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A SEMANTICS MODEL FOR IMPERATIVES

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To obtain an adequate semantics model for imperatives we will first consider changes, then human actions and finally human actions carried out in accordance with imperatives. Suppose that everything required by an imperative could be recorded on a motion picture reel which moved through a projector at the rate of ten frames per second. Thus we could think of the changes demanded by an imperative as a sequence of jumps from one interval of a tenth of a second during which nothing changed to another interval of a tenth of a second during which nothing changed. Let us call these static intervals elementary time intervals. What will be said in the sequel regarding elementary time intervals of a tenth of a second is true of elementary time intervals of any other length.

Let us take as our time scale the one tenth of a second time intervals before and after the beginning of the common era some 1977 years ago. We will make the simplifying assumption that no change can take place during any elementary time interval. Thus during each elementary time interval each of the present tense indicative sentences of English will have a constant truth value. What will be said in the sequel regarding English will be true of any other natural language.

We will call the state of affairs during one of these static elementary time intervals, an elementary world interval. Thus we can specify elementary world intervals by functions which assign 1 or 0 to each of the present tense indicatives of English—where we are to understand that when a function assigns 1 to a sentence, that sentence is true in the elementary world interval which the function specifies and when a function assigns the value 0, the sentence is false in the elementary world interval which the function specifies.

Let S be the set of present tense indicative sentences of English, then 2^S is the set of functions assigning truth values to these sentences. We will confine our attention to that subset T of 2^S in which the time indexing characteristics of English are conformed to, e.g., if $f \in T$ is an assignment of truth values to the present tense indicative sentences of English, then for exactly one integer n s.t. $-\infty < n < \infty$, f will assign 1 to 'it is the n'th one tenth of a second of the common era'; for exactly one integer n s.t.

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$1 \leqslant n \leqslant 12$, f will assign the value 1 to 'it is the n'th month of the current year', etc. Next we confine our attention to that subset L of T in which the ordinary rules of logic and indicative semantics are obeyed. Thus if $f \in L$ is an assignment of truth values to present tense indicative sentences of English and p, $q \in S$ are contradictory present tense indicatives, then $f(p) = 1 - f(q)$. If $f \in L$ is an assignment of truth values to present tense indicative sentences of English and p, $q \in S$ and r is the conjunction of p and q, i.e., r is 'p and q' for the standard truth functional use of 'and', then $f(r) = f(p)f(q)$. If $f \in L$ is an assignment of truth values to present tense indicative sentences of English and $p \in S$ is 'the door is open and the door is closed', then $f(p) = 0$, etc.

Let τ be the sequence of sentences such that: τ_n = it is the n'th one tenth of a second of the common era. Remembering that each $f \in L$ assigns 1 to exactly one term of τ, we can use this fact to set up an ordering relation on L: $f, g \in L \supset (f \leqslant g \equiv \bigwedge_{n,m} (f(\tau_n) = g(\tau_m) = 1 \supset n \leqslant m))$. On the other hand, for each τ_n there will be infinitely many $f \in L$ such that $f(\tau_n) = 1$. We use this to set up $\overline{T}$, a set of equivalence classes on L: $\overline{T} = \{\overline{f} \subseteq L: \bigwedge_{n,f,g} (f, g \in \overline{f} \supset f(\tau_n) = g(\tau_n))\}$.

By selecting exactly one element from each of these equivalence classes we obtain a possible history of the universe in so far as this can be described in our system. Let P be the set of these possible histories. Note that there is a 1-1 correspondence between the range of τ and each of the $H \in P$. Let $H_T \in P$ be the actual history of the universe in so far as it can be described in our system.

Change What we will call an elementary change can be characterised by the specification of one of the sentences which is true at some elementary time interval together with one of the sentences which is true at some other elementary time interval. Thus we might take an ordered quadruple (p_1, t_1, p_2, t_2) where p_1, p_2, are the sentences which are true at time intervals t_1, t_2 respectively to symbolise an elementary change. However, we will keep closer to the logic of change if we characterise a change by a sentence in English. As is well known, words such as 'and', 'or', 'implies', which are associated with the standard logical connectives, have varying uses in English. We will use the symbols $\wedge$, $\vee$, $\rightarrow$, $\sim$ for the standard truth functional uses of 'and', 'or', 'materially implies', 'it is not the case that'. With the aid of these logical connectives we will be able to characterise an elementary change by a single sentence.

Suppose the change is from p being true at τ_n to q being true at τ_m, then the change may be represented by $(\tau_n \rightarrow p) \wedge (\tau_m \rightarrow q)$. For if we take some $H \in P$ in which the change takes place, there will be exactly one $f \in H$ for which $f(\tau_n) = 1$ and for this f, $f(p) = 1$ since the change takes place in H; and there will be exactly one $g \in H$ for which $g(\tau_m) = 1$ and for this g, $g(q) = 1$ since the change takes place in H. Any change can be considered as a conjunction of elementary changes. Note that we are including under our definition of change continuing states of affairs on the one hand and contradictory states of affairs for the same elementary time interval on the other.

We obtain what we may call the extension of a change by finding the set of $H \in P$ in which the change can take place.

Let C be the change mentioned above, i.e., $(\tau_n \to p) \wedge (\tau_m \to q)$. Let $J(C)$ be the set of $H \in P$ in which C can take place. Thus

$$H \in J(C) \supset ((f \in H \;\&\; f(\tau_n) = 1 \supset f(p) = 1) \;\&\; (g \in H \;\&\; g(\tau_m) = 1 \supset g(q) = 1)).$$

Thus for changes C and D

$$\begin{aligned} J(C \wedge D) &= J(C) \cap J(D) \\ J(C \vee D) &= J(C) \cup J(D) \\ J(\sim C) &= P - J(C) \\ J(C \to D) &= (P - J(C)) \cup J(D) \end{aligned}$$

If $J(C) = 0$ for any change C then C is impossible because for some $p \in S$ the specification of the change requires that both p and its negation are true during the same elementary time interval. If $H_T \in J(C)$, the change actually takes place.

Human Actions For imperatives we will be concerned for the most part with those changes which are brought about by individual human agents. If we wanted to extend this study to automata or animals in so far as these are capable of carrying out imperatives, some of the terms used in the rest of the paper would be inappropriate.

Let A be the set of human agents. We will use (pC) to denote that change C is brought about by the agent $p \in A$. Let $V_c \in S^A$ be the intention of the predicate '- is effecting the change C' where we are using 'effecting' in the following sense. Suppose John Jones is building a boat over a period of a year. There will be times during the year when he is eating or sleeping. During these times he is not actively engaged in building the boat. Yet it would be true to say of him at any time during the year that he was effecting the change from wood, nails, etc. for a boat at the beginning of the year to the completed boat at the end of the year. Thus $C \wedge V_c(p)$ represents the human action in which the agent p brings about the change C. We will again use J for the extension of the human action. $J((pc)) = J(C) \cap J(V_c(p))$ where $J(V_c(p))$ gives the set of $H \in P$ in which $V_c(p)$ is true in the sense just discussed during the period relevant to the change.

Imperatives Let us return to our analogy of the frames of a motion picture reel. On reflection we see that these frames would in general record much that was irrelevant. The sketches of a cartoonist would give a better analogy of what is conventionally required by a given imperative. Also we would not want sketches for every one tenth second interval that goes to make up the time interval in which the imperative has to be carried out. We will begin by considering individual utterances of imperatives. Each of these will specify some action to be performed by or some state commanded or advised for some individual $p \in A$ under some circumstances. It will be important when the imperative is issued, because imperatives never request anything about the past. Thus the members $H \in P$ in which the uttered imperative can be carried out will coincide with H_T up until the time of utterance. We can always rephrase an imperative that is connected with necessary conditions for it to be in force, in such a way that we obtain an equivalent imperative which has sufficient conditions for it to be in force. We will neglect those connected with necessary conditions in the sequel (e.g., 'Shut the window only if it rains' is equivalent to 'If it does not rain, do not shut the window').

Let d be the conjunction of the sentences describing sufficient conditions for the imperative to be in force. Let $v \in S^A$ be the intention of the predicate describing the action or state requested by the imperative, i.e., if $p \in A$ is the person to whom the imperative is addressed $v(p)$ is the sentence of English saying that p is performing the action or bringing about the state demanded by the imperative. Let $n \in H_T$ be the world interval at which the utterance of the imperative has just been completed. Remember that n is an assignment of truth values to members of S. Let (n, p, v, d) denote the imperative, then its extension, $\mathcal{J}((n, p, v, d))$ is given by $\mathcal{J}((n, p, v, d)) = \{H \in P\colon \bigwedge_f (f < n \ \& \ f \in H \supset f \in H_T) \ \& \ \bigwedge_f (n \leqslant f \ \& f \in H \supset f(d) = 0$ or $f(v(p)) = 1)\}$. Thus the imperative is carried out if $H_T \in \mathcal{J}((n, p, v, d))$. Let $D = \{S_1, \ldots, S_m, (n_1, p_1, v_1, c_1), \ldots, (n_r, p_r, v_r, c_r)\}$ be a set of changes S_t and imperatives (n_j, p_j, v_j, c_j). Let $E = \mathcal{J}(S_1) \cap \ldots \cap \mathcal{J}(S_m) \cap \mathcal{J}((n_1, p_1, v_1, c_1)) \cap \ldots \cap \mathcal{J}((n_r, p_r, v_r, c_r))$. If $E = O$ then D is inconsistent. If $H_T \in E$ the changes S_t occur and the imperatives (n_j, p_j, v_j, c_j) are carried out.

The rules of inference proposed by Hare are verified if imperatives are modelled in this way, i.e., for a valid inference the intersection of the sets of possible histories in which the imperative premises are carried out and the change premises occur is a subset of the set of possible worlds in which the conclusion occurs if it is a change or is carried out if it is an imperative.

Obligation We now abstract from the occasion of utterance of an imperative and try to model the conventional meaning of imperative sentences. There is one feature of the context of an imperative utterance which we cannot abstract from completely and that is the time at which it is uttered. This is because imperatives are issued in an attempt to structure the future.

Let $O!(pC)$ symbolise that $p \in A$ is under an obligation to effect change C in virtue of an imperative $!(pC)$ issued in accordance with the conventions among members of A as to when, by whom and to whom an imperative may properly be issued in such a way that obliges an agent to bring about a change. Thus in accordance with what we have just said about time, one of the conditions that must obtain before an imperative obliges is that it be issued before the intended change is to take place. Another precondition for most imperatives is that the person issuing the imperative and the person to whom the imperative is addressed must be in the proper relation to each other for the type of imperative involved, e.g., parent and child, employer and employee, officer and enlisted man etc.

Another might be $\mathcal{J}((pC)) \neq 0$ if we are to accept the dictum 'ought implies can'. For $\mathcal{J}((pC) = 0$ means that there are no possible histories in which the agent p can carry out the change. However, as Lemmon shows in

"Deontic logic and the logic of imperatives" (*Logique et Analyse*, vol. 8 (1965), pp. 39-71), we sometimes find ourselves under different obligations to do contradictory things and if we are to have a logic of obligation, it seems likely that we should have that: an obligation to do C and an obligation to do D is equivalent to an obligation to do C and D. Thus we will leave it open as to whether a precondition for obligation is that the obligatory action be possible to perform. Let $W_c \in S^A$ be the intention of the predicate '- is under an obligation to effect change C because of an imperative issued in accordance with the conventions governing when an imperative obliges'.

Using the notions of David K. Lewis in *Convention: a philosophical study* (Harvard University Press, Cambridge, 1969), we can fill out what is meant here. If $O!(pC)$ is an imperative on the occasion of its utterance to $p \in A$ by $q \in A$ and p and q have—and it is common knowledge between them that they have—a common interest in making it possible for q to control p's actions within a certain range, and if there is some action within that range whereby p can effect the change C then p is under an obligation to effect the change C. Thus we can have 'ought implies can' in relation to the obligations arising from a single imperative whereas we most probably ought abandon it when we are considering conflicting obligations arising from different imperative acts.

The extension of the obligation $O!(pC)$ is given by $J(O!(pC))$ which is the intersection of $J((pC))$ with the set of possible histories $H \in P$ in which $W_c(p)$ is true during the time of the intended change C. Thus $J(O!(pC))$ is the set of possible histories in which p is effecting change C in accordance with the conventions governing imperatives. Thus $O!(pC)$ is represented by $C \wedge V_c(p) \wedge W_c(p) \in S$.

We will say that a sentence involving changes is valid if the extension of the sentence is P, the set of all possible histories. Under this definition, for changes B, C, D, the following are valid:

(01) $B \rightarrow (C \rightarrow D)$
(02) $(B \rightarrow (C \rightarrow D)) \rightarrow ((B \rightarrow C) \rightarrow (B \rightarrow D))$
(03) $(\sim B \rightarrow \sim C) \rightarrow (C \rightarrow B)$
(04) $O!((pB) \rightarrow (pC)) \rightarrow (O!(pB) \rightarrow O!(pC))$

With (01) - (04) as axioms and the rules

(R01) From B and $B \rightarrow C$, derive C.
(R02) From $(pB) \rightarrow (pC)$, derive $O!(pB) \rightarrow O!(pC)$ validity is preserved in inferences.

On our analysis the question as to whether one is under an obligation arising from an imperative is resolved by considering whether the conditions for obligations obtain or not. Thus, the obligations arising from conditional imperatives are best represented

$$B \rightarrow O!(pC)$$

i.e., whenever condition B obtains, p is under an obligation to effect change C. In this, B may be regarded as a change according to our definition.

'Take all the boxes to the station' becomes

$$\wedge x(Bx \to O!(pSx))$$

if p is obliged to carry out the imperative. Thus in relation to the invalid reference

Take all the boxes to the station
Make this one of the boxes
. . . .
Take this to the station

we note that 'all the boxes' refers to all the boxes indicated by the person issuing the imperative at the time the imperative was issued. Thus if the box ordered in the second premise is completed before the first order is issued and is one of the boxes included under the first order, the conclusion follows in virtue of the first premise alone; otherwise the conclusion does not follow.

In relation to

If he comes leave the file open
Do not leave the file open
. . . .
He will not come

the inference would be valid if $J(O!(p \sim F)) = P - J(O!(pF))$, i.e., if being under an obligation to refrain from doing F implied that one was not under an obligation to do F, but this is not the case as Lemmon shows.

The objectionable inference pattern

Post the letter
. . . .
Post the letter or burn it

is not valid since whereas

$$O!(pP) \vee O!(pB) \to O!(pP \vee pB)$$

is valid its converse is not. Thus from the tautology $(pB) \to (pP) \vee (pB)$ using (R02) & (04) and (R01) we derive $O!((pP) \vee (pB))$ from $O!(pP)$ but we cannot make the final step to $O!(pP) \vee O!(pB)$ which would be satisfied by burning the letter.

The examples Kenny uses in "Practical inference" (*Analysis*, vol. 26 (1965), pp. 65-75) which he takes from Aristotle deal with how a person chooses a course of action suitable for his purpose. The analogue for imperatives would be a description of the orders someone might give to implement a plan. The mental processes associated with the selection of this order rather than some other belong to a metalevel with respect to the considerations we have been dealing with in this paper.

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