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Russell, Presupposition, and the Vicious-Circle Principle

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Abstract Prompted by Poincaré, Russell put forward his celebrated vicious-circle principle (vcp) as the solution to the modern paradoxes. Ramsey, Gödel, and Quine, among others, have raised two salient objections against Russell's vcp. First, Gödel has claimed that Russell's various renderings of the vcp really express distinct principles and thus, distinct solutions to the paradoxes, a claim that gainsays one of Russell's positions on the nature of the solution to the paradoxes, namely, that such a solution be uniform. Secondly, Ramsey, Gödel, and Quine have protested that the vcp's proscription against impredicative specification is incompatible with a realistic conception of the domain of quantification, a conception that Russell certainly held. I examine Russell's vcp and defend it against these objections.

1 Introduction As a result of his exposure to the work of Peano and Moore, Russell produced his *Principles of Mathematics, Volume 1* [31]. In this volume, he put forward the beginnings of a logical theory whose precise details he intended to explain in a companion Volume 2. Before Russell could start to work on this companion volume, however, he discovered his class paradox and several logicians discovered other paradoxes soon afterward. Accordingly, he postponed the work on the companion volume and devoted himself instead to finding the solution to the paradoxes. After much effort, in 1906 he came upon what he took to be their solution, namely, his celebrated *vicious-circle principle*. Shortly afterward, Russell began work on the intended companion volume and, as is well known, this work culminated in the publication of *Principia Mathematica* [37]. It is noteworthy that the logical theory of this later volume—that is, what Russell called *the theory of types*—may be understood as the result of modifying the logical theory implicit in *Principles* in order to abide by the vicious-circle principle. In what follows, I shall examine this principle in detail. The examination is divided into three sections. Section 2 provides some historical background. Section 3 looks at two salient renderings of the vicious-circle principle

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as well as Russell's understanding of the relation of presupposition. And Section 4 considers two objections raised by Ramsey, Gödel, and Quine concerning these renderings.

2 Historical background In 1900, Russell put forward a very type-free characterization of logic, namely, the logical theory implicit in his *Principles of Mathematics*. For convenience, call this theory LP. As alluded to above, in 1901, he derived his class paradox within LP and other logicians derived several other paradoxes soon afterward. In the face of these paradoxes, Russell adopted two salient positions on the nature of their solution. First, the paradoxes arise from some mischaracterization of logic, and so their solution must consist in LP's reform. Secondly, all the paradoxes arise from the same error, and so their solution must be unitary. In this respect, the error must be a single mischaracterization of LP.[1]

As is well known, Russell entertained several possible solutions to the paradoxes before he arrived at his considered solution. Here we shall look at four. Russell's first solution, rather surprisingly, appears in an appendix to *Principles* under the rubric the *Doctrine of Types*: it is a version of simple type theory and, accordingly, it is a logical theory itself, different in spirit as well as in detail from LP. Qua simple type theory, it avoids giving rise to the set-theoretic paradoxes by failing to countenance structures like '$x \in x$' as well formed. Although it resembles standard versions of simple type theory such as that presented by Tarski in his "Wahrheitsbegriff" [36]—at least to the extent that it contains a hierarchy of types consisting of a bottom type of individuals and then a type of classes of individuals, and then a type of classes of these, and so on—the Doctrine of Types differs from these versions in several respects. Most notably, unlike these others, it distinguishes between the range of significance of, say, a monadic propositional function[2] and the type of an item that may fall under it. By its lights, such a range is a superset of such a type. The Doctrine of Types differs further in claiming that all the ranges of significance form a type, the numbers form a type, and the propositions form a type, where each of these types lies outside the hierarchy mentioned above. Because the propositions form a type, the Doctrine of Types is inconsistent, (see [31], §500) and for this reason alone, Russell rejected it as a viable solution.[3]

Russell offers the three other solutions in his "On some difficulties in the theory of transfinite numbers and order types" [33]. These solutions all concern LP's comprehension axioms. Many recognized early on that such axioms are appealed to in order to derive any of the paradoxes.[4] For this reason, in 1905, Russell suspects that they are responsible for the paradoxes. More precisely, since he in some sense takes such axioms as a required part of any correct formalization of logic, he suspects that they must be mischaracterized in some way and that such a mischaracterization is responsible for the paradoxes. If this is the case, then in what does such a mischaracterization consist? Consider how LP characterizes the comprehension axioms. Since it is very type-free, LP characterizes them as unrestricted and thus they all have the form

$$\ulcorner \exists f \forall v_1, \ldots, \forall v_n [f(v_1, \ldots, v_n) \longleftrightarrow \varphi] \urcorner$$

where 'φ' represents any formula not containing the variable f free but possibly containing other variables free among which of course may be any or all of $v_1, \ldots, v_n$.

Call such a formula an *s-formula* ('*s*' for specifying). In this regard, in 1905, Russell specifically suspects that the mischaracterization of the comprehension axioms consists in LP's countenancing all possible formulas—satisfying the constraint of not containing the variable f free—as *s*-formulas when only a very restricted collection of such formulas may be so countenanced.[5] To the extent that this suspicion is correct, one must determine which of the possible formulas are to be countenanced as *s*-formulas in order to arrive at the proper characterization of the comprehension axioms. Toward this end, Russell puts forward in [33] three possible theories that make such a determination. (I should note that rather than carrying out his discussion of these matters in [33] at the metalinguistic level, Russell, as always, does so at the object level. Thus, rather than asking which formulas are to be countenanced as *s*-formulas, he asks which propositional functions determine classes or whatever else are taken to be comprehended by the axioms in question. He calls the propositional functions that do determine these items 'predicative'. This is a word that acquires at least two other usages later on.)

The first of the three theories is the *zigzag theory*. Briefly, according to this theory, a formula is to be countenanced as an *s*-formula if and only if such a formula is *fairly simple*. This particular determination is motivated by the circumstance that appeals to comprehension axioms whose *s*-formulas are rather complicated or recondite such as '$\neg x \in x$' engenders paradox whereas appeal to those axioms whose *s*-formulas are in some sense simple never does. Of the three theories put forward in [33], the zigzag theory is the least worked out. Russell failed to arrive at anything like an explanation of the conditions for a formula's being "fairly simple." Indeed, no one since has succeeded in arriving at such an explanation. Some suggest that Quine is the one who has come the closest to doing so by means of his set theory *New Foundations* (see, for instance, Gödel [9], p. 125, and Quine [26]).

The second of the three theories is the *theory of the limitation of size*. According to this theory, a formula is to be countenanced as an *s*-formula if and only if the item such a formula specifies is not *too large*. This particular determination is motivated by the circumstance that the set-theoretic paradoxes all make reference to very large classes or "very large" propositional functions such as the class of all classes, the class of all ordinals, the class of all non-self-membered classes, and the propositional function that applies to all and only non-self-satisfying propositional functions. Unfortunately, Russell did not succeed in arriving at a clear explanation of the conditions for a formula's not specifying an item that is too large. Interestingly, some have understood Zermelo-Fraenkel set theory as providing such an explanation.[6] By their lights, its axioms of Aussonderung, pairing, power-set, union, and replacement determine collectively which formulas are to be countenanced as *s*-formulas in such a way that, given that certain sets are taken to exist, these axioms affirm the existence of other sets whose size is, roughly speaking, not very much larger than that of these given sets.

The third theory is the *substitutional theory*. According to this theory, *no* formula is to be countenanced as an *s*-formula. Clearly, this theory is the most revisionary of the three. Unlike LP, it does not countenance propositional functions or classes. In their stead, it countenances a primitive operation of substitution S which is such that given any items c and d and a proposition p containing c as constituent, $S(p, d, c)$

is the proposition that results from substituting d for c in p. The motivation behind this theory is simply that all of the paradoxes make reference to propositional functions or classes. After expending much effort to work out the theory, Russell eventually abandoned it for two reasons: the theory is technically very cumbersome and it leads to contradictions of its own (see Landini [16], [17], and [18] and Pelham and Urquhart [23]).

3 Vicious circularity In response to Russell's [33], Poincaré put forward his own account of which formulas are to be countenanced as s-formulas in his "Les mathématiques et la logique" [24]. According to this account, a formula is to be countenanced as an s-formula if and only if it is not what he designates *viciously circular*. Poincaré put forward this account because he took there to be some sort of pernicious circularity on a par with definitional circularity implicitly involved in the arguments to the paradoxes.[7] The circularity in question might be taken as the circularity pointed up by Russell in his discussion of *self-reproductive processes* at the beginning of [33].[8] In this light, it is not surprising that later, in his "On 'Insolubilia' and their solution by symbolic logic" [34], Russell accedes to Poincaré's account.

At this point, even if one has some understanding of the circularity that both Poincaré and Russell point up, one may be prompted to ask for an explanation of the conditions for a formula's being viciously circular. Russell to a certain extent offers an answer to this question when he states his vicious-circle principle. I employ the qualifier "to a certain extent" because such an explanation should, strictly speaking, speak at the metalinguistic-level whereas Russell states the principle at the object-level. The principle may, however, be construed metalinguistically.

Interestingly, Russell provides more than one rendering of the vicious-circle principle: at least ten appear in the corpus. Here are six of them.

> Whatever involves an apparent [i.e., bound] variable must not be among the values of that variable. ([34], p. 198)
>
> Whatever involves *all* of a collection must not be one of the collection. ([35], p. 63)
>
> Whatever contains an apparent variable must not be a possible value of that variable. (ibid., p. 75)
>
> all of [the paradoxes] arise from the fact that an expression referring to *all* of some collection may itself appear to denote one of the collection; (ibid., p. 101)
>
> given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. ([37], p. 37)
>
> If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total. (ibid., also Russell [35], p. 63)

Some have remarked that these renderings do not express a single principle. In [9], Gödel claims that "corresponding to the phrases 'definable only in terms of', 'involving', and 'presupposing', we have really three different principles . . . " (p. 127). Others have remarked that none of the renderings are particularly clear.

In order to arrive at a clear understanding of Russell's answer to the question about the conditions for a formula's being viciously circular, one must examine closely his vicious-circle principle. I do so by examining the last two of the above renderings: the rendering that employs ‘presuppose’ and the rendering that employs ‘definable’. For convenience, call these *VCP1* and *VCP2*. The examination will show that, although there is a sense in which Gödel’s claim is correct, it is not very significant.[9]

Two points should be noted before I start. First, as Quine has argued ([27], p. 242), definition is best understood as what occurs when a new notation is introduced as short for an old one. Thus, to the extent that VCP2 really concerns comprehension axioms, its employment of “definable” is misleading. That is, for any given comprehension axiom *c*, *c*’s *s*-formula is not a definiens, and so, *c* does not define anything. Rather, it affirms the existence of some item *i* which satisfies a certain formula—that is, the formula following *c*’s outer existential quantifier. Since it is usually the case that there can be only one such item, *c* may also be understood to *specify* the item *i*. In this respect, I shall talk about specification rather than definition in the following.

Secondly, although it may not yet be clear in what a formula’s being viciously circular consists, it should be rather clear what such circularity is not. In particular, as Quine has made clear, definition is not really at issue; rather, specification is. As such, the circularity in question is not the circularity of “smuggling the definiendum into the definiens” (ibid.). Also, the arguments to the paradoxes do not, on the face of it, commit *petitio principii*, and so it is not the circularity of smuggling a conclusion among premises. I should remark that because of these circumstances, Quine concluded that, to the extent that Russell and Poincaré thought that they had observed some circularity responsible for the paradoxes, they were confused (ibid., pp. 242–43). Quine, of course, did not attend to the details of LP and as a result, his assessment is not altogether fair—as we shall see.

3.1 VCP1 Recall Russell’s expression of VCP1:

> given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total.

In order to become clear about what Russell intends VCP1 to say, one should consider some of the expressions that it employs. To begin with, consider ‘set’. Russell clearly intends this expression to have a meaning that is more general in its application than that which it currently has (or may be taken to have), namely, the concept of set as explained by Zermelo-Fraenkel set theory. In addition to intending ‘set’ to mean such a concept, Russell intends it to mean the concepts collection, extension of a predicate, and proper class. To this extent, perhaps the expression ‘totality’ might suit his intentions more aptly.[10] Next, consider ‘member’. Russell intends this expression to have a meaning that is correlatively more general in its application than that which it may currently be understood to have, namely, the concept of member of ZF set. In addition to intending ‘member’ to mean such a concept, Russell intends it to mean the concepts part, component, constituent, subset, subclass, and so on. Next, consider the expression ‘to have a total’. In Chapter 2 of the Introduction to *Principia*, Russell writes:

> By saying that a set has “no total,” we mean, primarily, that no significant statement can be made about “all its members.” Propositions . . . must be a set hav-

> ing no total. The same is true . . . of propositional functions, even when these are restricted to such as can significantly have as argument a given object *a*. ([37], p. 37)

According to Russell, if a set has a total, then a significant statement may be made that has a quantifier ranging over it. Looking at later developments in Chapter 2 and elsewhere in *Principia*, however, one can see that Russell actually means something more when he says of an item like a collection that it has a total, namely, that such an item is an object in his ontology. In other words, any such item may play the logical role of subject, and as such, the quantifier and substitution rules apply to it in the usual way. Note by way of contrast that the universe of sets of Zermelo-Fraenkel set theory has a total in the sense that the theory's statements quantifiy over the entire universe but it does not have a total in the further sense that it may play the logical role of subject.[11] Finally, consider 'presuppose'. Russell intends this expression to mean a very important metaphysical relation—that of *presupposition*. He unfortunately does not write much that is explicit in order to explain the nature of this relation, but what he does write makes it clear that wholes must presuppose their parts. Thus, collections presuppose their members and propositions presuppose their constituents:

> *A and B* presupposes *A* and presupposes *B*. . . . [The proposition that] "*A* differs from *B*" . . . presupposes *A and difference and B*. ([31], §71, p. 71)
>
> A complex unit is a *whole*; its parts are other units, whether simple or complex, which are presupposed in it.[12]

The relation of presupposition is asymmetric and thus irreflexive:

> We cannot conclude that the parts of a whole are not really its parts, nor that the parts are not presupposed in the whole in the sense in which the whole is not presupposed in the parts . . . . ([31], §138)
>
> The first objection [to the claim that the relation of whole and part is that of logical priority] is, that logical priority is not a simple relation: implication is simple, but logical priority of *A* to *B* requires not only "*B* implies *A*," but also "*A* does not imply *B*." . . . This state of things, it is true, is realized when *A* is part of *B*; but it seems necessary to regard the relation of whole to part as something simple . . . [13]
>
> We require that the relation of whole and part should be always asymmetrical, *i.e.* if *A* is part of *B*, then *B* is never part of *A*. ([29], p. 38)
>
> The relation of whole and part is itself an asymmetrical relation, and the whole . . . is distinct from all its parts, both severally and collectively. ([31], §215, p. 225)
>
> When we say that "$\varnothing x$" ambiguously denotes $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., we mean that "$\varnothing x$" means one of the objects $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., though not a definite one, but an undetermined one. It follows that "$\varnothing x$" only has a well-defined meaning . . . if the objects $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., are well-defined. That is to say, a function is not a well-defined function unless all its values are *already* well-defined. It follows from this that no function can have among its values anything which *presupposes* the function, for if it had, we could not regard the objects ambiguously denoted by the function as definite until the function was definite, while conversely, as we have seen, the function cannot be definite until its values are definite. ([37], p. 39, emphasis added)

Presupposition is also transitive:

> We may observe . . . that a constituent of a constituent is a constituent of the unity, *i.e.* this form of the relation of part to whole . . . is transitive. ([31], §141, p. 144–45)

> [Collections] could only presuppose numbers in the particular case where the terms of the collection themselves presupposed numbers. ([31], §71, p. 69)

The relation of presupposition is one of supervenience:

> the simpler is always implied in the more complex, and therefore there can be no truth about the more complex unless there is truth about the simpler. Thus in the analysis of our infinite whole, we are always dealing with entities which would not be at all unless their constituents were. ([31], §143, p. 147)

> infinite wholes would not have Being at all, unless there were innumerable simple Beings whose Being is presupposed in that of the infinite wholes. ([31], §143, p. 148)

> it is plain to begin with that every complex term presupposes the being of the simple terms which compose it. Any one of these simple terms might be, without the complex term being; but if the complex term is, then the simple terms also are. ([29], p. 36)

Russell's usage of the expression 'presuppose' in the above statements and elsewhere makes it evident that presupposition is a relation of ontological dependence in the sense, for instance, that the definiteness of wholes (e.g., propositions) may be taken ontologically to depend on the definiteness of their parts and the definiteness of sets may be taken ontologically to depend on the definiteness of their elements. To repeat, it is asymmetric, transitive, and at least as strong as supervenience—for if c ontologically depends on d, c would not exist if d did not.

Having these glosses, one may rephrase VCP1 as follows: Given any totality T such that, if we suppose T to be an object, it will contain objects that ontologically depend on it, then T cannot be an object. Or better: No totality that is an object may contain an object which ontologically depends on it. Given the notion of well-foundedness from set theory,[14] one may rephrase VCP1 still more succinctly: all totalities that may be treated as objects are well-founded. Thus understood, VCP1 implies that no wholes may be proper parts of themselves and that no sets may be members of themselves or members of . . . members of themselves.

Admittedly, VCP1 so understood enjoys a certain intuitive appeal. Indeed, early versions of it occur in Russell's *Principles*:

> in the present work, it will be maintained that there are no contradictions peculiar to the notion of infinity, and that an endless process is not to be objected to unless it arises in the analysis of the actual meaning of a proposition. ([31], §55, p. 51)

> a complex whole can never be one of its own constituents. ([31], §70)

As may be expected, many in the history of philosophy have taken VCP1 as correct. Leibniz implicitly appealed to it in one of his arguments for the existence of so-called corporeal substances.[15] Kant implicitly appealed to it in his argument to the thesis of the second antinomy (Kant [14], A434/B462–A444/B472). Russell and Poincaré apart, more recently Gödel has claimed that VCP1 is plausible (Gödel [9], p. 127). Notwithstanding the intuitive appeal that it enjoys, consistent set theories have been developed—more precisely, consistent relative to ZF's consistency—that deny VCP1 by asserting the existence of non-well-founded sets. Moreover, since the axiom of regularity is independent of ZFC, ZFC itself has nonstandard models containing non-well-founded sets. To this extent, one might perhaps deny VCP1 without contradicting oneself. Needless to say, these points would not sway the conviction of anyone who was impressed by its intuitive appeal.

Importantly, Russell takes VCP1 thus understood as incompatible with many of the type-free features of LP. Indeed, he argues from VCP1 and certain LP claims about propositions to the conclusion that no propositional function may apply to itself *with sense*. The argument goes roughly as follows (see [37], p. 39, 40). Let $\lambda x.\varnothing x$ be a monadic propositional function. By Russell's lights, "a function is not a well-defined function unless all its values are *already* well-defined" ([37], p. 39, emphasis added). Pick an arbitrary item c and suppose that the proposition $\varnothing c$ is a value of $\lambda x.\varnothing x$. Since the well-definedness of $\lambda x.\varnothing x$ depends on the *prior* well-definedness of $\varnothing c$, $\lambda x.\varnothing x$ presupposes $\varnothing c$. Since c is a constituent of $\varnothing c$, $\varnothing c$ presupposes c. By the transitivity of presupposition, $\lambda x.\varnothing x$ presupposes c. By its irreflexivity, $\lambda x.\varnothing x$ and c are distinct. At this point, Russell concludes that $\varnothing(\varnothing)$ cannot be a value of $\lambda x.\varnothing x$ and thus that $\lambda x.\varnothing x$ may not apply to itself with sense. In other words, he concludes not that the complex $\varnothing(\lambda x.\varnothing x)$ is false but rather that it is ill-formed and thus that the judgment that $\lambda x.\varnothing x$ is $\varnothing$ cannot be expressed. On the surface, the validity of his argument might appear questionable since Zermelo-Fraenkel set theory with the axiom of regularity respects VCP1 in the sense that all the sets that it talks about are well-founded[16] but this theory does not realize in any way this last conclusion that Russell takes to follow from VCP1. There is a disanalogy, however, between the situation regarding propositional functions and that regarding sets. On the one hand, to countenance a proposition of the form $\varnothing(\varnothing)$ is to countenance a non-well-founded object and thus to violate VCP1. On the other hand, to countenance a proposition of the form $\ulcorner x \in x \urcorner$ is not to countenance any non-well-founded set since such a proposition may be taken as false (as ZF+Regularity does take it). Moreover, it is not to countenance a non-well-founded proposition since, by Russell's lights, the relation of membership may presuppose without violating VCP1 a given set (or ordered pair of sets) to which it may apply. Thus, the above consideration of set theory does not vitiate Russell's argument to the conclusion that no propositional function may apply to itself with sense.

Not surprisingly, several interesting claims also incompatible with the type-free features of LP follow in turn from Russell's conclusion. I mention three. The first is that, contrary to the position of LP that for any propositional function $\lambda x.\varnothing x$ and for any term c in Russell's ontology (I use 'term' in Russell's sense, that is, as meaning object in general), $\lambda x.\varnothing x$ may be predicated with sense of c and such a predication produces either a proposition or another propositional function $\varnothing(c)$, $\lambda x.\varnothing x$ may only be predicated with sense of those terms belonging to some proper subclass of all the terms in the ontology. As is well known, Russell calls such a proper subclass a 'type'. In this respect, he says:

> whatever function $\varnothing$ may be, there will be arguments x with which $\varnothing x$ is meaningless, *i.e.* with which as arguments $\varnothing$ does not have any value. The arguments with which $\varnothing$ has values form what we will call the "range of significance" of $\varnothing x$. A "type" is defined as the range of significance of some function. ([37], p. 161)

> A type is defined as the range of significance of some propositional function, i.e., as the collection of arguments for which the said function has values. ([35], p. 75)

The second claim is that, contrary to the position of LP that for any propositional function $\lambda x.\varnothing x$, the variable x that figures in the quantified claim $(x)\varnothing x$ is completely unrestricted, such an x must be restricted to range over and only over the proper subclass or type of terms of which $\varnothing$ may be predicated with sense. Thus Russell writes:

> Whenever an apparent value occurs in a proposition, the range of values of the apparent variable is a type, the type being fixed by the function of which 'all values' are concerned. (ibid.)
>
> we can speak of *all* of a collection when and only when the collection forms part or the whole of the *range of significance* of some propositional function, the range of significance being defined as the collection of those arguments for which the function in question is significant, i.e., has a value. (ibid.)

This claim, strictly speaking, only follows from the conjunction of Russell's conclusion that no propositional function may apply to itself with sense with the additional assumption that a variable figuring in a propositional function must be restricted to range over and only over those terms of which the function may be predicated with sense. Note that not only must the variable x in $(x)\varnothing x$ be restricted to range over at most the type of terms of which $\varnothing$ may be predicated with sense but it must also range over the entirety of the type. Indeed, in [35] (§3, pp. 71–73), Russell provides an argument—albeit a problematic one—that one cannot restrict by convention the range of a variable to a proper subset of the type since "restrictions naturally express themselves as hypotheses that the variable is of such and such a kind, and that when so expressed, the resulting hypothetical is free from the intended restriction" (ibid., p. 73). The third claim is that, contrary to the LP position that there is only one kind of variable, to the extent that there are propositional functions having different types of terms of which they may be predicated with sense, there will be variables of different kinds ranging over these different types.

Given that any propositional function $\lambda x.\varnothing x$ has associated with it some type—that is, the type of terms of which it may be predicated with sense—any term c in the ontology may be understood to fall under one or more types, namely, the types associated with the propositional functions which may be predicated of c with sense. In order to avoid confusion, for a given propositional function $\lambda x.\varnothing x$, call the type of terms of which it may be predicated with sense its *argument-type*, and for a given term c, call the type or types that c may fall under simply c's *type* or *types*.[17]

At this point, one may be prompted to ask how the types are all configured. By Russell's lights, VCP1 at least partly determines the answer to this question. Not only does he take VCP1 to require that the argument-type t of a given propositional function f be restricted so as not to contain, for instance, f itself, but he takes VCP1 to require that t not contain any other propositional function g whose own argument-type contains f. More generally, Russell takes VCP1 to require that the argument-types of propositional functions be configured so that no sequence of the following kind occur: f_1 applies to f_2 with sense, f_2 applies to f_3 with sense, . . . , f_{i-1} applies to f_i with sense, and f_i applies to f_1 with sense. In other words, the types must be configured so as to realize a well-founded ordering. There are, of course, many kinds of well-founded ordering, and so one may ask which particular ordering is so realized. Unfortunately, even by Russell's lights, VCP1 does not have anything to say to this question. Also, since the circumstance that the types realize a well-founded ordering does not determine whether or not the types are cumulative, one may ask whether or not they are. That is, one may ask whether or not the types are such that, for any two types t_1 and t_2 where t_1 is greater than (according to the ordering) t_2, some of the items falling under t_2 may also fall under t_1. Here, again, VCP1 even by Russell's lights does not have anything to say.[18]

It is noteworthy that many have taken the following configuration of types as the most plausible well-founded ordering (some have even taken it as the only possible one).[19] First, consider the collection of all terms in the ontology that Russell calls *individuals* in *Principia* (and *Things in Principles*). These are the terms that do not apply to anything. Take this collection as a type and call it *level-0*. Next, consider the collection of monadic propositional functions that apply to all and only level-0 items with sense. Take this collection as a type and call it *level-1*. Next, consider the collection of monadic propositional functions that apply to all and only level-1 items with sense. Take this collection as a type and call it *level-2*. Clearly, this procedure may be iterated indefinitely. Moreover, it may be extended without difficulty to cover polyadic propositional functions and propositions. The result of such an iteration and extension is a well-founded ordering in which there figures every type of every term in the ontology. This ordering resembles the simple hierarchy of types presented by Tarski in "Wahrheitsbegriff" [36] and by Gödel in [8][20] at least to the extent that its types are noncumulative, and since there is a least type—that is, level-0—as opposed to several minimal types, the types of the monadic propositional functions are *well-ordered* (not merely well-founded).

There are two points to make about the configuration of types described above. On the one hand, the description is only a sketch. In order to provide a more precise characterization of the configuration, one would have to specify how the configuration relates to the logical system that concerns it. For instance, suppose that for some fixed y, '$\lambda x.\varnothing xy$' means some monadic propositional function f in the configuration. Although the above description indicates roughly how the type of f relates to its argument-type, it says nothing about how the type of the parameter y relates to these two types. Needless to say, in order to provide a detailed characterization of the configuration one would have to specify other important features of it as well. On the other hand, since VCP1, even by Russell's lights, already underdetermines the particular kind of well-founded ordering that the types realize as well as whether or not the types are cumulative, it a fortiori underdetermines such relatively fine features of the configuration.

We should not see VCP1's underdetermining the configuration of types as gainsaying the vicious-circle principle in general. To the extent that the function of the principle is to exclude those *s*-formulas that give rise to the paradoxes, VCP1 (and VCP2) may best be read as a condition of adequacy for any proposed logical theory. Indeed, Russell says as much:

> It is important to observe that the vicious-circle principle is not itself the solution of the vicious-circle paradoxes, but merely the result which a theory must yield if it is to afford a solution of them. It is necessary, that is to say, to

> construct a theory of expressions containing apparent [bound] variables which will yield the vicious-circle principle as an outcome. ([34], p. 205)

> The [vicious-circle] principle is, however, purely negative in its scope. It suffices to show that many theories are wrong, but it does not show how the errors are to be rectified. ([35], p. 63)

Now that we have some understanding of VCP1, we should return to the question posed at the beginning of this section about an explanation of the conditions for a formula's being viciously circular (and hence, failing to be an *s*-formula). In the light of what we have seen, we may expect the explanation to go roughly as follows: a formula is viciously circular if and only if it attempts to express a predication that, according to VCP1 and the claims that Russell takes to follow from it, cannot be expressed. In this respect, formulas of the following kind whose variables are not restricted to types that form a well-founded ordering are viciously circular: $x(x), x(y) \wedge y(x), x(y) \wedge y(z) \wedge z(x), x \in x$.

It is noteworthy that, in this light, the comprehension axioms that are appealed to in the arguments to the set-theoretic paradoxes all involve *s*-formulas that are viciously circular. To the extent that no viciously circular formula is to be countenanced as a legitimate *s*-formula, the deduction of these paradoxes, at least by means of the standard arguments, is obstructed. To this extent, one may say that violation of VCP1 gives rise to such paradoxes, where such violation may be taken to consist in employing viciously circular formulas as *s*-formulas.[21]

3.2 VCP2 Recall Russell's statement of VCP2.

> If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.[22]

As with VCP1, in order to become clear about what Russell intends VCP2 to say, one should consider some of the expressions that it employs. To begin with, consider its expressions 'collection', 'member', and 'had a total'. Clearly, these may be understood in terms of the glosses at which we arrived when we looked at VCP1. Next, consider 'definable'. As we saw above, it is best to read this expression as meaning *specifiable*. Finally, consider 'in terms of'. Russell's application of VCP2 makes it clear that he intends this to mean something like *by quantifying over*. In this respect, one may rephrase VCP2 as follows: No totality T may contain an object that is specifiable only by quantifying over T.

VCP2 thus understood recommends rather clearly its own explanation for a formula's being viciously circular (and hence, failing to be an *s*-formula), namely, that a formula F is viciously circular when and only when, if F were to be countenanced as an *s*-formula, it would contain at least one quantified variable whose range contained the term whose existence would be affirmed by the comprehension axiom having F as its *s*-formula. (Again, I use 'term' here in Russell's sense, that is, as meaning object in general). To illustrate this explanation, consider the following familiar comprehension axiom from standard second-order logic which affirms the existence of the property of being a natural number ('0' denotes zero and '*s*' denotes the successor function).

$$\exists N \forall z [N(z) \longleftrightarrow \forall G\{(G(0) \wedge \forall u(G(u) \rightarrow G(s(u)))) \rightarrow G(z)\}]$$

The axiom's *s*-formula contains the quantified variable G that ranges over the term $\lambda x.Nx$ whose existence the axiom affirms. As such, by VCP2's lights, the axiom's *s*-formula is viciously circular and so is not to be countenanced as a legitimate *s*-formula.[23, 24]

Not surprisingly, VCP2 thus understood is inconsonant with many of the type-free features of LP. It is inconsonant in at least two respects. First, to the extent that there are *s*-formulas containing quantified variables, VCP2 requires that there be some restricted variables since, by its lights, only restricted variables may occur bound in *s*-formulas. Again, call the range of any of such restricted variables a type. Secondly, if for every *s*-formula containing a bound variable, there is another *s*-formula containing a bound variable that ranges over the item the first *s*-formula specifies, then, since the second bound variable must have a type different from that of the first, there must be an infinite number of restricted variables whose types are different one from the other. Note that this claim does not imply that every variable must be restricted—a situation different from that associated with VCP1.

Given that VCP2 requires types of its own, one may be prompted to ask how such types are all configured. VCP2 partly determines the answer to this question. For not only does VCP2 require that there be no specification of a term t_1 via some formula F containing quantified variables $v_1, \ldots, v_i$, some of whose types contain t_1, but it requires that there be no specification of a term t_1 via some formula F containing quantified variables $v_1, \ldots, v_i$, some of whose types contain a term t_2, which is in turn specified via a formula F' containing quantified variables $w_1, \ldots, w_j$, some of whose types contain t_1. More generally, VCP2 requires that its types be configured so that no sequence of the following kind occur: t_1 is specified 'in terms of' the quantified variables $v_1, \ldots, v_i$, some of whose types contain t_2, which is specified 'in terms of' the quantified variables $w_1, \ldots, w_j$, some of whose types contain $t_3, \ldots t_n$, which is specified 'in terms of' the quantified variables $z_1, \ldots, z_k$, some of whose types contain t_1. In other words, VCP2 requires that its types be configured so as to realize a well-founded ordering.[25] There are, of course, many kinds of well-founded ordering, and so one may ask which particular ordering is so realized. Unfortunately, as in the case of VCP1, VCP2 does not have anything to say in answer to this question. Also, since the circumstance that the types realize a well-founded ordering does not determine whether or not the types are cumulative, one may ask whether or not they are. Here again, VCP2 does not have anything to say.

Interestingly, many have taken the following configuration of types as the most plausible well-founded ordering. First, consider the collection of all terms in the ontology that Russell calls *individuals* in *Principia* (and *Things in Principles*). Take this collection as a type and call it *order-0*. Next, consider the collection of propositional functions that apply to all and only order-0 items with sense and whose existence need not be affirmed by comprehension axioms whose *s*-formulas contain quantified variables ranging over items not falling under order-0. The propositional functions of this collection must exist if any do. The collection is closed under the operation of specifying a propositional function by means of a comprehension axiom whose *s*-formula possibly contains bound variables ranging over order-0, free parameter variables ranging over the collection, but no bound variables ranging over it. Note that since such an *s*-formula contains no bound variables ranging over the collection, VCP2 does not require that a propositional function specified by means of it fall under a type different from that of any of its possible free parameter variables. Take the collection in question as a type and call it *order-1*. Next, consider the collection of propositional functions that apply either to all and only order-1 items with sense or to all and only order-0 items with sense or to both, and whose existence need not be affirmed by comprehension axioms whose *s*-formulas contain bound variables ranging over items not falling under order-0 or under order-1. This collection is closed under the operation of specifying a propositional function by means of a comprehension axiom whose *s*-formula possibly contains bound variables ranging over order-0 or order-1, free parameter variables ranging over the collection, but no bound variables ranging over it. Note that since such an *s*-formula contains no bound variables ranging over the collection, VCP2 does not require that a propositional function specified by means of it fall under a type different from that of any of its possible free parameter variables. Take the collection in question as a type and call it *order-2*. Clearly, this procedure may be iterated indefinitely. Moreover, it may be extended without difficulty to cover propositions. In this respect, a proposition or a propositional function of order-$i+1$, roughly speaking, may be specified by means of a comprehension axiom whose *s*-formula possibly contains bound variables of orders i or less but contains no bound variables of higher orders. The result of such an iteration and extension is a well-founded ordering in which there figures every type of every term in the ontology. The types are noncumulative and since there is a least type—that is, order-0—as opposed to several minimal types, the types of the propositional functions are *well-ordered*.[26]

Before I move on to discuss problems, I should note that the comprehension axioms that are appealed to in the arguments to the semantic paradoxes all involve *s*-formulas that are viciously circular in the sense of the explanation recommended by VCP2. For instance, the argument to the liar paradox requires the existence of a proposition that says of itself that it is false. This is affirmed by the following comprehension axiom:

$$\exists q(q \longleftrightarrow \exists p(\psi(p) \wedge \neg p))$$

where ψ is some propositional function true of q and only q. Here the quantified variable 'p' ranges over the proposition q and so the *s*-formula is viciously circular. In this respect, to the extent that no viciously circular formula is to be countenanced as a legitimate *s*-formula, the deduction of the semantic paradoxes, at least by means of the standard arguments, is obstructed. To this extent, one may say that violation of VCP2 gives rise to such paradoxes where such violation may be taken to consist in employing viciously circular formulas as *s*-formulas.

4 Problems I now consider two problems concerning VCP1 and VCP2. To provide a little context, I should summarize some of the above discussion. Around 1905–1906 Russell concluded, roughly speaking, that the modern paradoxes arise from a mischaracterization of the comprehension axioms and that this mischaracterization consists in its countenancing all possible formulas as *s*-formulas when only a very restricted collection of such formulas may be so countenanced. Following Poincaré, Russell took this collection to consist of those formulas that are not *viciously circular* and stated his vicious-circle principle in an effort to provide an explanation of the conditions for a formula's being viciously circular. He stated the principle in several renderings, and in order to arrive at a clear understanding of the explanation that he intended the principle to provide, I have examined two of these renderings, VCP1 and VCP2.

Now, the first problem: In the light of the examination of VCP1 and VCP2, we have seen that they offer different explanations of the conditions for a formula's being viciously circular. Roughly speaking, by VCP1's lights, a formula is viciously circular if and only if it is a formula of the following kind whose variables are not restricted to types that form a well-founded ordering:

$$x(x),\, x(y) \wedge y(x),\, x(y) \wedge y(z) \wedge z(x),\, x \in x.$$

By VCP2's lights, a formula F is viciously circular when and only when, if F were to be countenanced as an *s*-formula, it would contain at least one quantified variable whose range contained the term whose existence would be affirmed by the comprehension axiom having F as its *s*-formula. Thus, strictly speaking, Gödel's claim that Russell's various renderings of the vicious-circle principle express different principles is correct. Perhaps, one may construe VCP1 and VCP2 as offering explanations that are mutually compatible in the sense that each explanation only states a possible condition for a formula's being viciously circular and thus a possible condition for a formula's failing to be an *s*-formula. To this extent, one would have to construe VCP1 and VCP2 each as indicating a source of the modern paradoxes. This construal, however, appears incompatible with one of Russell's positions on the nature of the solution to the paradoxes, namely, that all the paradoxes arise from the same error and so their solution must be unitary. Although such an incompatibility may be resolved by countenancing a rather disjunctive notion of error, we shall see a more satisfactory resolution below.

The second problem: Whereas many are impressed by VCP1's intuitive appeal—as we saw above—many find it difficult to accept VCP2. Indeed, Ramsey, Gödel, and Quine argue against it. Thus, Ramsey writes:

> it will be objected . . . you cannot include $\lambda x.Fx = \lambda x.(\varnothing).\, f(\varnothing, x)$ among the $\varnothing$'s, for it presupposes the totality of the $\varnothing$'s. This is not, however, really a vicious circle. . . . to express [$\lambda x.Fx$] like this (which is the only way we can) is merely to describe it in a certain way, by reference to the totality of which it may be itself a member, just as we may refer to the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle. ([28], p. 192)

Gödel makes a similar remark in [9], pp. 127–28. In [27], Quine writes:

> For we are not to view classes literally as created through being specified—hence as dated one by one, and as increasing in number with the passage of time. . . . The doctrine of classes is rather that they are there from the start. . . . It is reasonable to single out a desired class by citing any trait of it, even though we chance thereby to quantify over it along with everything else in the universe. (p. 243)

By the lights of Ramsey, Gödel, and Quine, if propositions and propositional functions are construed *realistically*—that is, if they are construed as existing in Russell's ontology (i.e., his realm of being) independently of our activities—then they may be specified by means of *s*-formulas containing quantified variables whose ranges contain them. In other words, by their lights, VCP2 is incompatible with a realistic construal of propositions and propositional functions.

To illustrate the point, Quine considers the description 'the most typical Yale man', which echoes Ramsey's original 'the tallest in a group'. (ibid.) The logically explicit expression of this description involves quantified variables that range over all Yale scores including those of the person it specifies and as such it violates VCP2. However, to the extent that the relevant domain of quantification may be construed realistically, such a description is clearly innocuous.

Because Russell put forward VCP2, Ramsey, Gödel, and Quine conclude that Russell construed propositions and propositional functions *constructivistically*. That is, they conclude that he construed them as items that are in some sense "constructed" by us. In this respect, perhaps they are brought into being when we affirm their existence by means of comprehension axioms. If so, then of course we may not use *s*-formulas containing quantified variables whose ranges contain such items whose existence we are affirming.

Quine makes it clear that any attempt to give a coherent explanation of a constructivist construal of propositions and propositional functions would likely appeal to temporal notions and as such it would doubtfully succeed. For consider the difficulties encountered if, in order to affirm the existence of a propositional function by means of a comprehension axiom, one had to employ an *s*-formula containing quantified variables whose ranges contained only terms whose existence had *already* been affirmed.

Careful reading makes it clear, however, that Russell did not construe propositions and propositional functions constructivistically. On his conception, propositions and propositional functions are terms in the realm of being which are such that each has the nature that it does independently of the circumstances of every other term in the realm of being (see Goldfarb [10] and Hylton [12]). To this extent, Ramsey, Gödel, and Quine draw a false conclusion. However, their discussion still presents us with the following problem: to explain how VCP2 may be compatible with a realistic construal of propositions and propositional functions.

Interestingly, the two problems described above may be resolved by attending to the details of LP. Consider the second problem first. Ramsey, Gödel, and Quine may be taken to argue successfully that, on (what one may call for lack of a better nomenclature) the standard account of comprehension,[27] VCP2 *is* incompatible with a realistic construal of a given domain of quantification. Although LP's account of comprehension approaches very closely the standard account, it differs from it in two salient respects. By appealing to these two respects, one may explain how VCP2 may be compatible with a realistic construal of propositions and propositional functions.

The first respect in which LP's account of comprehension differs from the standard account is stated by the following claim.

Claim 4.1 *On an abstract level, for any proposition or propositional function specified by means of an s-formula, the form of such a specified term is the same as the form of the s-formula at least in the very weak sense that, if* t *is a segmentable item that figures in the s-formula, then there will be a term in the realm of being that is the meaning of* t *which figures in the specified term.*

By contrast, according to the standard account, for any item specified by means of an *s*-formula, there is no interesting sense in which the form of the item, to the extent that it may be taken to have a form, is the same as the form of the *s*-formula. Goldfarb may be read as pointing up this contrast in [10]:

> the comprehension axioms for propositions and propositional functions that are implicit in the system involve not so much the specification of these entities as the presentation of them. One is not characterizing a proposition or propositional function: one is giving it. ([10], p. 32; see also Hylton [13], p. 190)

The contrast may be highlighted by considering Zermelo-Fraenkel set theory. No one would claim that, for any set specified by means of an *s*-formula, the form or structure of the set is the same as that of the *s*-formula. At best, one would claim that the *s*-formula picks out all of the set's members and that these alone determine its form or structure. The claim that, by LP's lights, a proposition/propositional function and the respective *s*-formula that specifies it in a weak sense have the same form can be seen to follow from two features of LP. First, roughly speaking, although LP's implicit comprehension axioms for propositions and propositional functions may be taken as quantified biconditionals having the respective forms:

$$\exists p(p \longleftrightarrow A)$$

$$\exists f \forall v_1, \ldots, \forall v_n(f(v_1, \ldots, v_n) \longleftrightarrow B),$$

where 'A' represents any formula and 'B' represents any formula containing no free occurrence of f, these axioms may alternatively be taken as quantified identities having the respective forms:

$$\exists p(p = [A])$$

$$\exists f(f = \lambda v_1 \ldots \lambda v_n.B).$$

In this respect, 'p' and the formula represented by '$[A]$' are singular terms meaning the same term in the ontology. The same holds for f and the abstraction expression obtained by prefixing the formula represented by 'B' with the appropriate abstraction operators. That the comprehension axioms may be taken in this alternative way follows from the ontological commitments implicit in LP's operation of propositional function abstraction.[28]

Secondly, according to Russell, the formal language of the logical theory perspicuously mirrors the realm of being at least in the weak sense that, for any formula meaning some proposition, if t is a segmentable item figuring in the formula, then there will be a term that is the meaning of t figuring in the proposition and, further, that for any propositional function abstraction expression E meaning some propositional function, if t is a segmentable item figuring in the formula from whose abstraction E results, then there will be a term that is the meaning of t figuring in the propositional function.[29] It should be clear that this second feature of LP in conjunction with the first implies the claim in question, namely, that a proposition/propositional function and the respective *s*-formula that specifies it in a weak sense have the same form.

The second respect in which LP's account of comprehension differs from the standard account is stated by the following claim.

Claim 4.2 *A variable* qua *term in the realm of being presupposes in Russell's sense the terms belonging to the range over which it varies.*

Although, by modern lights, this claim is somewhat surprising, Russell's discussion of variables, denoting, classes, and propositional functions in his *Principles of Mathematics*, *Principia*, and elsewhere appears to commit him to it. This commitment can be seen in his further commitments to the three subclaims that compose Claim 4.2:

Subclaim 4.2.1 a variable is a term in the realm of being;
Subclaim 4.2.2 a variable presupposes its range (or is possibly identical with it); and
Subclaim 4.2.3 a totality/collection/complex presupposes its members.

Unlike Frege and later modern logicians, Russell takes variables not as linguistic symbols but as actual objects that are contained in complex propositions.[30] Thus, at the beginning of the Introduction to *Principia*, he blithely writes that "variables will be denoted by single letters" ([37], p. 5). Russell's commitment to Subclaim 4.2.3 is manifest in several places (some of which were already referred to above):

> *A and B* presupposes *A* and presupposes *B*. ([31], §71, p. 71)
>
> [the proposition that] "*A* differs from *B*" . . . presupposes *A and difference and B*. (ibid.)
>
> A complex unit is a *whole*; its parts are other units, whether simple or complex, which are *presupposed* in it. ([31], §133, emphasis added to *presupposed*)
>
> [Collections] could only *presuppose* numbers in the particular case where the terms of the collection themselves presupposed numbers. ([31], §130)

Finally, commitment to Subclaim 4.2.2 can be teased out of the following:

> From the [propositional] functions ψ, χ . . . we may proceed to form other functions of x, such as $(y).\psi(x,y)$, $(\exists y).\psi(x,y)$, $(y,z).\chi(x,y,z)$, $(y):(\exists z).\chi(x,y,z)$, and so on. All these presuppose no totality except that of individuals. ([37], p. 51)
>
> Consider a [propositional] function whose argument is an individual. This function presupposes the totality of individuals; but unless it contains functions as apparent variables, it does not presuppose any totality of functions.[31]

Clearly, the second remark strongly suggests the converse that, if a function contains a bound variable ranging over functions, then it presupposes some totality of functions. In this respect, both remarks express or strongly suggest the claim that a propositional function containing a bound variable presupposes its variable's range. Since such a propositional function must already presuppose its contained bound variable—as complexes must presuppose their constituents—we may explain this last claim by appealing to Subclaim 4.2.2 that a variable presupposes its range (or is possibly identical with it).[32]

Independently of such discussion, Goldfarb offers the following argument toward Claim 4.2 on Russell's behalf:

> I wish only to point to a consequence of having variables that lack complete generality. Once such variables are used, the question of the nature of the variable (as an entity) becomes far more urgent. Different variables can have different

> ranges; it then appears that our understanding of a proposition or a propositional function that contains quantified variables will depend quite heavily on an understanding of what those ranges are. The variable must carry with it some definite information; it must in some way represent its range of variation. Therefore, I would speculate, Russell takes a variable to presuppose the full extent of its range. ([10], p. 37)

Goldfarb's statement of the argument is terse. Perhaps, it may be expanded as follows. Russell has a compositional account of understanding according to which in order to understand a complex such as a proposition, one must understand its parts as well as how they are combined. In this light, consider a quantified propositional function, say, $\lambda y.(x).\varnothing xy$, in which the variable x's range is infinite. By the compositional account of understanding, someone j's understanding of $\lambda y.(x).\varnothing xy$ will depend on her understanding of its quantified variable x. This understanding in turn will depend on her understanding of the range over which x varies. Note that since VCP1 requires that the range of any variable be restricted, this latter dependence is nontrivial. j's understanding of this range must come by way of the circumstances that she understands x and that x represents the members of its range. To the extent that x does represent its range's members, x may be taken to presuppose them.[33]

Interestingly, Goldfarb's argument may also be expanded into a *de-epistomologized* form: According to Russell, the identity of the propositional function $\lambda y.(x).\varnothing xy$ supervenes on the identity of the variable x. The identity of x supervenes on the range over which it varies. The identity of this range supervenes on the identity of the terms that belong to it. Thus, the identity of the propositional function supervenes on the identity of the terms belonging to the range of x. To the extent that supervenience approximates presupposition, Claim 4.2 follows.

In any case, Claim 4.2 allows us to come to an interesting result. Briefly, suppose the negation of VCP2. Then, there is a comprehension axiom, say, of the following form whose s-formula contains a quantified variable g whose range contains the term f whose existence the axiom affirms:

$$\exists f \forall v_1, \ldots, \forall v_n \{ f(v_1, \ldots, v_n) \longleftrightarrow (g)(\text{---}g\text{---}) \}.$$

By the claim pertaining to the first respect—Claim 4.1—the propositional function f has the same form as the s-formula at least to the extent that the variable g must figure in f. Hence, f presupposes g. By the claim pertaining to the second respect—Claim 4.2—g presupposes the terms belonging to its range and so g presupposes f. By the transitivity of presupposition, f presupposes itself. Since presupposition is irreflexive, this consequence clearly impugns VCP1. Thus, insofar as VCP1, Claim 4.1, and Claim 4.2 are held, the original supposition must be rejected. In other words, VCP1 together with Claims 4.1 and 4.2 implies VCP2.

Let us return to the second problem posed above; namely, to explain how VCP2 may be compatible with Russell's realistic construal of propositions and propositional functions. As we just saw, VCP1 together with Claims 4.1 and 4.2 implies VCP2. Thus, insofar as VCP1 and Claims 4.1 and 4.2 may be taken as salient features of Russell's particular realistic construal, not only do we see how VCP2 is compatible with it, but we also see that such a construal indeed requires VCP2.

Next, turn to the first problem posed above; namely, to the extent that VCP1 and VCP2 may be construed each as indicating a source of the paradoxes, to explain how such a construal may be compatible with Russell's position that all the paradoxes arise from the same error. At this point, it should be clear how the explanation should go. By Russell's lights, the one error in question is violation of VCP1. When we looked at VCP1, we saw how such violation occurs in the arguments to the set-theoretic paradoxes. When we looked at VCP2, we saw how its violation occurs in the arguments to the semantic paradoxes. And just now, we saw how violation of VCP2 crucially involves violation of VCP1.

It is noteworthy that, in two places, Russell indeed may be read as saying that the single error that gives rise to all the paradoxes is violation of VCP1. The first instance occurs in a section of *Principia* entitled "The Vicious-Circle Principle." Immediately before this instance, Russell offers another expression of VCP2:

> An analysis of the paradoxes to be avoided shows that they all result from a certain kind of vicious circle. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. ([37], p. 37)

Then, Russell writes:

> *More generally*, given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. (ibid., emphasis added)

This more general case is just VCP1.

Immediately after another expression of VCP1 in a later section of *Principia*, we find the second instance:

> This is a particular case, but perhaps the *most fundamental* case, of the vicious-circle principle. ([37], p. 39, emphasis added)

VCP1 is the most fundamental in the sense that the other renderings of the vicious-circle principle can be derived from it.

Having just seen that VCP2 can be derived from VCP1, let us briefly consider some of the other renderings:

(i) Whatever involves an apparent [i.e., bound] variable must not be among the values of that variable. ([34], p. 198)

(ii) Whatever involves *all* of a collection must not be one of the collection. ([35], p. 63)

(iii) Whatever contains an apparent variable must not be a possible value of that variable. (ibid., p. 75)

Here, the crucial words are 'involves' and 'contains'. Careful reading makes it clear that 'contains' may best be read as 'has as a constituent' and, as such, the relation of containment implies that of presupposition. Thus, Russell says:

> any expression [i.e., propositional function] containing an apparent variable must not be in the range of that variable, *i.e.* must belong to a different type. Thus the apparent variables *contained* or *presupposed* in an expression are what determines its type. ([37], p. 161, emphasis added)

The verb 'involves' may best be read as 'contains'—in the case of quotation (i) above—or 'contains something (e.g., a bound variable) that presupposes'—in the case of quotation (ii) above—and, accordingly, the relation of involvement implies that of presupposition. Concerning the propositional function $(\varnothing)\, f(\varnothing\hat{z}, x)$, Russell writes:

> if x is a variable, we have a function of x; but as this function *involves* a totality of values $\varnothing\hat{z}$, it cannot itself be one of the values included in the totality, by the vicious-circle principle. It follows that the totality of values of $\varnothing\hat{z}$ concerned in $(\varnothing)\, f(\varnothing\hat{z}, x)$ is not the totality of all functions in which x can occur as argument, and that there is no such totality as that of all functions in which x can occur as argument.[34]

In this respect, VCP1 is the most fundamental in the sense that not only VCP2 but these other renderings of the vicious-circle principle can also be derived from it.

In closing, recall that I said earlier that, roughly speaking, because Russell took there to be some sort of circularity responsible for the paradoxes, he put forward his vicious-circle principle. However, because Quine did not see any such circularity, he concluded that Russell was confused about the matter. In the light of the above discussion, one should come to a conclusion different from Quine's. That is, one should conclude that, to the extent that the vicious-circle principle may in essence be identified with VCP1, there is a circularity to which Russell attended—namely, the circularity involved in non-well-founded structures.

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NOTES

1. See [31] §500, p. 528; [33], p. 144; [37], Chapter 2 of Introduction, §7, opening paragraph, p. 60.
2. A propositional function is, roughly speaking, a property.
3. Others have claimed that Russell rejected the Doctrine of Types also because it does not treat of the semantic paradoxes and because its ranges of significance appear rather ad hoc. See Chihara [4] and Copi [5].
4. See [31], §§102–4:

> The reason that a contradiction emerges . . . is that we have taken it as an axiom that any propositional function containing only one variable is equivalent to asserting membership of a class defined by the propositional function. Either this axiom, or the principle that every class can be taken as one term, is plainly false, and there is no fundamental objection to dropping either. But having dropped the former, the question arises: Which propositional functions define classes which are single terms as well as many, and which do not? And with this question our real difficulties begin. ([31], §102)
>
> We took it as axiomatic that the class as one is to be found wherever there is a class as many; but this axiom need not be universally admitted, and appears to have been the source of the contradiction. By denying it, therefore, the whole difficulty will be overcome. ([31], §104)

5. Indeed, as early as May 1901, Russell specifically suspects this mischaracterization. In an unpublished draft of Part 1 of *Principles*, Russell writes after having presented his paradox:

> It follows from the above [paradox] that not every definable collection of terms forms a class defined by a common predicate. This fact must be borne in mind, and we must endeavour to discover what properties a collection must have in order to form a class. ([30], p. 195)

6. Levy [19] (pp. 7 and18) is one. See also Fraenkel [7] (pp. 32 and 135). However, Hallett ([11], Chapter 5) argues against such an understanding. In particular, he argues effectively against the claim that the powerset operation abides by the theory of the limitation of size.

7. Thus, Poincaré writes:

> *The definitions which must be considered nonpredicative are those which contain a vicious-circle.* . . . A definition containing a vicious-circle defines nothing. ([25], p. 481)

   See also de Rouilhan [6], pp. 131–39.

8. There Russell concludes:

> the contradictions result from the fact that, according to current logical assumptions, there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property. ([33], p. 144)

9. Near the paper's end, we shall see that the examination will shed light on the renderings of the vicious-circle principle that employ 'involves' and 'contains'.

10. Here and just below I use the expressions 'totality' and 'item' as Russell in places uses 'class as many'—that is, I treat them as a Boolos plural (see Boolos [1] and [2]).

11. Perhaps all formal theories are such that any of their items having a total in the second sense has one in the first sense.

12. [31], §133. An earlier version occurs in [29], p. 35: "A complex unit I define as a *whole*; all units, complex or simple, which it presupposes, I call its *parts*.

13. [31], §134, p. 138. An earlier version occurs in [29], p. 38.

14. Roughly speaking, x is well-founded if and only if each of the branches of its membership tree is finite in length. Thus, each begins with either the null set or an urelement.

15. See Leibniz's letter of April 1867 to Arnault, [1875–90], vol. 2, p. 96.

16. I am only considering the theory's standard model here.

17. For the sake of simplicity, I have been focusing on monadic propositional functions. However, analogous claims apply to polyadic propositional functions.

18. Interestingly, although Russell's theory of types is in general noncumulative, one may see the variable "$\varnothing!\hat{x}$" that Russell speaks about in §V, Chapter 2 of the Introduction to *Principia* (p. 51) as ranging over a cumulative type—the type of first-order functions which includes the mutually exclusive subtypes of first-order matrices (e.g., $Fxyz$), single quantifications of first-order matrices (e.g., $(z)Fxyz$), double quantifications of first-order matrices (e.g., $(y)(z)Fxyz$), and so on.

19. Indeed, Hylton appears to take VCP1 to necessitate it. See [12], pp. 301–2.

20. See also Carnap [3], Part 3.

21. One should conclude further that insofar as a viciously circular formula attempts to express a predication that it cannot express, such a formula—far from being considered as a legitimate *s*-formula—cannot even be considered as well-formed.

22. [37], p. 37. Also Russell [35], p. 63.

23. One should note that the range of the variable G in the above *s*-formula is just the totality T of which VCP2 forbids existence.

24. One may recognize that contemporary logicians call an *s*-formula that is viciously circular in this sense (as well as its containing comprehension axiom) *impredicative*. They call an *s*-formula that is not so viciously circular (as well as its containing comprehension axiom) *predicative*. Note that this is a sense of 'predicative' that differs from Russell's usage in [33]. (Russell uses 'predicative' in yet another way in *Principia*.)

25. Recall that VCP2 was spelled out above as follows: a formula F is viciously circular when and only when, if F were to be countenanced as an *s*-formula, it would contain at least one quantified variable whose range contained the term whose existence would be affirmed by the comprehension axiom having F as its *s*-formula. Strictly speaking, this spelling out does not require that the further conditions just mentioned be met and, thus, does not require that the types be configured so as to realize a well-founded ordering. Later, however, we shall see that Russell construes VCP2 as following from VCP1 and that under this construal VCP2 does require that these further conditions be met. In any case, one could argue that the less precise spelling out of VCP2 offered above—that is, no totality T may contain an object that is specifiable only by quantifying over T—does require that the further conditions be met.

26. The remarks made above concerning the sketch of a configuration of types fitting VCP1 apply to this sketch *mutatis mutandis*.

27. Here, for the purposes of exposition, I am supposing there is such a thing as a standard account of comprehension or, more precisely, a standard account of what is involved in the affirmation of comprehension axioms. What such an account would be should become clear from the discussion of that with which it will be contrasted: the LP account of comprehension.

28. Strictly speaking, for any formula F and for any variables $u_1, \dots, u_i$ occurring free in F, there will be the abstraction expression represented by $\lambda u_1 \dots \lambda u_i.F$. Then, from the identity having the form '$\lambda u_1 \dots \lambda u_i.F = \lambda u_1 \dots \lambda u_i.F$', one may existentially generalize to obtain the statement having the form: $\exists f(f = \lambda u_1 \dots \lambda u_i.F)$.

29. To spell out this claim precisely, more care would be needed. As Linsky makes clear (see for instance [20], [21], and [22]), if $\lambda x.\varnothing x$ is a propositional function and $\varnothing c$ is a Russellian proposition, then $\varnothing c$ is a value of $\lambda x.\varnothing x$ and, so, $\lambda x.\varnothing x$ cannot be a constituent of $\varnothing c$. This circumstance figured in the account offered above of Russell's argument to the conclusion that no propositional function may apply to itself with sense. In any case, the above claim stated in its rough form is suitable for our purposes.

30. See Hylton [12], pp. 198 (n. 34), 216–19, 292, 297 (n. 17); Goldfarb [10], p. 34 f.

31. [37], p. 54; also, in §62 of [31], concerning the proposition 'I met a man' Russell says that "the whole human race is involved in my assertion." Later I argue that the verb 'involve' as used by Russell in such contexts expresses a relation that implies the relation of presupposition.

32. Unfortunately, what Russell says about the relation between a variable and its range does not allow us to decide between the two disjuncts of Subclaim 4.2.2. On the one hand, he says in several places (e.g., [31], §72, p. 73; §141, p. 145.; [37], pp. 50, 53) that any *humanly* comprehensible proposition can have only finite complexity. This claim gainsays the notion that a variable is identical with its range. On the other hand, one may ask how it is possible to entertain propositions about infinite objects as one supposedly does in mathematical discourse. Russell offers an answer to this question in Principles, namely, his theory of denoting. Briefly, consider the proposition that all numbers are even or odd. This proposition is both humanly comprehensible and about a term having infinite complexity, namely, the class of natural numbers. According to the theory of denoting, such a proposition is possible in virtue of the following: the proposition contains as a constituent of finite complexity the *meaning* of the phrase 'all numbers'—what Russell calls a 'denoting concept'; this denoting concept stands in a relation of representation or 'denoting' to the class of natural numbers; as a result, the proposition in question is not about the denoting concept but about this infinite class, a circumstance that constitutes an exception to the general claim that a proposition contains as a constituent what it is about.

    With regard to infinite classes, say the class of numbers, it is to be observed that the concept *all numbers*, though not itself infinitely complex, yet denotes an infinitely complex object. This is the inmost secret of our power to deal with infinity. An infinitely complex concept, though there be such, can certainly not be manipulated by the human intelligence; but infinite collections, owing to the notion of denoting, can be manipulated without introducing any concepts of infinite complexity. ([31], §72)

    So, perhaps the relation between a variable and its range is that of denoting rather than that of identity—that appears to be the *Principles* view. If so, then since by Russell's lights (see note 33) the relation of denoting or representation in general implies that of presupposition, the view would speak for the first disjunct of Subclaim 4.2.2.

    There is a difficulty, however, with this view that the relation between a variable and its range is that of denoting. In [32], Russell presents arguments against the coherence of the denoting relation, the most celebrated of which is the Grey's Elegy argument. Ricketts (in "Russell's Grey's Elegy Argument," an unpublished manuscript) has provided a persuasive analysis according to which these arguments show that, on Russell's conception of a proposition, there can be no such relation of denoting. (See also Hylton [12], Chapter 6 and Kremer [15].)

    At this point, one can plausibly say that either Russell must give up his position that any humanly comprehensible proposition is of only finite complexity (and so a variable may be identical with its range), or he must hold that there is some other relation of representation other than denoting that relates a variable to its range and to which the [32] arguments do not apply (and so a variable may presuppose—irreflexively—its range). Whichever is the case, the above discussion does not turn on this matter.

33. Clearly, the texts cited above in support of Claim 4.2 support in particular this last claim that the relation of representation implies that of presupposition. In addition, consider:

    When we say that "$\varnothing x$" ambiguously denotes $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., we mean that "$\varnothing x$" *means* one of the objects $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., though not a definite one, but an undetermined one. It follows that "$\varnothing x$" only has a well-defined meaning . . . if the objects $\varnothing a$, $\varnothing b$, $\varnothing c$, etc., are well-defined. That is to say, a function is not a well-defined function unless all its values are already well-defined. It follows from this that no function can have among its values anything which *presupposes* the function, for if it had, we could not regard

> the objects ambiguously *denoted* by the function as definite until the function was definite, while conversely, as we have seen, the function cannot be definite until its values are definite. ([37], p. 39, emphases added)
>
> a function in which $\varnothing\hat{z}$ appears as argument requires that "$\varnothing\hat{z}$" should not stand for *any* function which is capable of a given argument, but must be restricted in such a way that none of the functions which are possible values of "$\varnothing\hat{z}$" should involve any *reference* to the totality of such functions. ([37], p. 49, emphasis added to *reference*)

The latter remark may be glossed as follows. A propositional function $\varnothing$ should not contain anything that refers to (and hence presupposes) members of the type of the function $\varnothing$ for otherwise we would have the non-well-founded circle: $\varnothing$ presupposes a constituent c and c presupposes the members of the type of $\varnothing$, including $\varnothing$ itself.

34. [37], p. 49, emphasis added; consider also:

> A [propositional] function is what ambiguously denotes some one of a certain totality, namely the values of the function; hence this totality cannot contain any members which *involve* the function, since, if it did, it would contain members *involving* the totality, which, by the vicious-circle principle, no totality can do. ([37], p. 39, emphasis added)

Here, Russell appeals to the transitivity of presupposition. Further:

> a predicative function of a variable argument is one which *involves* no totality except that of the possible values of the argument, and those that are *presupposed* by any one of the possible arguments. ([37], p. 54, emphasis added)

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