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A Note on the Axiomatization of Equational Classes of n-Valued Łukasiewicz Algebras

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Abstract It is shown that each nontrivial equational subclass of n-valued Łukasiewicz algebras is determined by a canonical equation that has the least possible number of variables.

By means of Jónsson's lemma on congruence distributive varieties it is possible to prove that Λ_n, the lattice of equational subclasses of n-valued Łukasiewicz algebras, is isomorphic to the free bounded distributive lattice with $[(n-1)/2]$ free generators $F_D([(n-1)/2])$, where $[x]$ denotes the integral part of the number x.

The aim of this note is to give a canonical way of writing, for each nontrivial element in Λ_n, an equation which determines it and which has the smallest possible number of variables. The result cited above then follows immediately.

We assume that the reader is familiar with the theory of n-valued Łukasiewicz algebras as it is given in [1] or [2]. (Note that our operations D_i correspond to the s_{n-i} of [2], $1 \leq i \leq n-1$.)

We begin with some notation. The least element of Λ_n (namely, the trivial equational class) will be denoted by $\mathbf{T}$ and the greatest element of Λ_n (namely, the equational class of all n-valued Łukasiewicz algebras) by $\mathbf{L}_n$. The n-element chain $0 < 1/(n-1) < \ldots < (n-2)/(n-1) < 1$, with the natural lattice structure and the operations $\sim$ and D_i, $1 \leq i \leq n-1$, $n \geq 2$, defined as $\sim(j/(n-1)) = (n-1-j)/(n-1)$ and $\mathrm{D}_i(j/(n-1)) = 1$ if $i \leq j$ and $\mathrm{D}_i(j/(n-1)) = 0$ if $j < i$, will be denoted by C_n. It is well-known that C_n and its subalgebras are the subdirectly irreducible algebras of $\mathbf{L}_n$, and that they are simple.

Let $S_2 = \varnothing$ and, for $n \geq 3$, let $S_n = \{1,2,\ldots,(n-2)/2\}$ when n is even and $\{1,2,\ldots,(n-1)/2\}$ when n is odd. If $J \subseteq S_n$, then

$$A_J = \{0\} \cup \{j/(n-1): j \in J\} \cup \{(n-1-j)/(n-1): j \in J\} \cup \{1\}$$

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is a subalgebra of C_n, and it is easy to check that the correspondence $J \to A_J$ establishes an isomorphism from the Boolean algebra 2^{S_n} onto the lattice of subalgebras of C_n (cf. [2], Lemma 5.9). In particular, $A_\varnothing = \{0,1\}$ and $A_{S_n} = C_n$.

We introduce in all $A \in \mathbf{L}_n$ n unary operators $H_0, H_1, \ldots, H_{n-1}$ by the following formulas: $H_0(x) = D_1(x)$, $H_{n-1}(x) = \sim D_{n-1}(x)$, and $H_i(x) = \sim D_i(x) \vee D_{i+1}(x)$ for $0 < i < n-1$. Note that in the algebra C_n we have that $H_i(j/(n-1)) = 0$ when $i = j$ and $H_i(j/(n-1)) = 1$ when $i \neq j$, for all $0 \leq i, j \leq n-1$.

For each $J \subseteq S_n$, $J \neq \varnothing$, choose a bijection $'$ from J onto $\{0,1,\ldots,|J|-1\}$ and let (E_J) be the following equation in $|J|$ variables:

$$(E_J) \quad \bigvee_{i \in J} H_i(x_{i'}) = 1.$$

Our main tool for proving the results announced above is the following:

Lemma 1 *Let $\varnothing \neq J \subseteq S_n$. Then the equation (E_J) fails in the algebra A_J but it holds in all algebras A_I with $J \nsubseteq I$.*

Proof: For each $i \in J$, let $x_{i'} = i/(n-1)$. Then $\bigvee_{i \in J} H_i(i/(n-1)) = 0$ and (E_J) fails in A_J. Suppose now that $J \nsubseteq I \subseteq S_n$, and take $j_0 \in J \setminus I$. Since $H_{j_0}(x_{j_0'}) = 1$ for each value of $x_{j_0'}$ in A_I, and $\bigvee_{i \in J} H_i(x_{i'}) \geq H_{j_0}(x_{j_0'})$, we have that (E_J) holds in A_I.

Lemma 2 *For each $\mathbf{V} \in \Lambda_n \setminus \{\mathbf{T}, \mathbf{L}_n\}$ let $S'(\mathbf{V}) = \{J \subseteq S_n : A_J \notin \mathbf{V}\}$, and let $S(\mathbf{V})$ denote the set of all minimal sets (with respect to inclusion) belonging to $S'(\mathbf{V})$. Then the equation:*

$$(E_{\mathbf{V}}) \quad \bigwedge_{J \in S(\mathbf{V})} \bigvee_{i \in J} H_i(x_{i'}) = 1$$

determines the equational class $\mathbf{V}$.

Proof: Suppose that $A_I \in \mathbf{V}$. If $J \in S'(\mathbf{V})$ then $J \nsubseteq I$ and, by Lemma 1, (E_J) holds in A_I. Therefore (E_J) holds in A_I for each $J \in S(\mathbf{V})$ and, hence, $(E_{\mathbf{V}})$ also holds in A_I. Conversely, suppose that $A_I \notin \mathbf{V}$. Then $I \in S'(\mathbf{V})$ and there is a $J_0 \in S(\mathbf{V})$ such that $J_0 \subseteq I$. By Lemma 1, (E_{J_0}) fails in $A_{J_0} \subseteq A_I$. Therefore, there is an evaluation of $x_{i'}$, $i \in J_0$ in A_I, such that $\bigvee_{i \in J_0} H_i(x_{i'}) = 0$, and since

$$\bigwedge_{J \in S(\mathbf{V})} \bigvee_{i \in J} H_i(x_{i'}) \leq \bigvee_{i \in J_0} H_i(x_{i'}),$$

we have that $(E_{\mathbf{V}})$ fails in A_I. Since the subdirectly irreducibles in $\mathbf{L}_n$ are the algebras A_I, the equational class $\mathbf{V}$ is determined by the equation $(E_{\mathbf{V}})$.

Let $\mathrm{Ant}(S_n)$ be the set of all antichains formed by subsets of S_n, ordered by the following relation: If R, Q are in $\mathrm{Ant}(S_n)$, then $R \leq Q$ if and only if, for each $I \in Q$, there is a $J \in R$ such that $J \subseteq I$. Note that, under this relation, $\{\varnothing\}$ is the least element of $\mathrm{Ant}(S_n)$ and $\varnothing$ is the greatest.

Put $\mathrm{Ant}^*(S_n) = \mathrm{Ant}(S_n)\setminus\{\{\varnothing\},\varnothing\}$ and $\Lambda_n^* = \Lambda_n\setminus\{\mathbf{T},\mathbf{L}_n\}$. For each $R \in \mathrm{Ant}^*(S_n)$, let (E_R) be the equation

$$(\mathrm{E}_R) \qquad \bigwedge_{J\in R} \bigvee_{i\in J} \mathrm{H} = {}_{\mathrm{i}}(\mathrm{x}_{\mathrm{i}'}) = 1$$

and let $\mathbf{V}_R$ be the equational subclass of $\mathbf{L}_n$ determined by the equation (E_R). Then we have:

Theorem 1 *The correspondence $R \to \mathbf{V}_R$ establishes an order isomorphism from* $\mathrm{Ant}^*(S_n)$ *onto* Λ_n^*.

Proof: Since $S(\mathbf{V}) \in \mathrm{Ant}^*(S_n)$ for each $\mathbf{V} \in \Lambda_n^*$, it follows from Lemma 2 that $R \to \mathbf{V}_R$ is an onto mapping. To see that it is an order isomorphism, suppose first that $R \leq Q$ and take $A_K \in \mathbf{V}_R$. Since (E_R) holds in A_K, we have that all the equations (E_J), for $J \in R$, hold in A_K. Hence, by Lemma 1, $J \not\subseteq K$ for each $J \in R$. If $A_K \notin \mathbf{V}_Q$, then there would be an $I \in Q$ such that (E_I) fails in A_K, and again by Lemma 1 we would have that $I \subseteq K$. Since $R \leq Q$, there would be a $J \in R$ such that $J \subseteq I \subseteq K$, a contradiction. Hence $R \leq Q$ implies $\mathbf{V}_R \subseteq \mathbf{V}_Q$. Suppose now that $Q \not\leq R$. Then there is an $I \in Q$ such that $J \not\subseteq I$ for each $J \in R$, and, by Lemma 1, the equation (E_J) holds in A_I for each $J \in R$. Thus (E_R) holds in A_I and so $A_I \in \mathbf{V}_R$. But since, by Lemma 1, (E_I) fails in A_I, (E_Q) also fails in A_I and so $A_I \notin \mathbf{V}_Q$. Therefore $\mathbf{V}_R \subseteq \mathbf{V}_Q$ implies $R \leq Q$ and we have completed the proof.

Note that we can extend the correspondence of Theorem 1 to an order isomorphism from $\mathrm{Ant}(S_n)$ onto Λ_n by putting $\mathbf{V}_{\{\varnothing\}} = \mathbf{T}$ and $\mathbf{V}_\varnothing = \mathbf{L}_n$.

Corollary 1 *For each $n \geq 2$, Λ_n is isomorphic to* $\mathrm{F}_D([(n-1)/2])$.

Theorem 2 *Let $R \in$* $\mathrm{Ant}^*(S_n)$. *Then the equational class $\mathbf{V}_R$ cannot be determined by an equation having a smaller number of variables than those occurring in the equation* (E_R).

Proof: Let $\mathrm{m}_R = \mathrm{Max}\{|J| : J \in R\}$ and $R' = \{J \in R\colon |J| = \mathrm{m}_R\}$. Since $\{\varnothing\} \neq R \neq \varnothing$, $0 < \mathrm{m}_R \in \omega$. If $\mathrm{m}_R = 1$, then there is nothing to prove. Assume then that $\mathrm{m}_R \geq 2$. We are going to show that there is a $\mathbf{V} \in \Lambda_n$ with the following two properties: (i) $\mathbf{V}_R \subset \mathbf{V}$ and (ii) if $|I| < \mathrm{m}_R$, then $A_I \in \mathbf{V}$ if and only if $A_I \in \mathbf{V}_R$. Property (ii) implies that the varieties $\mathbf{V}$ and $\mathbf{V}_R$ have the same subdirectly irreducibles generated by less than m_R elements, and hence that the free algebras with $\mathrm{m}_R - 1$ generators in $\mathbf{V}$ and $\mathbf{V}_R$ are isomorphic. But this together with property (i) imply that no equation with less than m_R variables can characterize $\mathbf{V}_R$. To find a $\mathbf{V}$ satisfying conditions (i) and (ii) we consider the following cases:

Case 1: $R \neq R'$. Let $Q = R\setminus R'$. Since $R \leq Q$ and $R \not\geq Q$, by Theorem 1 we have that $\mathbf{V}_R \subset \mathbf{V}_Q$. If $I \subseteq S_n$ and $|I| < \mathrm{m}_R$, then $J \not\subseteq I$ for each $J \in R'$ and it follows from Lemma 1 that all equations (E_J), for $J \in R'$, hold in A_I. Consequently (E_Q) holds in A_I if and only if (E_R) holds, and we can take $\mathbf{V} = \mathbf{V}_Q$.

Case 2: $R = R'$ and $\mathrm{m}_R = |S_n|$. In this case we must have that $R = \{S_n\}$. Hence, by Lemma 1, $A_I \in \mathbf{V}_R$ for each $I \subset S_n$ and $A_{S_n} = \mathrm{C}_n \notin \mathbf{V}_R$. Then we can take $\mathbf{V} = \mathbf{L}_n$.

Case 3: $R = R'$ and $\mathrm{m}_R < |S_n|$. Take $J_0 \in R$, $k \in S_n \backslash J_0$, and let $Q = \{J_0 \cup \{k\}\}$. Since $Q \in \mathrm{Ant}^*(S_n)$, $R \leq Q$, and $R \not\geq Q$, we have that $\mathbf{V}_R \subset \mathbf{V}_Q$. Let $I \subseteq S_n$, $|I| < \mathrm{m}_R$ (recall we are considering $\mathrm{m}_R \geq 2$). Since $J \not\subseteq I$ for each $J \in R$, *a fortiori* $J_0 \cup \{k\} \not\subseteq I$, so we have again by Lemma 1 that both (E_R) and (E_Q) hold in A_I. Hence we can take $\mathbf{V} = \mathbf{V}_Q$.

REFERENCES

[1] Balbes, R. and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Missouri, 1974.

[2] Cignoli, R., *Moisil Algebras*, Notas de Lógica Matemática 27, Instituto de Matemática, Universidad Nacional del Sur, Bahia Blanca, 1970.

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