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THE SUBSTITUTION INTERPRETATION AND THE EXPRESSIVE POWER OF INTENSIONAL LOGICS

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The substitution interpretation may be employed in place of the objectual interpretation in giving the semantics for first-order logic, without affecting the class of formulas defined as valid. If the usual definition of satisfaction is given, namely that a set is satisfiable just in case its members are all true on some model, and the substitution interpretation is used, the notion of satisfaction is no longer compact. So notions of semantic entailment and satisfaction differ from those generated by the standard account. However, with a simple adjustment to the definition of satisfaction, compactness is restored, and notions of satisfiability and semantic entailment match exactly those of the standard account, at least as far as their extensions go. An adjusted definition of satisfaction suitable for the substitution interpretation looks something like this: a set of formulas is satisfiable just in case there is a syntax for first-order logic which has those formulas among its well-formed formulas, and a model for that syntax such that every formula of the set is ruled true. In a sense to be made somewhat clearer below, (semantics for) first-order logic has substitution interpretation invariance (sii), at least when the definition of satisfaction is adjusted correctly.

The same is not true of intensional logics. The results of Garson [2] show that a semantics for topological logic is not sii, and one of the results of this paper will be that Thomason's **Q2** is not sii either [4]. We need to present some of the details of these two systems. We give here a definition of u-semantics for topological logic. A u-model is a triple $\langle C, F, u\rangle$ where C is a non-empty set (of possible worlds, contexts, etc.), F is the set of all transformations on C, and u is an interpretation function which assigns intensions to the terms and predicates as we would expect:

$u(n) \epsilon F$, for each term n
$u(P^j) \epsilon \{f{:}f{:}C \rightarrow \mathcal{P}(C^j)\}$, for each j-ary predicate P^j where '$\mathcal{P}$' indicates power set.

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The truth value of a formula A at $c\,(U_c(A))$ on model U is defined to be either 1 or 0 according to the following:

(P) $U_c(P^j n_1 \ldots n_j)$ is 1 iff $\langle u(n_1)(c), \ldots, u(n_j)(c)\rangle \in u(P^j)(c)$
($\sim$) $U_c(\sim A)$ is 1 iff $U_c(A)$ is 0
($\supset$) $U_c((A \supset B))$ is 1 iff $U_c(A)$ is 0 or $U_c(B)$ is 1
(T) $U_c(\mathrm{T}nA)$ is 1 iff $U_{u(n)(c)}(A)$ is 1
(0) $U_c(\forall xA)$ is 1 iff $U^f/x_c(A)$ is 1 for all $f \in F$, where U^f/x is the model identical to U save that $u(x)$ is f.

We have assumed here that our language has $\sim$, $\supset$, and $\forall$ as its logical primitives, and that it also contains a symbol T (read 'at') which takes a term n and a sentence A into a new sentence $\mathrm{T}nA$. We will assume for the rest of this paper that the definition of satisfaction appropriate for the substitution interpretation is used, for it makes no difference which is used when the standard interpretation of the quantifiers is used. Validity is defined from satisfaction as usual.

When we change this semantics so that (0) is replaced by

(S) $U_c(\forall xA)$ is 1 iff $U_c(A^n/x)$ is 1 for each term n

the set of formulas ruled valid (hence the sets of formulas ruled satisfiable) changes. It was shown in [2] that the semantics using (S) is easily axiomatized, and that the semantics is equivalent to one which employs (0), but where F is *a* set of transformations on C, rather than the set of *all* transformations on C. However, u-semantics, as we have first defined it (using (0)) has not been axiomatized and there are formulas which it validates which cease to be valid when (S) is used. Clearly u-semantics is not sii.

The same is true of **Q2**. A **Q2**-model u is a sextuple[1] $\langle C, R, F, I, \psi, u\rangle$ such that C is as before, R is a binary relation on C, F is the set of all functions from C to I, I is a (non-empty) set of individuals, ψ is a function that takes each $c \in C$ into a subset $\psi(c)$ of I, and u is as before. The truth value of a formula at c on U is defined by (P), ($\sim$), ($\supset$) and the following:

($\Box$) $U_c(\Box A)$ is 1 iff $U_d(A)$ is 1 for all d such that Rcd
(**Q20**) $U_c(\forall xA)$ is 1 iff $U^f/x_c(A)$ is 1 for all f such that $f(c) \in \psi(c)$.

Thomason reports that Kaplan has shown that this semantics is not axiomatizable. He takes this failure as a sign that there is something fundamentally wrong with **Q2**, and that it points out the need for introducing the concept of a substance into the formal theory.

But **Q2** has intuitive appeal, and so it is satisfying to know that it can by rescued simply by adopting the substitution interpretation. When

(**Q2S**) $U_c(\forall xA)$ is 1 iff $U_c(A^n/x)$ is 1 for all n such that $u(n)(c) \in \psi(c)$

is replaced for (**Q20**), the resulting semantics is easily axiomatized. Completeness can be demonstrated along the lines of Garson [1]. The system which is adequate, and the main definitions and lemmas of the completeness proof, appear in the appendix. It is interesting to note that when we retain (**Q20**) and relax the condition that F is the set of *all* functions from C into I, we arrive at the same system, just as was the case in topological logic. So we can rescue **Q2** without resorting to the substitution interpretation. We probably want to rescue **Q2** in one or the other of these ways, for there are formulas such as $\exists x\Box\exists yy = x$ (read 'something exists necessarily') which are intuitively unacceptable, but valid on the original definition of **Q2** by virtue of the fact that the domain of quantification is the set of all functions from C into I. The original definition, then, makes unwarranted ontological assumptions.

The property of sii is a property of a semantics, not a model. It has to do, primarily, with how the truth clause for the quantifier affects the final definition of satisfaction. This property, then, is not the proper study of model theory. Model theory studies models, and during this study, it is presumed that the definition of truth on a model is held fixed. Intensional languages have the property that the definition of satisfaction is sensitive to the techniques used in the truth definition. So for these languages, it is worthwhile to open a new branch of study: semantics theory, the study of the effects of the use of different styles of truth definition within the same class of models. It would be interesting to know under what circumstances a semantics is sii. Perhaps we can connect this property with some purely model-theoretic result. The Lowenheim-Skolem Theorem comes immediately to mind and, as we shall see, this intuition is correct.

Before we come to that, however, I would like to engage in a small diversion that may put that result in a slightly different light, and suggest areas for further research. We might guess that the reason first-order logic is sii but certain intensional logics are not is that the latter have more expressive power, given their semantics. We know, for example, that when the quantifiers are deleted from first-order logic every formula has a finite model. When the quantifiers are restored, however, we can construct a formula that has no finite model. We might express this by saying that the quantifiers allow us to express the property that our domain is (denumerably) infinite. We know by the Lowenheim-Skolem Theorem that first-order logic cannot express that the domain is superdenumerably infinite, but if we add the quantifier $>$ so that $(>x\Phi x)\Theta x$ is interpreted as claiming that the cardinality of the set of things that satisfies Φ is greater than the cardinality of the set of things that satisfies Θ, then we can find a formula which is true only on superdenumerable models. We can write a sentence to ensure that the cardinality of the set of things that satisfies Θ is denumerably infinite. When this sentence is conjoined to $(>x\Phi x)\Theta x$ the result will be satisfied on a model when the cardinality of the set of things that satisfies Φ is greater than omega. In a sense, $>$ adds expressive power to first-order logic.

Let us formalize this notion of expressive power. A *language* is composed of a syntax (recursive definition of the wffs) and a semantics. The *semantics* defines a class of models and a recursive definition for 'formula A is true at c on model U'. We assume that the truth definition is intensional for generality, but for extensional languages the mention of c

may be ignored or deleted. A language is recursive iff the set of formulas defined valid on its semantics is recursive. A *language* $\mathcal{L}$ *expresses property* P (of a model) just in case there is a set of formulas of that language which is satisfied on one of its models with P, and in no model without P. The *definiens* is equivalent to 'It is not the case that every set of formulas satisfied on a model of $\mathcal{L}$ with P is satisfied on some model without P', so we have the following results about expressive power:

(1) *First-order logic cannot express that the domain of a model is superdenumerably infinite.*

Proof: By the Lowenheim-Skolem Theorem

(2) *A recursive language is decidable if it cannot express that the domain of a model is infinite.*

Proof: If the language cannot express infinity, then it has the finite model property, and so is decidable.

It would be interesting to know in what sense intensional operators add expressive power to first-order logic. In intensional languages, there is a rich variety of topics to investigate, since the domain F of quantification is a set of functions. So we may investigate such standard properties as whether F is closed under composition, or whether it contains inverses, etc. Some results about expression of some properties of this kind appear in [2], but much more needs to be done, especially for modal languages. We may also investigate the properties which are familiar from model theory for first-order logic, such as questions about the cardinality of F. The next thing we want to do is give a result of this kind. We will show how to connect a property of semantics theory (namely, sii) with one of model theory (namely whether superdenumerability can be expressed). We will be showing in effect that a language is sii just in case the Lowenheim-Skolem Theorem holds for it, given that it satisfies some minimal properties.

We will call a language *standard* just in case its semantics is defined so that the intension of any complex expression is a function of the intension of its immediate subexpressions, and mention of the domain of quantification appears only in the clause giving the intension of a formula whose main operator is a quantifier, and the clause for the quantifier has the shape:

$$(\mathbf{Q})\ U_c(\forall xA) \text{ is } 1 \text{ iff } U^{f}/x_c(A) \text{ is } 1 \text{ for all } f \text{ such that } \Theta(f,c)$$

where $\Theta(f,c)$ is some property expressed in the metalanguage whose only parameters are f and c. We have had to pick a notation for (**Q**) for the sake of concreteness, which limits our definition of a standard language somewhat. Still, the theorem to follow will not depend on any of the notational details. We should note, however, that there are intensional languages with quantifier clauses which do not have the form (**Q**), for example, those of the form

$$(\mathbf{Q}')\ U_c(\forall xA) \text{ is } 1 \text{ iff } U^{f(c)}/x_c(A) \text{ is } 1 \text{ for all } f \text{ such that } \Theta(f,c).$$

Hintikka [3] has investigated languages of this kind, but their axioms are quite cumbersome. The results of this paper do not apply to such systems.

Before we state and prove our theorem, we need to define sii more exactly. Let us assume for the rest of this paper that $\mathbf{S}$ is the semantics of a standard language, and that $\mathbf{S}_s$ is the semantics like $\mathbf{S}$ save that where $\mathbf{S}$ has a truth clause for the quantifier of the form (Q), $\mathbf{S}_s$ has a clause of the form:

$$\text{(QS)}\quad U_c(\forall xA) \text{ is 1 iff } U_c(A^n/x) \text{ is 1 for all } n \text{ such that } \Theta(u(n),c).$$

A standard language with semantics $\mathbf{S}$ is sii just in case any set of formulas is satisfiable on $\mathbf{S}$ exactly when it is satisfiable on $\mathbf{S}_s$.

Theorem *A standard language is* sii *just in case it cannot express superdenumerable infinity.*

Proof: The proof depends on two simple lemmas. We will write '$\mathrm{Sat}^{U}_{\mathbf{S}}(W)$' for 'for some c in the set C of U, and every $A \in W$, $U_c(A)$ is 1 on semantics $\mathbf{S}$'.

Lemma 1 *For any models U, U' which differ only in their domains F, F' of quantification,* $\mathrm{Sat}^{U}_{\mathbf{S}_s}(W)$ *iff* $\mathrm{Sat}^{U'}_{\mathbf{S}_s}(W)$.

Proof: Obvious, since semantics $\mathbf{S}_s$ will not mention the domain in any of its truth clauses.

Lemma 2 *If the domain of quantification F of U is $\{u(n){:}n$ is a term$\}$ where u is the interpretation function of U, then* $\mathrm{Sat}^{U}_{\mathbf{S}_s}(W)$ *iff* $\mathrm{Sat}^{U}_{\mathbf{S}}(W)$.

Proof: Show $U_c(A)$ is 1 on $\mathbf{S}_s$ iff $U_c(A)$ is 1 on $\mathbf{S}$ by induction on the structure of A. The case for the quantifier is guaranteed because F for U is $\{u(n){:}n$ is a *term*$\}$. We should note, however, that the proof does depend on the fact that $U_c^{u(n)}/x(A)$ is $U_c(A^n/x)$ in semantics $\mathbf{S}$. That follows because a standard language's semantics calculates the intension of an expression only on the basis of the intensions of its subexpressions.

Now we are ready to prove the Theorem. To prove it from left to right, suppose that a standard language with $\mathbf{S}$ as its semantics is sii, and suppose that there is a model that satisfies W on $\mathbf{S}$ which has a superdenumerable domain. It follows that for some model U, $\mathrm{Sat}^{U}_{\mathbf{S}_s}(W)$. By Lemma 1, the model U' like U save that its domain is $\{u(n){:}n$ is a term$\}$ is such that $\mathrm{Sat}^{U'}_{\mathbf{S}_s}(W)$. By Lemma 2, $\mathrm{Sat}^{U'}_{\mathbf{S}}(W)$, and so W is satisfiable on a model with a denumerable domain, and the language cannot express superdenumerable infinity. For the proof in the other direction, suppose that a standard language with semantics $\mathbf{S}$ cannot express superdenumerable infinity. Suppose that W is satisfiable in $\mathbf{S}_s$. Then there is a model U such that $\mathrm{Sat}^{U}_{\mathbf{S}_s}(W)$ and by Lemma 1 the model like U with $\{u(n){:}n$ is a term$\}$ as its domain satisfies W. By Lemma 2, W is satisfiable in $\mathbf{S}$. Now suppose that W is satisfiable in $\mathbf{S}$. We know that W has a model with denumerable domain F. Form an extension of the language by adding a new rank of terms $t_1, \ldots, t_j, \ldots$. We order the members of F, and we extend the model U so that $u(t_i) = f_i$, and call it U'. Clearly $\mathrm{Sat}^{U'}_{\mathbf{S}}(W)$. But by

Lemma 2, $\mathsf{Sat}^{U'}_{\mathbf{S_s}}(W)$ in this new language. Since we are using the definition of satisfaction appropriate for the substitution interpretation, it follows that W is satisfiable in $\mathbf{S_s}$. It follows that the language is sii, and the Theorem is proved.

This theorem is useful in showing that an intensional semantics lacks the property of the Lowenheim-Skolem Theorem, i.e., for showing it can express superdenumerability. Ordinarily this would be done by finding a formula which is true on a superdenumerable model, which is false on all smaller models. By our theorem, it is enough to find *any* formula which is valid on **S** but not on its $\mathbf{S_s}$, for then the language is not sii. It won't matter to the proof whether this formula is true on superdenumerable models and not on smaller models, and so it will be much easier to locate a formula that proves what we want. For example, it is easy to see that $\exists x\Box\exists yy = x$ is not valid on $\mathbf{Q2_s}$ but valid on **Q2**, hence the Lowenheim-Skolem Theorem fails for **Q2**, even though $\exists x\Box\exists yy = x$ is true on denumerable models.

APPENDIX

We will show that $\mathbf{Q2_s}$ is complete with respect to the following system (soundness is easily shown):

(Prop) Axioms and rules for propositional logic

(K) Axiom and rule for K

($\forall$I) $\forall xA \supset (\mathrm{E}n \supset A^n/x)$, where '$\mathrm{E}n$' abbreviates '$\exists xx = n$'

($\forall$G) $\dfrac{B \supset (\mathrm{E}n \supset A^n/x)}{B \supset \forall xA}$, where n does not appear in

(=) $n = n$

(AS) $n = n' \supset (A \supset A^{n'}//n)$, where $A^{n'}//n$ is the result of replacing n' properly for one or more occurrences of n in A, each of which is not in the scope of $\Box$.

Definition 1: N_G is the set of terms that appear in the set of formulas G.

Definition 2: $\mathsf{Wff}(N)_{\mathcal{L}}$ is the set of formulas of a syntax like $\mathcal{L}$ save that N is its set of terms. (We omit $\mathcal{L}$ when the meaning is clear without it.)

Definition 3: A set G of formulas of $\mathcal{L}$ is a $\mathbf{Q2_s}$*-model set*$_{\mathcal{L}}$ iff for each $A \in \mathsf{Wff}(N_G)$, $G \nvdash \bot$, if $A \notin G$ then $G \cup \{A\} \vdash \bot$, and if $\sim\forall xA \in G$, then for some term $n \in N_G$, $(\mathrm{E}n \,\&\, \sim A^n/x) \in G$.

The next lemma shows that $\mathbf{Q2_s}$-model sets obey the standard properties for maximally consistent omega complete sets, when attention is restricted to the appropriate set of formulas.

Lemma 3 *If* $A, B \in \mathsf{Wff}(N_c)$ and c is a $\mathbf{Q2_s}$-model set, then

P$\vdash$. $c \vdash A$ *iff* $A \in c$

P$\sim$. $\sim A \in c$ *iff* $A \notin c$

P$\supset$. $(A \supset B) \in c$ *iff* $A \notin c$ or $B \in c$

P$\forall$. $\forall xA \in c$ *iff* $(\mathrm{E}n \supset A^n/x) \in c$ *for all terms* n *of* $\mathcal{L}$

Definition 4: The canonical model $\mathfrak{U}^{\mathsf{Can}}$ for $\mathbf{Q2_s}$ in language $\mathcal{L}$ is defined so that $\mathfrak{U}^{\mathsf{Can}} = \langle C, R, F, I, \psi, u\rangle$ and $c \in C$ iff c is a $\mathbf{Q2_s}$-model set and there is an infinite set of terms of $\mathcal{L}$ foreign to c, (R) Rdc iff for all formulas A of $\mathcal{L}$, if $\Box A \in d$ then $A \in c$. Now let n_d be the least term t of $\mathcal{L}$ such that $n = t \in d$. The existence of n_d is guaranteed by (=). Now $I = \{n_c : c \in C\}$, $u_c(n)$ is n_c, $u_c(P^j)$ is $\{\delta : P^j\delta \in c\}$, when P^j is a j-ary predicate and δ is a string of terms, and

$$(\psi)\ i \in \psi_c \text{ iff for some term } t,\ t_c \text{ is } i \text{ and } \mathrm{E}t_c \in c.$$

Lemma 4 *If* $A \in \mathsf{Wff}(N_c)$, *then* $A \in c$ *iff* $\mathfrak{U}^{\mathsf{Can}}_c(A)$ *is* 1.

The proof makes use of the following sublemmas:

Sublemma a *If* E$n \in \mathsf{Wff}(N_c)$, *then* E$n \in c$ *iff* $n_c \in \psi_c$.

Proof: By (ψ) and (AS).

Sublemma b *If* $\forall xA \in \mathsf{Wff}(N_c)$, *then* $\forall xA \in c$ *iff* $\mathfrak{U}_c^{\mathrm{Can}}(\forall xA)$ *is* 1.

Proof: By P$\forall$., (ψ), Sublemma a, and the hypothesis of the induction.

Sublemma c *If* $\Box A \in \mathsf{Wff}(N_c)$, *then* $\Box A \in c$ *iff* $\mathfrak{U}_c^{\mathrm{Can}}(\Box A)$ *is* 1.

Proof: If $\Box A \in c$ then $\mathfrak{U}_c^{\mathrm{Can}}(\Box A)$ is 1 by (R) and the hypothesis of the induction. Now suppose $\mathfrak{U}_c^{\mathrm{Can}}(\Box A)$ is 1, and suppose for *reductio* that $c_\Box = \{B: \Box B \in c\} \nVdash A$. Then $c \cup \{\sim A\} \nVdash \perp$. Since $c \in C$ there is an infinite set N of terms foreign to c, and since members of $c_\Box \cup \{\sim A\}$ are all in $\mathsf{Wff}(N_c)$, no member of N appears in $c_\Box \cup \{\sim A\}$. Form two infinite disjoint sets N_1, N_2 from N. By reasoning similar to the Lindenbaum Lemma, there is a $\mathbf{Q2_s}$-model set d such that $c_\Box \cup \{\sim A\} \subseteq d$, and $d \subseteq \mathsf{Wff}(N \cup N_1)$. There is an infinite set of terms (namely N_2) foreign to N_d, so $d \in C$. So for some $d Rcd$ and $\sim A \in d$, which, with the hypothesis of the induction, conflicts with our initial assumption. We conclude that $c_\Box \cup \{\sim A\} \Vdash \perp$, in which case $c_\Box \Vdash A$, and by principles of K, $d \Vdash \Box A$. Since $\Box A \in \mathsf{Wff}(N_c)$, $\Box A \in c$.

Theorem 2 $\mathbf{Q2_s}$ *is complete*.

Proof: We show that every $\mathbf{Q2_s}$-consistent set G is satisfiable. Add an infinite set N of terms new to $\mathsf{Wff}(N_G)$, and divide it into two infinite disjoint sets, N_1, N_2. Construct a $\mathbf{Q2_s}$-model set M which is an extension of G, using set N_1. Now consider $\mathfrak{U}^{\mathrm{Can}}$ for the language enlarged with N. There is an infinite set of terms (namely N_2) foreign to M, so $M \in C$, and by Lemma 4, $A \in M$ iff $\mathfrak{U}_M^{\mathrm{Can}}(A)$ is 1, and so G is satisfiable.

Notice that the definition (R) of R in the canonical model is standard. This means that standard results about how axioms of modal logic correspond to conditions on R carry through this proof. Completeness of a wide variety of modal logics which use condition (**Q2S**) can be carried out in the same format.

NOTE

1. We have adjusted the notation, and some aspects of the formal definition of **Q2** to make its parallels with the situation with topological logic clearer, and to square with the notation of the rest of this paper. No crucial changes have been made, however.

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