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THE EXTENDED CALCULUS OF INDICATIONS INTERPRETED AS A THREE-VALUED LOGIC

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1 *Introduction* The point of view of indication, as a foundational notion for mathematical thinking, was introduced by G. Spencer Brown in 1969 [1]. Taking as primitive only the intuitive notion of distinction or indication, he presented a simple yet amazingly powerful calculus, the Calculus of Indications (**CI**), whose full import is slowly being recognized [2]. In its two values, indicated, '⏋', and not indicated, ' ', this calculus embodies the general form of any two-valued situation. Many possible interpretations of **CI** are thus possible, but a particularly interesting one is for classical propositional logic, where statements can be true or false (*cf.* [1], Appendix 2). I have taken the Calculus of Indications as a starting point in an attempt to produce adequate tools to deal with self-referential situations.* Self-reference is, of course, of great historical importance; it was responsible for a major crisis in mathematical thinking at the turn of the century. More recently, with the development of cybernetics and systems theory, other aspects of self-referential situations have become apparent, namely, the fact that many highly relevant systems have a self-referential organization. The key character of self-production in living systems is, perhaps, the most obvious instance; examples from the neurological, cognitive, and social domains also abound [3,4,5,6]. With his motivation I developed an Extended Calculus of Indications (**ECI**), capable of dealing with the basic forms of self-reference, and thus, providing a foundation to interpret any possible instance of them [7]. The point of view of indication greatly simplifies the discussion of self-referential situations, by simply having an expression indicate itself. Expressions where self-indication is allowed, are called boolean expressions of higher degree by Spencer Brown, and in his [1] he hinted at their possible applications. In [7] I showed that **CI** is not consistent with self-indicating expressions and derived **ECI**, where not two but three values exist: indicated, not indicated, and self-referring or

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autonomous, '◻'. It can be showed that, although all self-referring forms are allowed in **ECI**, their diversity can essentially be reduced to the atomic case of the autonomous value.

These introductory statements do not attempt to be a recapitulation of all these results, but only a background motivation for the rest of this work. I will not discuss here the varieties of self-referential forms that can be accomodated in **ECI** (i.e. expressions of higher degree) and their interpretation in a propositional logic (*cf.* Section III.B). My intention here, is only to *interpret* the values of **ECI** as truth values and thus to produce a three-valued logic from **ECI**. The interest of such three-valued logic is twofold. It is on the one hand theoretical, since it introduces into the theory of many-valued logic an approach for calculations of self-referential linguistic forms. On the other hand, it is of interest in some applications of the **ECI** to deal with self-referential systems, since the present logic, being derived from **ECI** itself, can serve as a language to provide access to computer modeling and simulation.

2 *Interpretation* I will define language $\mathcal{L}$(**E**), to be understood as **ECI** interpreted for a propositional logic. In $\mathcal{L}$(**E**) there are three possible truth-values: ⌉, true; , false; and ◻, autonomous (i.e. self-naming). The following tables define negation and disjunction in $\mathcal{L}$(**E**):

'not *p*': *p*⌉

p	*p*⌉
⌉	
	⌉
◻	◻

'*p* or *q*': *pq*

p \ *q*	⌉		◻
⌉	⌉	⌉	⌉
	⌉		◻
◻	⌉	◻	◻

The rest of the connectives are defined as follows:

'*p* implies *q*': *p*⌉*q*

p \ *q*	⌉		◻
⌉	⌉		◻
◻	⌉	⌉	⌉
◻	⌉	◻	◻

'*p* and *q*': *p*⌉*q*⌉⌉

p \ *q*	⌉		◻
⌉	⌉		◻
◻	◻		◻

'*p* if and only if *q*': *p*⌉*q*⌉*q*⌉*p*⌉⌉

p \ *q*	⌉		◻
⌉	⌉		◻
		⌉	◻
◻	◻	◻	◻

The truth value mark ⌉ can be seen to act also as a logical operator in $\mathcal{L}$(**E**). In other words, we chose to view true as 'not false', and false as 'not true'; similarly, autonomous is taken to be 'not itself', since if *p* is ◻, then *p* is identical to *p*⌉, thence the symbol ◻.

In order to construct expressions in $\mathcal{L}$(**E**) we adopt the following rules of formation: (i) ⌉, , ◻, are expressions; (ii) if *p* is an expression so is *p*⌉; (iii) if *p, q* are expressions, then *pq* is an expression. Expressions *p, q* which can be shown to simplify (via the tables) to the same value are taken to be identical in which case we write '*p* = *q*'. For example, from the tables and rules of formation, we have the identities of the following expressions:

$\overline{\rceil}\rceil = \quad,$

$\overline{\square}\rceil = \square,$

$\square\,\square = \square,$

$\rceil = \rceil v, \qquad\qquad v$ any value: $\rceil$, $\;$, $\square$; etc.

We adopt the rules of substitution as usual, that is, identical expressions can be replaced for one another. For detachment, however, we shall not adopt a modus ponens but an equivalence rule, by detaching expressions at their point of equivalence. The reasons for this choice will be apparent below.

The axioms of the $\mathcal{L}$(**E**) are the following:

A1 $\overline{\overline{p}\rceil \overline{q}\rceil}\rceil p = p$

A2 $\overline{\overline{pr}\rceil \overline{qr}\rceil}\rceil = \overline{\overline{p}\rceil \overline{q}\rceil}\rceil r$

A3 $\overline{\square p}\rceil p = p\square$

Within the above interpretation, the main theorems of **ECI** [7] can be rendered thus:

(i) If in an expression all the variables take a value, then the expression has a value.

(ii) For any choice of values for the variables of an expression, the value of the expression is unique. In other words, $\mathcal{L}$(**E**) is consistent.

(iii) The axioms of $\mathcal{L}$(**E**) are complete. That is, if p has the same value as q, then '$p = q$' can be derived from the axioms.

(iv) $\mathcal{L}$(**E**) is strongly complete. If a non-derivable expression is added as an axiom, the $\mathcal{L}$(**E**) becomes inconsistent.

3 *Discussion* In this section I wish to compare $\mathcal{L}$(**E**) with other well-known three-values systems. I will discuss separately those issues pertaining to the purely formal aspects or 'syntax', and those issued pertaining to the intended use and meaning or 'semantics'.

A. *Syntax* The truth tables of $\mathcal{L}$(**E**) are identical to those first described by Kleene [8], and redefined as the variant-standard system S_3 in the study of Dienes [9]. This is easily seen by changing '$\rceil$' into '**T**' (or '1'), ' ' into '**F**' (or '0') and '$\square$' into '**U**' (or '2') in the preceeding tables. Further, any expression of $\mathcal{L}$(**E**) can be immediately transcribed into an expression in Kleene's system (**K**) simply by rewriting the connectives as defined in section **2**. The real difference between these classic systems and $\mathcal{L}$(**E**) lies in the latter's use of the '='-equivalence. In this connection it should be noticed that an expression which is derivable from **K**'s axioms, must have either the value 'true' or 'undefined', since these are the tautologies of **K** [9]. Let us transcribe '$\vdash_{\mathbf{K}} p$' in **K** for '$p = \square p$' in $\mathcal{L}$(**E**). Since $\mathcal{L}$(**E**) is complete, all identities of the form '$p = \square p$' are derivable; therefore all derivable expressions p in **K** are recovered in $\mathcal{L}$(**E**) as exactly those expressions satisfying $p = \square p$. Conversely, if an expression in $\mathcal{L}$(**E**) is such that $p = \square p$, then we must have $\vdash_{\mathbf{K}} p$, since it has a designated value, and thus it must be derivable in **K**. This justifies the choice of an equivalence form for $\mathcal{L}$(**E**): by doing so, not only do we recover all expressions derivable with implication, but have also access to many *other* classes of expressions. By taking an implication rule for detachment, we force ourselves to confuse expressions which are '**T**' or '**U**' in **K**. In this sense, $\mathcal{L}$(**E**) is richer than the classical systems with which it has coincidence in the truth-tables.

B. *Semantics* The expressions of $\mathcal{L}$(**E**) were derived as an interpretation of **ECI** for the domain of logical discourse. Thus, by necessity they carry an immediate interpretation, and the semantic rendering for the truth-tables poses no problem. The real semantic issues stem from the fact that the intended use of $\mathcal{L}$(**E**) is to deal with self-referential linguistic forms. When interpreted for classical logic, self-reference engenders the paradoxes of self-denying sentences. To this problem several authors have addressed themselves, in an attempt to solve it by means of a three-valued logic. Moh Shaw-Kwei [10] used the three-valued system of Łukasiewicz, interpreting the third value as 'paradoxical'. He showed that in this case, as in all the family of lukasiewiczian systems, paradoxes will recurr; however, his results do not apply for systems of the **K**-type, where 'p implies p' is untrue. Later, Asenjo [11] proposed a calculus of antinomies, where the third value is taken as 'antinomic', with truth-tables of the **K**-type; his axioms, however, are incomplete. The present work extends these author's attempts by providing a consistant, complete system that can accept self-denying statements as non-paradoxical. We depart from them in various aspects. First, $\mathcal{L}$(**E**) is an interpretation of indicative expressions which stand for more general forms; conversely all logical forms in $\mathcal{L}$(**E**) have access to an underlying simple calculus of indications not available in other logics. Second, this interpretation for a three-valued logic is a first step in an attempt to deal with self-reference in general, not to circumvent paradoxes, which are seen at this light as only a particular case of atomic self-reference or autonomous value. Third, this logic can be a linguistic carrier for description, modeling, and simulation of self-referential systems other than logic; in this sense $\mathcal{L}$(**E**) is, so to speak, constructive.

Beyond these considerations, let us look more in detail the autonomous value as a paradigm for self-reference. As it now stands in $\mathcal{L}$(**E**) it is only a third value which can deal with self-reference in a very loose way, namely, insofar self-referring statements require a value which is identical to its negation. In [10,11] these paradoxical statements are taken for granted, or borrowed from the calculus of classes in the form of Russell's paradox. To study self-reference in a deeper and more rigorous way, we must stay at the level of the calculus used, propositional logic in the case at hand, and introduce self-referential forms only through the means available in it. A classic treatment is [12] through the use of the 'quotation' of a proposition. In a simpler form, this can be done by defining a two-place connective $\mathsf{Val}(v,p)$, to be taken as 'the value of p is v'. Let 'Val' be defined by the following table:

$v \backslash p$	⌝		□
⌝	⌝		
		⌝	□
□	□	□	□

A self-referring statement can then be constructed, asserting its own value as in

$$p = \mathrm{Val}(\ \ ,p)$$
$$p = \mathrm{Val}(\square,p),$$

both of which are autonomous. This is, of course, not surprising since indeed both statements refer to themselves, and thus are self-indicatory. No "paradoxical" results are produced through the use of this metaliguistic connective. However, it is highly unpalatable to have a statement

$$q = \mathrm{Val}(\square,p)$$

to be autonomous, since if p is not identical to q, q is not asserting anything about itself, but merely the fact that another statement is self-referring. This clearly points to the weakness of approaching the problem of recovering self-referential forms in language from the point of view of truth-tables. Several modern logicians, for this very reason, have tried to find other ways of dealing with self-referential statements, notably van Frasser [13] and Skyrms [14]; these new attempts, to be sure, imply a departure from the customary form of valuation and/or substitution. I will not discuss these results here. The purpose of this paper was only to present the basic values of **ECI** as interpreted for logic, leaving aside higher degree or re-entering expressions. Consequently let me only suggest that the access in $\mathcal{L}(\mathbf{E})$ to an underlying calculus where these expressions can be accomodated, might be an alternative and powerful way of circumventing the pitfalls of truth-tables and of dealing with self-reference more directly. I will expand on this at length in a forthcoming work.

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