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Some Remarks on (Weakly) Weak Modal Logics

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The weakest modal logics have not come in for much attention from logicians or philosophers principally, it seems, because they are supposedly incapable of supporting interpretations of much philosophical succulence. But from the point of view of general semantic theory they deserve more attention than they get, for it is only by a study of weak modal logics that we come to appreciate many of the limitations of the now standard semantical methods. A multitude of examples bears this out. In the theory of first-order definability in modal logic valuable insights would have been lost had we restricted our attention to extensions of *S4*. The McKinsey formula M_0, $\Box\Diamond p \rightarrow \Diamond\Box p$, is characterized by a first-order condition on transitive binary relations, but KM_0 is not defined by any first-order condition. Similarly, looking to logics weaker than K reveals limitations of first-order definability which would otherwise go unnoticed. Here it is that we see that while D, $\Box p \rightarrow \Diamond p$, and G, $\Diamond\Box p \rightarrow \Box\Diamond p$, are definable in a first-order language with a single binary predicate, neither is definable in a first-order language with a single ternary or n-ary ($n \geqslant 3$) predicate (see [3]).

Furthermore, weak modal logics preserve philosophically significant distinctions which are lost in stronger logics. If a proposed interpretation requires even so obvious a distinction as that between Con, $\neg\Box\bot$, and D, $\Box p \rightarrow \Diamond p$, or that between D', $\Box(\Box p \rightarrow \Diamond p)$, and D^*, $\Box\Box p \rightarrow \Box\Diamond p$, then a logic weaker even than K is required. These two facts are not unrelated. Taken together they amount to this: formulas like D and G are first-order definable only if we restrict ourselves to a first-order language so crude that it cannot

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make distinctions of obvious philosophical significance. Any first-order language sensitive enough to do justice to those elementary distinctions is a language in which these formulas are not definable.

Nor are the present authors the first to feel this concern. H. MacColl, C. I. Lewis, and H. B. Smith, all important early figures in modern modal logic, were eager to develop modal logics which preserved a maximum number of modal and other distinctions. In light of the concerns of those pioneers, the recent (post-Kripke) concern with reduction principles in strong modal logics seems a mere fad. C. West Churchman, a student of Smith, put the matter admirably in an analogy which cuts against the claims of those who suppose that *T* ($\Box p \rightarrow p$), *B* ($p \rightarrow \Box\Diamond p$), *S4* ($\Box p \rightarrow \Box\Box p$), *S5* ($\Diamond p \rightarrow \Box\Diamond p$) represent crucially important intuitions about necessity. Following Churchman's account, the usual informal motivation for the adoption of some fairly strong set of modal reduction principles may be put in this way:

> Most of us have some intuitive understanding of expressions like "$\Box\alpha$" and "$\Diamond\alpha$", perhaps even of those like "$\Box\Box\alpha$", "$\Box\Diamond\alpha$" and "$\Diamond\Box\alpha$". Those with highly developed modal intuitions may indeed even be able to fathom expressions of the form "$\Diamond\Box\Diamond\alpha$". However not even those with the most exquisite modal sensitivity can dredge up intuitions which answer to something like "$\Box^{42}\Diamond^{19}\Box^{23}\Diamond^{98}\alpha$". We may go further: not only does nobody have any intuitions about such expressions, modal assertions of any but the most unassuming complexity are counterintuitive. Since nothing comprehensible can be expressed by such assertions we should take care to see that their content reduces to exactly that of some more comprehensible one. That is, we must ensure that complex iterations of modal operators are logically equivalent to simpler iterations or perhaps even to uniterated modalities. Since it can be shown that the principles which accomplish such a reduction are consistent, nothing stands in the way of carrying out this programme which so recommends itself to our intuitions.

Compare this informal line with the following one:

> Every successful student of the calculus has an intuitive understanding of expressions like: "$\frac{df}{dt}$" and "$\frac{d^2f}{dt^2}$" since we have all had experience of their physical counterparts, viz., velocity and acceleration. However our intuitions drop off very sharply after this. Many perhaps can appreciate, at some intuitive level, the notion of a change in the rate of acceleration, i.e., "$\frac{d^3f}{dt^3}$", while those capable of extremely abstract thought may indeed find some way of making a useful distinction by means of "$\frac{d^4f}{dt^4}$". However even the most sophisticated must find an expression like "$\frac{d^{87}f}{dt^{87}}$" completely bewildering. In order to avoid what are, in effect, contentless assertions we ought to introduce into our analysis a restriction to the effect that all higher-order derivatives are equal to some sufficiently lower-order derivative, i.e., we must require something along the lines

of $\frac{d^nf}{dt^n} = \frac{df}{dt}$ or $\frac{d^nf}{dt^n} = \frac{d^2f}{dt^2}$ or $\frac{d^nf}{dt^n} = \frac{d^3f}{dt^3}$ for all n. Since such a restriction may be shown to be consistent this approach will do much to free mathematical analysis from its overly abstract and counterintuitive elements.

Of course this last proposal would be universally scorned by working mathematicians. It would so restrict the class of functions appropriate to mathematical analysis that most of modern mathematics would collapse. The moral here is that an intuitive understanding of our formal machinery is both admirable and desirable but, after all, the function of formal theories is to extend beyond the realm of the intuitive. By insisting upon staying always within sight of intuitions, be they linguistic or physical, we lose the ability to make crucial distinctions. Crucial, that is, for the sake of generality but perhaps crucial also for the application of the formalism. It is not just mathematics which would suffer upon the advent of naive intuitions of this sort. Virtually all of the areas in which mathematics finds useful applications would be similarly impoverished. How can we say then that there do not exist important applications of modal logic which would be similarly thrown out if strong modal reduction principles became universally adopted?

A second related strain of discomfiture also afflicts us. We (modal logicians) have an interest in noting the smallest logics amenable to particular sorts of semantic analysis. So, for example, we have an interest in E which is determined by the universal class of neighbourhood or Scott-Montague frames, and we have an interest in K which is determined by the class of all normal binary relational frames. E, we could say, is the fundamental logic for neighborhood semantics and K the fundamental logic for binary relational semantics. It is for this reason that, for example, Segerberg [5] moves so briskly from the logic E to the logic K, paying little attention initially to the logic C or other logics that lie between the two. A quick perusal will show that disregarding for a moment the fact that E and K are the fundamental representatives of neighbourhood and binary relational semantics, there are several logics available to us according as we adopt or reject different principles. The four principles in question are:

RE	$\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \Box\alpha \leftrightarrow \Box\beta$
RR	$\vdash \alpha \rightarrow \beta \Rightarrow \vdash \Box\alpha \rightarrow \Box\beta$
RN	$\vdash \alpha \Rightarrow \vdash \Box\alpha$
K	$\vdash \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$.

The logics obtained may be seen to form a lattice:

$$\begin{tikzcd}
& E \arrow[d] & \\
& N_q \arrow[dl] \arrow[dr] & \\
N \arrow[d, dashed] & & C \arrow[ddl] \\
K_n \arrow[dr, dashed] & & \\
& K &
\end{tikzcd}$$

where E adopts RE; N_q adopts RR; N adopts RR and RN; C adopts RR and K; and K adopts RR, RN, and K. K_n represents not one but a countably infinite sequence of logics in which K is taken to be K_1 and each $K_n \subseteq K_{n-1}$. We have used generalized relational frame techniques in [4], to analyze, in first-order language, the sequence K_n of logics weaker than K, a nonmathematical exposition of which is given in [2]. For present purposes it is sufficient to remark that K_n is the logic determined by the class of $(n + 1)$-ary relational frames where the following truth condition is adopted:

$$\overset{\mathcal{M}}{\underset{u}{\models}} \Box\alpha \Leftrightarrow \forall x_1, \ldots, x_n, uRx_1, \ldots, x_n \Leftrightarrow \overset{\mathcal{M}}{\underset{x_1}{\models}} \alpha \text{ or}, \ldots, \text{or } \overset{\mathcal{M}}{\underset{x_n}{\models}} \alpha.$$

This result shows, in effect, that for each $n \geqslant 1$, K_n is a fundamental logic for a kind of semantics in just the sense in which E and $K_{(1)}$ are known to be, viz., that each is determined by an unrestricted class of structures. Here we concern ourselves with more global features of this sequence of logics. In particular, we show that the logic N defined above appears as the limit of this sequence.

Locale frames A locale frame is a pair $\mathcal{F} = \langle U, \mathcal{L} \rangle$ where $U \neq \phi$ and $\mathcal{L}$: $U \rightarrow 2^{2^U}$ satisfies the condition:

[Minimality] $\forall u \in U, \forall a \subseteq U, a \in \mathcal{L}(u) \Rightarrow \forall b, b \subseteq a \Rightarrow b \notin \mathcal{L}(u)$.

$\mathcal{L}(u)$ is called the *family of locales* of u.

Models on locale frames A *locale model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a locale frame and $V: At \rightarrow 2^u$ is an assignment to atomic formulas. Truth conditions for PC formulas are classical and for formulas of the form $\Box\beta$:

$$\overset{\mathcal{M}}{\underset{u}{\models}} \Box\beta \Leftrightarrow \exists a \in \mathcal{L}(u) : a \subseteq \|\beta\|^{\mathcal{M}} \text{ (where } \|\beta\|^{\mathcal{M}} = \{x \in U : \overset{\mathcal{M}}{\underset{x}{\models}} \beta\}).$$

Note on locale semantics The minimality condition on $\mathcal{L}(u)$ together with the truth condition for modal formulas make locale semantics distinct from neighbourhood semantics. To illustrate this we remark that the neighbourhood frame restriction associated with the logic K is that each neighbourhood family forms a filter. By contrast, the corresponding locale frame restriction is that each locale family contains exactly one element. In view of the specific ends of this study, in what follows we consider only locale frames in which every locale family is nonempty. The most general account of locale frames would be one in which this restriction is not observed and in which, consequently, RN fails to preserve validity. We call this restriction the *normality condition*.

Theorem 1 *N is determined by the class of all normal locale frames.*

Proof: The proof follows from the following two lemmas.

Lemma 1.1 *N is sound with respect to the class of all normal locale frames.*

Proof: Trivial.

Lemma 1.2 *N is complete with respect to the class of all normal locale frames.*

Proof: By a Henkin construction.

The canonical locale model The *canonical locale model* $\mathcal{M}_L$ of a modal logic L is a triple:

$$\langle U_L, \mathcal{L}_L, V_L \rangle$$

where U_L is the set of all L-maximally consistent sets, and V_L, the canonical assignment, is given by $V_L(p_n) = |p_n|_L$ where $|p_n|_L$ is the set of all L-maximally consistent sets containing p_n.

The definition of $\mathcal{L}_L$ requires some preliminary definitions. Initially, we define $X(u)$ for each $u \in U_L$:

$$X(u) = \{|\alpha|_L : \Box\alpha \in u\}.$$

We note that $X(u)$ is partially ordered by set inclusion. A maximal chain C in $(X(u))^2 \cap \subseteq$ is a totally ordered subset of $X(u)$ of which no proper superset is a totally ordered subset of $X(u)$. Clearly,

$$\bigcup_{i \in I} \{C_i : C_i \subseteq X(u)\} = X(u).$$

We now define $\mathcal{L}_L(u)$ by:

$$\mathcal{L}_L(u) = \{\cap C : \phi \neq C \subseteq X(u)\}.$$

We shall have shown completeness when we have shown that for any natural modal logic L,

$$\models^{\mathcal{M}_L}_{u} \alpha \Leftrightarrow \alpha \in u.$$

The proof is by induction on the length of α. For α of length l, the theorem holds by the definition of V_L. We prove only the hard direction of the inductive step for α of the form $\Box\beta$.

Suppose that $\models^{\mathcal{M}_L}_{u} \Box\beta$. Then $\exists a \in \mathcal{L}(u) : a \subseteq \|\beta\|^{\mathcal{M}_L}$. By the hypothesis of induction, $\exists a \in \mathcal{L}(u) : a \subseteq |\beta|_L$. But $a = \cap C_i$ for some totally ordered subset C_i of $X(u)$. So we may infer that $\{\gamma : \Box\gamma \in u \;\&\; |\gamma|_L \in C_i\} \vdash_L \beta$ and, therefore, that for some finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of that set,

$$\vdash_L \gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \beta.$$

But the set $\{|\gamma_i|_L\}_{(1 \leq i \leq n)}$ is itself totally ordered by inclusion and $\gamma_1, \ldots, \gamma_n$ may therefore be ordered in some such way for each γ_j, $\vdash_L \gamma_j \rightarrow \gamma_{j+1}$. Then there is some $\gamma_k \rightarrow \gamma_1 \wedge \ldots \wedge \gamma_n$. Therefore, $\vdash_L \gamma_k \rightarrow \beta$. By RR, $\vdash_L \Box\gamma_k \rightarrow \Box\beta$. But $\Box\gamma_k \in u$. Therefore $\Box\beta \in u$. This completes the proof.

We are now in a position to prove the main result of this essay, which answers a question of Hans Kamp's.

Theorem 2 *N is the intersection of the K_n logics.*

Proof: This result is obtained from the following two lemmas.

Lemma 2.1 $N \subseteq \bigcap_{i \in \mathbf{N}} K_i$.

Proof: Trivial.

Lemma 2.2 $\bigcap_{i \in \mathbf{N}} K_i \subseteq N$.

Proof: Since N is determined by the class $\mathcal{C}_{\mathcal{L}}$ of all normal locale frames and for each $i \in \mathbf{N}$, K_i is determined by the class $\mathcal{C}_i$ of all relational frames of rank i, it suffices to show that $\not\models_{\mathcal{C}_{\mathcal{L}}} \alpha \Rightarrow \not\models_{\mathcal{C}_i} \alpha$ for some i. This we show by induction on the length of α, once more proving only the inductive step for α of the form $\Box\beta$.

Let $|\delta|$ for a well-formed formula δ designate the set of variables of δ. Let card$|\Box\beta| = n$. Now suppose that for some point x and some locale model $\mathcal{M} = \langle U, \mathcal{L}, V\rangle$, we have $\not\models_x^{\mathcal{M}} \Box\beta$. We define an n-rank relational model $\mathcal{M}' = \langle U', R, V'\rangle$ as follows:

$$U' = U \text{ and } V' = V.$$

$R \subseteq U^{n+1}$ is defined in terms of its corresponding function r as follows: for each $u \in U$, $r(u)$ is the union of Cartesian products of elements $(\mathcal{L}(u))^n$. That is, $\langle x_1, \ldots, x_n\rangle \in r(u) \Leftrightarrow \langle x_1, \ldots, x_n\rangle \in \prod_n (a_1, \ldots, a_n)$ for some $(a_1, \ldots, a_n) \in (\mathcal{L}(u))^n$. Finally, $uRx_1 \ldots x_n \Leftrightarrow \langle x_1, \ldots, x_n\rangle \in r(u)$. Clearly,

$$\not\models_x^{\mathcal{M}} \Box\beta \Rightarrow \not\models_x^{\mathcal{M}'} \Box\beta.$$

Make the assumption. Then $\forall a \in \mathcal{L}(x)$, $\exists y \notin a: y \in \|\beta\|^{\mathcal{M}}$. Then $\exists y_1, \ldots, y_n$: $xRy_1 \ldots y_n \;\&\; y_1 \notin \|\beta\|^{\mathcal{M}'} \;\&\; \ldots \;\&\; y_n \notin \|\beta\|^{\mathcal{M}'}$. Therefore $\not\models_x^{\mathcal{M}'} \Box\beta$.

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