) Suppose that μ_1 is a term of $\mathfrak{D}^1_{n+1}$. Then, since τ_1 is a term of $\mathfrak{M}$, there are the following three subcases:

(13) Assume that τ_1 is a term of $\mathbf{A}$. Hence, since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$ and, therefore, since μ_1 is a term of $\mathfrak{D}_{n+1}$, μ_1 is a term of $\mathbf{V}^{*1}_{2\mathbf{E}}$. Therefore, since $\{\rho_1, \mu_1\}\vdash_{\overline{\mathrm{R2}}}\mu$ and ρ_1 is a term of $\mathfrak{D}^1_{n+1}$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$. Hence, we are allowed to construct an augmentation $\mathbf{S}^{+*1}_m$ of $\mathbf{S}^{+*}_m$ adding μ to $\mathbf{S}^{+*}_m$ as its last term. And, consequently, since ρ_1 and μ_1 are terms of $\mathfrak{D}^1_{n+1}$, to replace $\mathfrak{D}^1_{n+1}$ by its augmentation

$$\mathfrak{D}^2_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*1}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ \ldots;\ \mathbf{S}^{+*}_{m-1};\ \mathbf{S}^{+*1}_m;\ \mathbf{S}^{+*}_{m+1};\ \ldots;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^2_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ is a term of $\mathbf{S}^{+*1}_m$. Hence, in $\mathfrak{D}^2_{n+1}$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$.

(14) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, clearly, *cf.*, point (2) in **6.6.1**, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$ and, therefore, since μ_1 is a term of $\mathfrak{D}^1_{n+1}$, μ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Hence, obviously, we can solve this eventuality in the same way as in point (13) above.

(15) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{2\mathbf{E}}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$ and μ_1 is a term of $\mathfrak{D}^1_{n+1}$, such a case is impossible because otherwise μ_1 would be a term of $\mathbf{E}$. But, in $\mathfrak{D}^1_{n+1}$, $\mathbf{E}$ is empty.

(d) Suppose that μ_1 is not a term of $\mathfrak{D}^1_{n+1}$. Then:

(16) Assume that τ_1 is a term of $\mathbf{A}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$. Therefore, preserving the condition established in Remark I, *cf.*, point (10) in **6.6.1**, we are able to construct an augmentation $\mathbf{V}^{*2}_{1\mathbf{E}}$ of $\mathbf{V}^{*1}_{1\mathbf{E}}$ adding μ_1 to $\mathbf{V}^{*1}_{1\mathbf{E}}$ as its term and consequently, to replace $\mathfrak{D}^1_{n+1}$, defined in this section by its augmentation

$$\mathfrak{D}^3_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*2}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ \ldots;\ \mathbf{S}^{+*}_m;\ \ldots;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^3_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ_1 is a term of $\mathbf{V}^{*2}_{1\mathbf{E}}$. Hence, in $\mathfrak{D}^3_{n+1}$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$. Since $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\{\rho_1, \mu_1\}$ and $\{\rho_1, \mu_1\}\vdash_{\overline{\mathrm{R2}}}\mu$, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$. Therefore, we can construct an augmentation $\mathbf{S}^{+*1}_m$ of $\mathbf{S}^{+*}_m$ adding μ to $\mathbf{S}^{+*}_m$ as its last term and, consequently, to replace $\mathfrak{D}^3_{n+1}$ by its augmentation

$$\mathfrak{D}^4_{n+1} = \{\mathbf{A};\ \mathbf{V}^{*2}_{1\mathbf{E}};\ \mathbf{V}^{*1}_{2\mathbf{E}};\ \mathbf{S}^{+*}_1;\ \ldots;\ \mathbf{S}^{+*}_{m-1};\ \mathbf{S}^{+*1}_m\ \mathbf{S}^{+*}_{m+1};\ \ldots;\ \mathbf{S}^{+*}_n;\ \mathbf{S}^{+}_{n+1};\ \mathbf{S}_{n+2};\ \ldots;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}^4_{n+1}$ is a proof sequence of $\mathbf{b}$ in which μ is a term of $\mathbf{S}^{+*1}_m$. Hence, in $\mathfrak{D}^4_{n+1}$ $\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\mu$.

(17) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{1\mathbf{E}}$. Since $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, clearly, *cf.*, point (2) in **6.6.1**, $\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}}\mu_1$. Therefore, since μ_1 is not a term of $\mathfrak{D}^1_{n+1}$, it is self-evident that we can solve this subcase exactly in the same way as in point (16) above.

(18) Assume that τ_1 is a term of $\mathbf{V}^{*1}_{2\mathbf{E}}$. Obviously, this, together with the assumption that $\{\tau_1\}\vdash_{\overline{\mathrm{R1}}}\mu_1$, yields that although μ_1 is not a term of $\mathfrak{D}^1_{n+1}$ we can apply to it the same methods of deduction which were used in regard to formula ρ_1 at the beginning of this section, *cf.*, also points (3), (6), and (12)

in **6.6.1**. Therefore, preserving the condition established in Remark I and using exactly the same reasonings as those which were presented in section **4**, we are able to construct the suitable augmentations $\mathbf{V}_{1\mathbf{E}}^{*3}$ and $\mathbf{V}_{2\mathbf{E}}^{*3}$ of $\mathbf{V}_{1\mathbf{E}}^{*1}$ and $\mathbf{V}_{2\mathbf{E}}^{*1}$ respectively, *cf.*, **4.1**, such that $\mathbf{V}_{2\mathbf{E}}^{*3}$ will contain μ_1 as its term. And, consequently, *cf.*, **4**, we are able to replace $\mathfrak{D}_{n+1}^1$, defined in this section by its augmentation

$$\mathfrak{D}_{n+1}^5 = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^{*3};\ \mathbf{V}_{2\mathbf{E}}^{*3};\ \mathbf{S}_1^{+*};\ .\ .\ .;\ \mathbf{S}_m^{+*};\ .\ .\ .;\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ .\ .\ .;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^5$ is a proof sequence of **b** in which μ_1 is a term of $\mathbf{V}_{2\mathbf{E}}^{*3}$. Hence, in $\mathfrak{D}_{n+1}^5$, $\mathbf{A} \vdash_{\overline{R1^*,R2}} \mu_1$. Therefore, since $\mathbf{A} \vdash_{\overline{R1^*,R2}} \{\rho_1, \mu_1\}$ and $\{\rho_1, \mu_1\} \vdash_{\overline{R2}} \mu$, $\mathbf{A} \vdash_{\overline{R1^*,R2}} \mu$. Hence, due to this we are allowed to construct an augmentation $\mathbf{S}_m^{+*1}$ of $\mathbf{S}_m^{+*}$ adding μ to $\mathbf{S}_m^{+*}$ as its last term and, consequently, to replace $\mathfrak{D}_{n+1}^5$ by its augmentation

$$\mathfrak{D}_{n+1}^6 = \{\mathbf{A};\ \mathbf{V}_{1\mathbf{E}}^{*3};\ \mathbf{V}_{2\mathbf{E}}^{*3};\ \mathbf{S}_1^{+*};\ .\ .\ .;\ \mathbf{S}_{m-1}^{+*};\ \mathbf{S}_m^{+*1};\ \mathbf{S}_{m+1}^{+*};\ .\ .\ .\ \mathbf{S}_n^{+*};\ \mathbf{S}_{n+1}^{+};\ \mathbf{S}_{n+2};\ .\ .\ .;\ \mathbf{S}_y\}$$

such that $\mathfrak{D}_{n+1}^6$ is a proof sequence of **b** in which μ is a term of $\mathbf{S}_m^{+*1}$. Hence, in $\mathfrak{D}_{n+1}^6$, $\mathbf{A} \vdash_{\overline{R1^*,R2}} \mu$.

(x) It is obvious that the eventualities discussed in points (13)-(18) above exhaust all possible subcases generated by the assumption that in $\mathfrak{D}_{n+1}$, σ_1 is a term of $\mathbf{V}_{2\mathbf{E}}^{*}$ and, moreover, that these eventualities are mutually disjoint. Therefore, the deductions presented in this section show that if Case 1 holds for its instance analyzed above, then Lemma 1 is proved.

6.6.4 Since in sections **6.6.1**-**6.6.3** all subcases of Case 1 are solved, we can conclude that if Case 1 holds for $\mathfrak{D}_{n+1}$, then Lemma 1 is proved. Moreover, since all discussed subcases are mutually disjoint, we know that the solution obtained for Case 1 is unique.

To be continued.

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