Michigan Math. J. 62 (2013), 39–57

Determinantal Facet Ideals

Viviana Ene, Jürgen Herzog,
Takayuki Hibi, & Fatemeh Mohammadi

Introduction

Let K be a field, $X = (x_{ij})$ an $m \times n$ matrix of indeterminates, and $S = K[X]$ the polynomial ring over K in the indeterminates x_{ij}. We assume that $m \leq n$. Classically the ideals $I_t(X)$ generated by all t-minors of X have been considered. Hochster and Eagon [15] proved that the rings $S/I_t(X)$ are normal Cohen–Macaulay domains. A standard reference on the classical theory of determinantal ideals, including the study of the powers of $I_t(X)$, is the book by Bruns and Vetter [4]. In addition, the study of a more general class of ladder determinantal ideals has been motivated by geometrical considerations [6]. A new aspect to the theory of determinantal ideals was introduced by Sturmfels [17] and Caniglia et al. [5], who showed that the t-minors of X form a Gröbner basis of $I_t(X)$ with respect to any monomial order that selects the diagonals of the minors as leading terms. This technique yields a new proof of the Cohen–Macaulayness of the determinantal rings $S/I_t(X)$ and was subsequently used also to compute important numerical invariants of these rings—including the a-invariant, the multiplicity, and the Hilbert function (see [2; 7; 13]). Bruns and Conca [1] have written an excellent survey on the theory of determinantal ideals with regard to the Gröbner basis aspect that includes many references to more recent work.

Applications in algebraic statistics prompted the study of determinantal ideals generated by quite general classes of minors, including ideals generated by adjacent 2-minors [11; 16] or ideals generated by an arbitrary set of 2-minors in a $2 \times n$ matrix [12]. Thus one may raise the following questions. Given an arbitrary set of minors of X, what can be said about the ideal they generate? When is such an ideal a radical ideal, and when is it a prime ideal? What is its primary decomposition, when is it Cohen–Macaulay, and what is its Gröbner basis? Apart from the classical cases mentioned before, satisfactory answers to some of these questions are known for ideals generated by arbitrary sets of 2-minors of a $2 \times n$

Received August 18, 2011. Revision received November 1, 2012.

This paper was written while J. Herzog and F. Mohammadi were staying at Osaka University. They were supported by the JST (Japan Science and Technology Agency) CREST (Core Research for Evolutional Science and Technology) research project *Harmony of Gröbner Bases and the Modern Industrial Society* in the frame of the Mathematics Program "Alliance for Breakthrough between Mathematics and Sciences". The first author was supported by the grant UEFISCDI, PN-II-ID-PCE- 2011-3-1023.

matrix of indeterminates. For these ideals—all of which are radical—the primary decomposition and the Gröbner basis are known (see [12]).

The purpose of this paper is to extend some of the results shown in [12] to ideals generated by an arbitrary set of maximal minors of an $m \times n$ matrix of indeterminates. For any sequence of integers $1 \leq a_1 < a_2 < \cdots < a_m \leq n$, we denote by $[a_1 a_2 \ldots a_m]$ the maximal minor of X with columns $a_1, a_2, \ldots, a_m$. The set of integers $\{a_1, a_2, \ldots, a_m\}$ may be viewed as a facet of a simplex on the vertex set $[n]$. We are thus led to the following definition. Let Δ be a pure simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ of dimension $m - 1$. With each facet $F = \{a_1 < a_2 < \cdots < a_m\}$ we associate the minor $\mu_F = [a_1 a_2 \ldots a_m]$, and we call the ideal

$$J_\Delta = (\mu_F : F \in \mathcal{F}(\Delta))$$

the *determinantal facet ideal* of Δ. Here $\mathcal{F}(\Delta)$ denotes the set of facets of Δ.

If $m = 2$, then (i) Δ may be identified with a graph G and (ii) the m-minors are binomials. In this case, the determinantal facet ideal coincides with the binomial edge ideal of [12].

In the first section of this paper we establish when the maximal minors generating J_Δ form a Gröbner basis of J_Δ. In order to explain this result, we must introduce some notation. Let Γ be a simplicial complex, and denote by $\Gamma^{(i)}$ the i-skeleton of Γ. The simplicial complex $\Gamma^{(i)}$ is the collection of all simplices of Γ whose dimension is at most i.

Now let Δ be a pure $(m-1)$-dimensional simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$. We denote by $\mathcal{S}$ the set of simplices Γ with vertices in $[n]$ for which $\dim \Gamma \geq m - 1$ such that $\Gamma^{(m-1)} \subset \Delta$. Let $\Gamma_1, \ldots, \Gamma_r$ be the maximal elements in $\mathcal{S}$ (with respect to inclusion) and set $\Delta_i = \Gamma_i^{(m-1)}$. Then $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$. The simplicial complex whose facets are the Γ_i is called the *clique complex* of Δ, the Δ_i are the *cliques* of Δ, and $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$ is the *clique decomposition* of Δ. For example, let Δ be the 2-dimensional simplicial complex on the vertex set [7] with facets $F_1 = \{1, 2, 3\}$, $F_2 = \{1, 2, 4\}$, $F_3 = \{1, 3, 4\}$, $F_4 = \{2, 3, 4\}$, $F_5 = \{3, 4, 5\}$, and $F_6 = \{5, 6, 7\}$. Then Δ has the clique decomposition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where $\Delta_1 = \Gamma_1^{(2)}$ for Γ_1 is the 3-dimensional simplex on the set [4], $\Delta_2 = \Gamma_2^{(2)}$ for Γ_2 the 2-dimensional simplex on the set $\{3, 4, 5\}$, and $\Delta_3 = \Gamma_3^{(2)}$ for Γ_3 the 2-dimensional simplex on the set $\{5, 6, 7\}$.

Note that if $m = 2$ (i.e., if Δ is a graph), then the Δ_i are exactly the cliques of Δ as they are known in graph theory and $\Gamma_1, \ldots, \Gamma_r$ are the facets of the clique complex of the graph Δ.

The complex Δ is called *closed* (with respect to the given labeling) if, for any two facets $F = \{a_1 < \cdots < a_m\}$ and $G = \{b_1 < \cdots < b_m\}$ with $a_i = b_i$ for some i, the $(m-1)$-skeleton of the simplex on the vertex set $F \cup G$ is contained in Δ. In terms of its clique decomposition, the property of Δ of being closed can be expressed in two different ways.

(1) Δ is closed if and only if, for all $i \neq j$ and for all $F = \{a_1 < a_2 < \cdots < a_m\} \in \Delta_i$ and $G = \{b_1 < b_2 < \cdots < b_m\} \in \Delta_j$, we have $a_\ell \neq b_\ell$ for all ℓ.

(2) Δ is closed if and only if, for all $i \neq j$ and for all $\{a_1, \dots, a_m\} \in \Delta_i$ and $\{b_1, \dots, b_m\} \in \Delta_j$, the monomials $\mathrm{in}_<[a_1 \dots a_m]$ and $\mathrm{in}_<[b_1 \dots b_m]$ are relatively prime; here $<$ is the lexicographical order induced by the natural order of indeterminates

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{mn}$$

(row by row from left to right).

The main result (Theorem 1.1) of Section 1 states that the minors generating the facet ideal J_Δ form a quadratic Gröbner basis—with respect to the lexicographic order induced by the natural order of the variables—if and only if Δ is closed. We also show that, whenever Δ is closed, J_Δ is Cohen–Macaulay and the K-algebra generated by the minors that generate J_Δ is Gorenstein (see Corollary 1.3 and Corollary 1.4).

In Section 2 we discuss when a determinantal facet ideal is a prime ideal. As a main result we show in Theorem 2.2 that if Δ is closed and if J_Δ is a prime ideal, then the clique complexes Δ_i of Δ satisfy the following intersection property: for all $2 \leq t \leq m = \dim \Delta + 1$ and for any pairwise distinct cliques $\Delta_{i_1}, \dots, \Delta_{i_t}$,

$$|V(\Delta_{i_1}) \cap \cdots \cap V(\Delta_{i_t})| \leq m - t.$$

We expect that this intersection property actually characterizes closed simplicial complexes whose determinantal facet ideal is prime, but we have not yet proved this. In Theorem 2.4 we can give only a partial converse of Theorem 2.2.

We show in Example 2.5 that, for determinantal facet ideals satisfying the intersection condition just described, primality can be expected only in the case of closed simplicial complexes. For nonclosed simplicial complexes, the primality problem is difficult.

In Section 3 we study primality of J_Δ for a closed simplicial complex under the following strict intersection condition. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ. We require that

(i) $|V(\Delta_i) \cap V(\Delta_j)| \leq 1$ for all $i < j$ and
(ii) $V(\Delta_i) \cap V(\Delta_j) \cap V(\Delta_k) = \emptyset$ for all $i < j < k$.

For $m = 3$, this is exactly the necessary condition for primality formulated in Theorem 2.2.

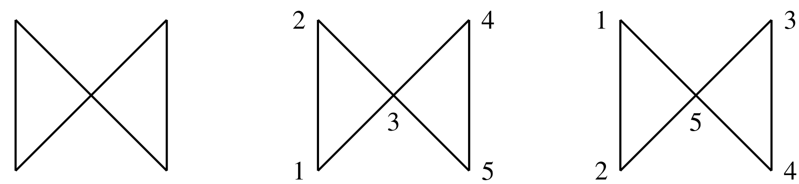
Assuming (i) and (ii), we let G_Δ be the simple graph with vertices $v_1, \dots, v_r$ and edges $\{v_i, v_j\}$ for all $i \neq j$, where $V(\Delta_i) \cap V(\Delta_j) \neq \emptyset$. We would like to identify the graphs G_Δ for which the determinantal facet ideal J_Δ is a prime ideal; this is the case when Δ is closed and G_Δ is a forest or a cycle (see Theorem 3.2 and Theorem 3.3). Finally, we show in Theorem 3.4 that for any graph G there is a closed simplicial complex Δ, with $G = G_\Delta$, whose cliques are all simplices.

1. Determinantal Facet Ideals Whose Generators Form a Gröbner Basis

In this section we seek to classify those ideals generated by maximal minors of a generic $m \times n$ matrix X whose generating minors form a Gröbner basis. As explained in the Introduction, we identify each m-minor $[a_1a_2\dots a_m]$ of X with the $(m-1)$-simplex $F=\{a_1,a_2,\dots,a_m\}$. Thus an arbitrary collection of m-minors of X can be indexed by the facets of a pure $(m-1)$-dimensional simplicial complex Δ on the vertex set $[n]$. The ideal generated by these minors will be denoted J_Δ and is called the *determinantal facet ideal* of Δ. In other words, if $\mathcal{F}(\Delta)$ denotes the set of facets of Δ then $J_\Delta=(\mu_F : F\in\mathcal{F}(\Delta))$, where $\mu_F=[a_1a_2\dots a_m]$ for $F=\{a_1,a_2,\dots,a_m\}$.

In analogy to the case of 2-minors as considered in [12], we say that Δ is *closed with respect to the given labeling* if, for any two facets $F=\{a_1<\dots<a_m\}$ and $G=\{b_1<\dots<b_m\}$ with $a_i=b_i$ for some i, the $(m-1)$-skeleton of the simplex on the vertex set $F\cup G$ is contained in Δ. Note that Δ is called *closed* if there is a labeling of its vertices such that Δ is closed with respect to it.

For example, let Δ be the 2-dimensional simplicial complex of Figure 1(a). The cliques of Δ are two simplices of dimension 2. The complex Δ is closed with respect to the labeling given in Figure 1(b) but not with respect to the labeling given in Figure 1(c). Indeed, with the first labeling, the facets $\{1,2,3\}$ of the first clique and $\{3,4,5\}$ of the second clique have no common label in the same position whereas, with the second labeling, the facets $\{1,2,5\}$ and $\{3,4,5\}$ both have the label 5 in the last position. In terms of initial monomials, in the first case $\mathrm{in}_<[123]=x_{11}x_{22}x_{33}$ and $\mathrm{in}_<[345]=x_{13}x_{24}x_{35}$ are relatively prime; in the second case, $\mathrm{in}_<[125]=x_{11}x_{22}x_{35}$ and $\mathrm{in}_<[345]=x_{13}x_{24}x_{35}$ are not relatively prime. Nonetheless, the simplicial complex is closed because one may find at least one labeling of its vertices with respect to which Δ is closed.



(a) (b) (c)

Figure 1

The main result of this section is the following statement.

THEOREM 1.1. *The set $\mathcal{G}=\{[a_1\dots a_m] : \{a_1,\dots,a_m\}\in\Delta\}$ is a Gröbner basis of J_Δ with respect to the lexicographical order induced by the natural order of indeterminates if and only if Δ is closed.*

Before proving this theorem we recall some notation that is often used in the classical theory of determinantal ideals. If $r<m$, then the minor corresponding to the submatrix of X with rows $a_1,\dots,a_r$ and columns $b_1,\dots,b_r$ is denoted by $[a_1\dots a_r|b_1\dots b_r]$. Proving Theorem 1.1 will require the following technical result.

LEMMA 1.2. *Let $m \le n-1$. For any $m-1$ rows $c_1, c_2, \ldots, c_{m-1}$ and any $m+1$ columns $d_1, d_2, \ldots, d_{m-2}, e_1, e_2, e_3$ of X, we have*

$$\begin{aligned}(-1)^k&[c_1 \ldots c_{m-1}|d_1 \ldots d_{m-2}e_3][d_1 \ldots d_{m-2}e_1e_2]\\ &+(-1)^j[c_1 \ldots c_{m-1}|d_1 \ldots d_{m-2}e_2][d_1 \ldots d_{m-2}e_1e_3]\\ &+(-1)^i[c_1 \ldots c_{m-1}|d_1 \ldots d_{m-2}e_1][d_1 \ldots d_{m-2}e_2e_3]=0\end{aligned}$$

provided that $d_1 < d_2 < \cdots < d_{i-1} < e_1 < d_i < \cdots < d_{j-2} < e_2 < d_{j-1} < \cdots < d_{k-3} < e_3 < d_{k-2} < \cdots < d_{m-2}$ for some $1 \le i < j < k \le m$.

Proof. Our assumption on the sequence of integers means that e_1 is the ith term, e_2 the jth term, and e_3 the kth term in the preceding sequence.

Now consider the matrix

$$M = \begin{pmatrix} x_{1d_1} & \ldots & x_{1d_{i-1}} & x_{1e_1} & \ldots & x_{1e_2} & \ldots & x_{1e_3} & \ldots & x_{1d_{m-2}} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{md_1} & \ldots & x_{md_{i-1}} & x_{me_1} & \ldots & x_{me_2} & \ldots & x_{me_3} & \ldots & x_{md_{m-2}} \\ g_{d_1} & \ldots & g_{d_{i-1}} & g_{e_1} & \ldots & g_{e_2} & \ldots & g_{e_3} & \ldots & g_{d_{m-2}} \end{pmatrix},$$

where g_ℓ is the minor $[c_1 \ldots c_{m-1}|d_1 \ldots d_{m-2}\ell]$ of X for each $\ell \in \{d_1, d_2, \ldots, d_{m-1}, e_1, e_2, e_3\}$. Expanding g_ℓ by the last column yields

$$g_\ell = \sum_{i=1}^{m-1}(-1)^{m-1+i}[c_1 \ldots c_{i-1}c_{i+1} \ldots c_{m-1}|d_1 \ldots d_{m-2}]x_{c_i\ell}$$

for each ℓ. Hence the last row of M is a linear combination of the rows $c_1, \ldots, c_{m-1}$ of M and so the determinant of M is zero. On the other hand, $g_\ell = 0$ for $\ell = d_1, \ldots, d_{m-2}$ because, for these ℓ, the polynomial g_ℓ is the determinant of a matrix with two equal columns. Now computing the determinant of M by expanding its last row, we obtain the desired identity. □

Proof of Theorem 1.1. Assume that Δ is closed. We show that all S-pairs,

$$S([a_1 \ldots a_m], [b_1 \ldots b_m]),$$

reduce to zero. If $a_i \ne b_i$ for all i, then $\text{in}_<[a_1 \ldots a_m]$ and $\text{in}_<[b_1 \ldots b_m]$ have no common factor. Therefore, $S([a_1 \ldots a_m], [b_1 \ldots b_m])$ reduces to zero.

Let $a_i = b_i$ for some i. Since Δ is closed, all m-subsets of $\{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$ belong to Δ. As a result, $S([a_1 \ldots a_m], [b_1 \ldots b_m])$ reduces to zero with respect to the m-subsets of $\{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$ and hence with respect to $\mathcal{G}$. It then follows from Buchberger's criterion that $\mathcal{G}$ is a Gröbner basis of J_Δ.

Assume that $\mathcal{G}$ is a Gröbner basis for the ideal J_Δ. Let $[a_1a_2 \ldots a_m]$ with $a_1 < a_2 < \cdots < a_m$ and $[b_1b_2 \ldots b_m]$ with $b_1 < b_2 < \cdots < b_m$ belong to $\mathcal{G}$, and assume that $a_i = b_i$ for some i. We will show that Δ is closed. The proof is by descending induction on

$$k = |\{a_1, \ldots, a_m\} \cap \{b_1, \ldots, b_m\}|.$$

Case 1: $k = m-1$. Then there exists an integer ℓ such that $a_1 = b_1, \ldots, a_{\ell-1} = b_{\ell-1}$ and $a_\ell \ne b_\ell$. We may assume $b_\ell < a_\ell$, so

$$\{b_1 < \cdots < b_m\}$$
$$= \{a_1 < a_2 < \cdots < a_{\ell-1} < b_\ell < a_\ell < \cdots < a_{\ell'-1} < a_{\ell'+1} < \cdots < a_m\}$$

for some $\ell' \geq \ell$. In this case, proving that Δ is closed requires showing that

$$\{a_1, \dots, a_m, b_\ell\} \setminus \{a_r\} \in \Delta$$

for all r.

Since $a_i = b_i$ for some i we have that either $\ell' < m$ or $1 < \ell$. First assume that $\ell' < m$, and choose an integer r with $\ell' < r \leq m$. We can use the determinantal identity of Lemma 1.2, whereby $\{d_1 < \cdots < d_{m-2}\}$ is equal to

$$\{a_1 < \cdots < a_{\ell-1} < a_\ell < \cdots < a_{\ell'-1} < a_{\ell'+1} < \cdots < a_{r-1} < a_{r+1} < \cdots < a_m\}$$

and $\{e_1 < e_2 < e_3\} = \{b_\ell < a_{\ell'} < a_r\}$, to obtain

$$\begin{aligned}(-1)^{\ell'+1}&[1 \dots m-1 | a_1 \dots \hat{a}_{\ell'} \dots a_m][a_1 \dots a_{\ell-1} b_\ell a_\ell \dots \hat{a}_r \dots a_m]\\ &+ (-1)^{r+1}[1 \dots m-1 | a_1 \dots \hat{a}_r \dots a_m][b_1 \dots b_m]\\ &+ (-1)^{\ell}[1 \dots m-1 | a_1 \dots a_{\ell-1} b_\ell a_\ell \dots \hat{a}_{\ell'} \dots \hat{a}_r \dots a_m][a_1 \dots a_m] = 0.\end{aligned}$$

Since the last two terms are in J_Δ and since $\mathcal{G}$ is a Gröbner basis for J_Δ, it follows that the initial monomial of the first term is divisible by the initial monomial of a minor in $\mathcal{G}$.

The initial monomial of the first term is

$$\begin{aligned}u = &(x_{1a_1} \cdots x_{\ell'-1 a_{\ell'-1}} x_{\ell' a_{\ell'+1}} \cdots x_{m-1 a_m})\\ &\times (x_{1a_1} \cdots x_{\ell-1 a_{\ell-1}} x_{\ell b_\ell} x_{\ell+1 a_\ell} x_{\ell+2 a_{\ell+1}} \cdots x_{r a_{r-1}} x_{r+1 a_{r+1}} \cdots x_{m a_m}).\end{aligned}$$

Hence $\mathrm{in}_<[a_1 \dots a_{\ell-1} b_\ell a_\ell \dots \hat{a}_r \dots a_m]$ is the only initial monomial of a maximal minor of X that divides the displayed monomial. Indeed, in order to find the initial monomial of a maximal minor that divides u, we must choose an increasing subsequence of $a_1 < \cdots < a_{\ell-1} < b_\ell < a_\ell < a_{\ell+1} < \cdots < a_m$ with m elements. Observe that we have a unique choice for the first $\ell - 1$ elements and the last $m - r$ elements; that choice is $a_1 < \cdots < a_{\ell-1}$ and, respectively, $a_{r+1} < \cdots < a_m$. We must therefore choose a subsequence with $r - \ell + 1$ elements of $b_\ell < a_\ell < a_{\ell+1} < \cdots < a_r$. Now we see that x_{ra_r} does not divide u, so we cannot keep a_r in the preceding sequence. As a result, the unique choice of the subsequence is $b_\ell < a_\ell < a_{\ell+1} < \cdots < a_{r-1}$. Hence we deduce that

$$[a_1 \dots a_{\ell-1} b_\ell a_\ell \dots \hat{a}_r \dots a_m] \in \mathcal{G}$$

and so $\{a_1, \dots, a_{\ell-1}, b_\ell, a_\ell, \dots, \hat{a}_r, \dots, a_m\}$ is in Δ for all $r > \ell'$.

Next we assume that $1 < \ell$. Then we deduce as in the $\ell' < m$ case that

$$\{a_1, \dots, \hat{a}_r, \dots, a_{\ell-1}, b_\ell, a_\ell, \dots, a_m\}$$

is in Δ for $r < \ell$. More precisely, we again use Lemma 1.2 but now for

$$[c_1 \dots c_{m-1}] = [2 \dots m];$$

this leads to the following identity:

$$(-1)^r[2\dots m|a_1\dots\hat{a}_r\dots a_m][b_1\dots b_m]$$
$$+(-1)^{\ell-1}[2\dots m|a_1\dots\hat{a}_r\dots b_\ell a_\ell\dots\hat{a}_{\ell'}\dots a_m][a_1\dots a_m]$$
$$+(-1)^{\ell'-1}[2\dots m|a_1\dots\hat{a}_{\ell'}\dots a_m][a_1\dots\hat{a}_r\dots b_\ell a_\ell\dots a_m]=0.$$

The last term in this identity belongs to J_Δ, so its initial monomial is divisible by the initial monomial of a minor in $\mathcal{G}$. By using similar arguments as before, we get the claim.

Finally we show that $\{a_1,\dots,a_m,b_\ell\}\setminus\{a_r\}\in\Delta$ for *all* r. Toward this end, we may assume that $\ell'<m$ and choose $r=\ell'+1$ to obtain (arguing as before) that $\{a_1,\dots,a_{\ell-1},b_\ell,a_\ell,\dots,\hat{a}_{\ell'+1},\dots,a_m\}$ is a facet of Δ. When we compare this facet with the facet $\{a_1,\dots,a_{\ell-1},b_\ell,a_\ell,\dots,\hat{a}_{\ell'},\dots,a_m\}$ of Δ, it follows from the previous considerations that $\{a_1,\dots,a_{\ell-1},b_\ell,a_\ell,\dots,a_m\}\setminus\{a_r\}\in\Delta$ for all $r\le\ell'$.

Case 2: $k<m-1$. Let s be the number of integers i such that $a_i=b_i$. By our assumption, $s\ge 1$ and of course $s\le k$. Assume that $a_1=b_1,\dots,a_s=b_s$ and $a_{s+1}<b_{s+1}$. Then

$$\begin{aligned}&\mathrm{in}_<([s+1\dots m|b_{s+1}\dots b_m][a_1\dots a_m]-[s+1\dots m|a_{s+1}\dots a_m][b_1\dots b_m])\\&\quad=(x_{s+1b_{s+1}}\cdots x_{mb_m})(x_{1a_1}\cdots x_{s-1a_{s-1}}x_{sa_{s+1}}x_{s+1a_s}x_{s+2a_{s+2}}\cdots x_{ma_m})=u;\end{aligned}$$

this is because the monomials greater than u (in the expression whose initial monomial we compute) cancel. Hence there exists a minor $[c_1\dots c_m]$ in $\mathcal{G}$ with $c_1<c_2<\cdots<c_m$ such that $\mathrm{in}_<[c_1\dots c_m]$ divides the monomial

$$(x_{s+1b_{s+1}}\cdots x_{mb_m})(x_{1a_1}\cdots x_{s-1a_{s-1}}x_{sa_{s+1}}x_{s+1a_s}x_{s+2a_{s+2}}\cdots x_{ma_m}),$$

from which it follows that

$$c_1=a_1,\ \dots,\ c_{s-1}=a_{s-1},\ c_s=a_{s+1},\ c_{s+1}=b_{s+1}$$
$$\text{and}\quad c_\ell\in\{a_\ell,b_\ell\}\ \text{ for } \ell\ge s+2.$$

First consider the case $s=k$. Then c_m is either a_m or b_m, and we may assume that $c_m=a_m$. Therefore, $|\{c_1,\dots,c_m\}\cap\{a_1,\dots,a_m\}|>k$. Applying the inductive hypothesis for the facets $\{c_1,\dots,c_m\}$ and $\{a_1,\dots,a_m\}$ of Δ, we conclude that all m-subsets of

$$\{a_1,\dots,a_m\}\cup\{c_1,\dots,c_m\}$$

belong to Δ.

Note that there exists some c_i such that $c_i\notin\{a_1,\dots,a_m\}$, given $a_s\notin\{c_1,\dots,c_m\}$. It follows that $c_i=b_i$ and consequently $b_i\notin\{a_1,\dots,a_m\}$. Moreover, since $k<m-1$ there exist two integers j_1 and j_2 such that

$$a_{j_1},a_{j_2}\notin\{b_1,\dots,b_m\}.$$

Since $\{a_1,\dots,\hat{a}_{j_1},\dots,a_m,b_i\}$ and $\{a_1,\dots,\hat{a}_{j_2},\dots,a_m,b_i\}$ are m-subsets of

$$\{a_1,\dots,a_m\}\cup\{c_1,\dots,c_m\},$$

both of them belong to Δ. Now applying the inductive hypothesis to the sets $\{b_1,\dots,b_m\}$ and $\{a_1,\dots,\hat{a}_{j_1},\dots,a_m,b_i\}$ that intersect in $k+1$ elements, we obtain all m-subsets of

$$\{a_1, \dots, \hat{a}_{j_1}, \dots, a_m, b_i\} \cup \{b_1, \dots, b_m\}$$

in Δ. By the same argument we deduce that all m-subsets of

$$\{a_1, \dots, \hat{a}_{j_2}, \dots, a_m, b_i\} \cup \{b_1, \dots, b_m\}$$

belong to Δ.

Now assume that F is an arbitrary subset of $\{a_1, \dots, a_m\} \cup \{b_1, \dots, b_m\}$ such that $a_{j_1}, a_{j_2} \in F$ and $b_j \notin F$ for some j. By the foregoing statements we have $(F \setminus \{a_{j_1}\}) \cup \{b_j\}$ and $(F \setminus \{a_{j_2}\}) \cup \{b_j\}$ in Δ. Comparing these two facets then allows us to deduce that $F \in \Delta$, since their intersection has cardinality $m-1$.

The proof is similar in the more general case where $a_{\ell_1} = b_{\ell_1}, \dots, a_{\ell_s} = b_{\ell_s}$. We simply consider the minor

$$[1 \dots \hat{\ell}_1 \dots \hat{\ell}_s \dots m | a_1 \dots \hat{a}_{\ell_1} \dots \hat{a}_{\ell_s} \dots a_m]$$

instead of $[s+1 \dots m | a_{s+1} \dots a_m]$ and the minor

$$[1 \dots \hat{\ell}_1 \dots \hat{\ell}_s \dots m | b_1 \dots \hat{b}_{\ell_1} \dots \hat{b}_{\ell_s} \dots b_m]$$

instead of $[s+1 \dots m | b_{s+1} \dots b_m]$ to get the desired minors in $\mathcal{G}$. Therefore, the assertion of the theorem is proved if $s = k$.

Now assume that $s < k$ and that the results hold for every two sets in Δ with k common elements at least $s+1$ of which have the same position in both sets. Let $a_{\ell_1} = b_{t_1}, \dots, a_{\ell_{k-s}} = b_{t_{k-s}}$ for some integers $\ell_1 < \dots < \ell_{k-s}$ and $t_1 < \dots < t_{k-s}$, where $t_r \neq \ell_r$ for $r = 1, \dots, k-s$. Assume that

$$\{a_{\ell_{\sigma_1}}, \dots, a_{\ell_{\sigma_p}}\} \subset \{c_{s+2}, \dots, c_m\} \quad \text{and} \quad \{a_{\ell_{\tau_1}}, \dots, a_{\ell_{\tau_q}}\} \not\subset \{c_{s+2}, \dots, c_m\}$$

for $\{\sigma_1, \dots, \sigma_p, \tau_1, \dots, \tau_q\} = \{\ell_1, \dots, \ell_{k-s}\}$.

We begin by assuming that $p = k-s$. We remark that, since $k < m-1$, there exists some index j with $j \notin \{1, \dots, s+1, \ell_1, \dots, \ell_{k-s}\}$. If $c_j = a_j$ for some $j \notin \{1, \dots, s+1, \ell_1, \dots, \ell_{k-s}\}$ then $|\{a_1, \dots, a_m\} \cap \{c_1, \dots, c_m\}| > k$ and, by the inductive hypothesis, we derive all m-subsets of $\{a_1, \dots, a_m\} \cup \{c_1, \dots, c_m\}$ in Δ. Otherwise, $|\{b_1, \dots, b_m\} \cap \{c_1, \dots, c_m\}| > k$ and so, again by the inductive hypothesis, all m-subsets of $\{b_1, \dots, b_m\} \cup \{c_1, \dots, c_m\}$ belong to Δ. In both cases we can apply the same argument as in the case $s = k$ and thereby deduce that all desired m-subsets are in Δ.

Next assume that $p < k-s$. We claim that

$$c_{\ell_r} = b_{\ell_r} \quad \text{for } r = \tau_1, \dots, \tau_q;$$

in particular, we have $\{b_{\ell_{\tau_1}}, \dots, b_{\ell_{\tau_q}}\} \subset \{c_1, \dots, c_m\}$. Indeed, suppose that $a_{\ell_r} \notin \{c_{s+2}, \dots, c_m\}$. Then $c_{\ell_r} = b_{\ell_r}$ and $c_{t_r} = a_{t_r}$.

Since $a_{s+1} < b_{s+1} < \dots < b_m$, we have $\ell_r > s+1$ for all r. Therefore,

$$c_1 = b_1, \dots, c_{s-1} = b_{s-1}, c_{s+1} = b_{s+1}, \qquad c_{\ell_{\tau_1}} = b_{\ell_{\tau_1}}, \dots, c_{\ell_{\tau_q}} = b_{\ell_{\tau_q}},$$

$$c_{\ell_{\sigma_1}} = a_{\ell_{\sigma_1}} = b_{t_{\sigma_1}}, \dots, c_{\ell_{\sigma_p}} = a_{\ell_{\sigma_p}} = b_{t_{\sigma_p}}.$$

These expressions show that $\{c_1, \dots, c_m\}$ and $\{b_1, \dots, b_m\}$ have at least k common elements and that $s+q \geq s+1$ of them have the same position in both sets. Now applying the result of the first case to these two sets, we deduce that all m-subsets of

$$\{b_1, \dots, b_m\} \cup \{c_1, \dots, c_m\}$$

are in Δ. Finally, the same argument as in the case $k = s$—but for $\{b_1, \dots, b_m\} \cup \{c_1, \dots, c_m\}$ instead of $\{a_1, \dots, a_m\} \cup \{c_1, \dots, c_m\}$—implies that all desired m-subsets belong to Δ. □

For determinantal facet ideals of closed simplicial complexes, we may compute important numerical invariants.

COROLLARY 1.3. *Let Δ be a closed simplicial complex of dimension $(m-1)$ and let $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$ be its clique decomposition. For $1 \le \ell \le r$, let n_ℓ be the number of vertices of Δ_ℓ. Then*:

(a) height $J_\Delta = \sum_{\ell=1}^{r}$ height $J_{\Delta_\ell} = \sum_{\ell=1}^{r} n_\ell - (m-1)r$;
(b) *J_Δ is Cohen–Macaulay*;
(c) *the Hilbert series of S/J_Δ has the form*

$$H_{S/J_\Delta}(t) = \frac{\prod_{\ell=1}^{r} Q_\ell(t)}{(1-t)^{mn - \sum_{\ell=1}^{r} n_\ell + (m-1)r}},$$

where

$$Q_\ell(t) = \frac{\det\left(\sum_k \binom{m-i}{k}\binom{n_\ell - j}{k}\right)_{1 \le i, j \le m-1}}{t^{\binom{m-1}{2}}}$$

for $1 \le \ell \le r$;
(d) *the multiplicity of S/J_Δ is*

$$e(S/J_\Delta) = \prod_{\ell=1}^{r} \binom{n_\ell}{m-1}.$$

Proof. It follows from characterization (2) of closed simplicial complexes that the initial ideals $\mathrm{in}_<(J_{\Delta_\ell})$ are monomial ideals in disjoint sets of variables; hence the first equality in (a) is obvious. The second equality follows from the formula for the height of determinantal ideals (see e.g. [8, Thm. 6.35]).

By [10, Cor. 3.3.5], S/J_Δ and $S/\mathrm{in}_<(J_\Delta)$ have the same Hilbert series. By [7, Cor. 1] or by [1, Thm. 6.9] and [13, Thm. 3.5], we have formulas for the Hilbert series and know the multiplicity of determinantal rings defined by maximal minors. Hence (c) and (d) follow once we observe that, by characterization (2) of closed simplicial complexes,

$$S/\mathrm{in}_<(J_\Delta) \cong \bigotimes_{i=1}^{r} S_i/\mathrm{in}_<(J_{\Delta_i}); \tag{1.1}$$

here the S_i are polynomial rings in disjoint sets of variables whose union is the set of all the variables of X. Another application of (1.1) reveals that, since all factors in the right-hand side are Cohen–Macaulay (see [5] and [17]), $\mathrm{in}_<(J_\Delta) = \mathrm{in}_<(J_{\Delta_1}) + \cdots + \mathrm{in}_<(J_{\Delta_r})$ is also Cohen–Macaulay. This, in turn, implies that J_Δ is Cohen–Macaulay (see e.g. [10, Cor. 3.3.5]). □

Corollary 1.4. *Suppose that Δ is closed and has clique decomposition $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$. Then the K-algebra*

$$A = K[\{[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta\}]$$

is Gorenstein and of dimension $r + \sum_{i=1}^{r} m(n_i - m)$, where n_i is the cardinality of the vertex set of Δ_i.

Proof. We first observe that

$$\begin{aligned} B &:= K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta\}] \\ &\cong \bigotimes_{i=1}^{r} K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]. \end{aligned}$$

We use the Sagbi basis criterion (see [8, Thm. 6.43]), which asserts that the minors $[a_1 \dots a_m]$ with $\{a_1, \dots, a_m\} \in \Delta$ form a Sagbi basis of A; in other words, the monomials $[a_1 \dots a_m]$ with $\{a_1, \dots, a_m\} \in \Delta$ generate the initial algebra $\mathrm{in}_<(A)$ provided a generating set of binomial relations of the algebra B can be lifted. It follows from the tensor presentation of B that a set of binomial relations of B is obtained as the union of the binomial relations of each of the algebras $K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]$. Because these algebras are known to admit a set of liftable relations, we have $B = \mathrm{in}_<(A)$.

Next we note that, for each i, the K-algebra $K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]$ is the Hibi ring associated to the distributive lattice $\mathcal{L}_i$ of all maximal m-minors $[a_1 \dots a_m]$ with $\{a_1, \dots, a_m\} \in \Delta_i$. The partial order of this lattice is given by

$$[a_1 \dots a_m] \le [b_1 \dots b_m] \iff a_i \le b_i \text{ for } i = 1, \dots, m.$$

The distributive lattice $\mathcal{L}_i$ is graded, which by a theorem of Hibi [14] implies that

$$K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]$$

is Gorenstein. It follows from [1, Thm. 3.16] that A is Gorenstein.

Finally, we observe that

$$\begin{aligned} \dim A = \dim \mathrm{in}_<(A) &= \sum_{i=1}^{r} \dim K[\{\mathrm{in}_<[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}] \\ &= \sum_{i=1}^{r} \dim K[\{[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]. \end{aligned}$$

The desired formula for the dimension of A follows: $K[\{[a_1 \dots a_m] : \{a_1, \dots, a_m\} \in \Delta_i\}]$ is the algebra of all maximal minors of an $m \times n_i$ matrix of indeterminates, so its dimension is $m(n_i - m) + 1$ (see e.g. [8, Thm. 6.45]). □

2. Primality of Determinantal Facet Ideals

In this and the following section we discuss the conditions under which a determinantal facet ideal is a prime ideal. In general, J_Δ need not be a prime ideal even when Δ is closed. For example, if Δ is the simplicial complex with facets $\mathcal{F}(\Delta) = \{\{1,2,3\},\{2,3,4\}\}$ or $\mathcal{F}(\Delta) = \{\{1,2,3\},\{2,3,6\},\{3,4,5\}\}$, then J_Δ is not a prime ideal. Indeed, in the first case we have height $J_\Delta = \operatorname{height} \operatorname{in}_<(J_\Delta) = 2$ since $\operatorname{in}_<(J_\Delta)$ is generated by a regular sequence of length 2 and since $P = (x_2y_3 - x_3y_2, x_2z_3 - x_3z_2, y_2z_3 - y_3z_2)$ is a prime ideal of height 2 that, it is clear, strictly contains J_Δ. We denote the variables of the first row of a matrix (here, X) by x, of the second row by y, and of the third row by z together with appropriate indices. In the second case we have height $J_\Delta = \operatorname{height} \operatorname{in}_<(J_\Delta) = 3$ and $J_\Delta \subsetneq (x_3, y_3, z_3)$, so clearly J_Δ is not prime. Even in these relatively simple examples we see that the primary decomposition of determinantal facet ideals is far more complicated than that for binomial edge ideals.

The main result of this section, Theorem 2.2, explains why J_Δ is not a prime ideal in the examples just given. The proofs of primality that will follow depend on localization with respect to nonzero divisors, a technique that allows for the use of induction arguments. Indeed, suppose we want to show that $J \subset S$ is a prime ideal. Then we are looking for an element $f \in S$ that is regular modulo J, whose existence would imply that the natural map $S/J \to (S/J)_f$ is injective. If we can find a prime ideal $L \subset S$ such that $(S/L)_f \cong (S/J)_f$ then $(S/J)_f$ (and consequently S/J) is a domain, which implies that J is a prime ideal. This procedure often allows us to use inductive arguments, as in many cases L is of a simpler structure.

The next lemma explicates the effect of localization when we are dealing with ideals generated by minors of a matrix.

LEMMA 2.1. *Let K be a field, X an $m \times n$ matrix of indeterminates, and $I \subset S = K[X]$ an ideal generated by a set $\mathcal{G}$ of minors. Let x_{ij} be an entry of X. We assume that, for each minor $[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ $(t \geq 1)$, there exists an ℓ such that $a_\ell = i$ and so every minor of $\mathcal{G}$ involves the i*th *row.*

Then $(S/I)_{x_{ij}} \cong (S/J)_{x_{ij}}$, where J is generated by the minors

$$[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$$

for $b_\ell \neq j$ with $\ell \in \{1, \dots, t\}$ and by the minors $[a_1 \dots \hat{a}_\ell \dots a_t | b_1 \dots \hat{b}_k \dots b_t]$ when $[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ for $a_\ell = i$ and $b_k = j$.

Proof. We assume for simplicity (and without loss of generality) that $i = 1$ and $j = 1$. We apply the automorphism $\varphi \colon S_{x_{11}} \to S_{x_{11}}$ with

$$x_{ij} \mapsto x'_{ij} = \begin{cases} x_{ij} + x_{i1}x_{11}^{-1}x_{1j} & \text{if } i \neq 1 \text{ and } j \neq 1, \\ x_{ij} & \text{if } i = 1 \text{ or } j = 1. \end{cases}$$

Let $I' \subset S_{x_{11}}$ be the ideal that is the image of $IS_{x_{11}}$ under the automorphism φ. Then $(S/I)_{x_{11}} \cong S_{x_{11}}/I'$. The ideal I' is generated in $S_{x_{11}}$ by the elements $\varphi(\mu_M)$, where $\mu_M \in \mathcal{G}$. Note that if $\mu_M = [a_1 \dots a_t | b_1 \dots b_t]$ then $\varphi(\mu_M) = \det(x'_{a_i b_j})_{i,j=1,\dots,t}$.

We can safely assume hereafter that $a_1 < a_2 < \cdots < a_t$ and $b_1 < b_2 < \cdots < b_t$ for $\mu_M = [a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$. Then our assumption implies that $a_1 = 1$. First consider the case $b_1 \neq 1$; then $\varphi(\mu_M)$ is the determinant of the matrix

$$\begin{pmatrix} x_{1b_1} & x_{1b_2} & \cdots & x_{1b_t} \\ x_{a_2b_1} + x_{a_21}x_{11}^{-1}x_{1b_1} & x_{a_2b_2} + x_{a_21}x_{11}^{-1}x_{1b_2} & \cdots & x_{a_2b_t} + x_{a_21}x_{11}^{-1}x_{1b_t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_tb_1} + x_{a_t1}x_{11}^{-1}x_{1b_1} & x_{a_tb_2} + x_{a_t1}x_{11}^{-1}x_{1b_2} & \cdots & x_{a_tb_t} + x_{a_t1}x_{11}^{-1}x_{1b_t} \end{pmatrix}.$$

After subtracting suitable multiples of the first row from the other rows, we see that

$$\varphi(\mu_M) = \det(x_{a_i b_j})_{i,j=1,\dots,t} = \mu_M.$$

If instead $b_1 = 1$ then the element $\varphi(\mu_M)$ is the determinant of the matrix

$$\begin{pmatrix} x_{11} & x_{1b_2} & \cdots & x_{1b_t} \\ x_{a_2 1} & x_{a_2 b_2} + x_{a_2 1} x_{11}^{-1} x_{1b_2} & \cdots & x_{a_2 b_t} + x_{a_2 1} x_{11}^{-1} x_{1b_t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_t 1} & x_{a_t b_2} + x_{a_t 1} x_{11}^{-1} x_{1b_2} & \cdots & x_{a_t b_t} + x_{a_t 1} x_{11}^{-1} x_{1b_t} \end{pmatrix};$$

applying suitable row operations, we obtain the matrix

$$\begin{pmatrix} 1 & x_{11}^{-1} x_{1b_2} & \cdots & x_{11}^{-1} x_{1b_t} \\ 0 & x_{a_2 b_2} & \cdots & x_{a_2 b_t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{a_t b_2} & \cdots & x_{a_t b_t} \end{pmatrix}.$$

It follows that $\varphi(\mu_M) = \det(x_{a_i b_j})_{i,j=2,\dots,t}$. These calculations show that $I' = JS_{x_{11}}$, as desired. □

Now we are ready to prove this section's main result.

Theorem 2.2. *Let $m \le n$, let Δ be a pure $(m-1)$-dimensional closed simplicial complex on the vertex set $[n]$, and let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ. If J_Δ is a prime ideal then, for all $2 \le t \le \min(m, r)$ and for any pairwise distinct cliques $\Delta_{i_1}, \dots, \Delta_{i_t}$,*

$$|V(\Delta_{i_1}) \cap \cdots \cap V(\Delta_{i_t})| \le m - t.$$

Proof. We proceed by induction on m. The initial step, $m = 2$, is already known [12].

Let us make the inductive step. We first consider $t < m$. Let us assume that there exist $\Delta_{i_1}, \dots, \Delta_{i_t}$ such that $|V(\Delta_{i_1}) \cap \cdots \cap V(\Delta_{i_t})| > m - t$. Without loss of generality, we may assume that $V(\Delta_1) \cap \cdots \cap V(\Delta_t) = \{a_1, a_2, \dots, a_\ell\}$ with $\ell \ge m - t + 1$ and $1 \le a = a_1 < \cdots < a_\ell \le n$. We may further assume that there exists an $s \ge t$ such that $a \in V(\Delta_i)$ for $1 \le i \le s$ and $a \notin V(\Delta_i)$ for $s + 1 \le i \le r$. Since J_Δ is prime, it follows that x_{ma} is regular on J_Δ and that $J_\Delta S_{x_{ma}}$ is also a prime ideal in the localization $S_{x_{ma}}$ of S. Thus $(S/J_\Delta)_{x_{ma}}$ is a domain. Then by Lemma 2.1 we have $(S/J_\Delta)_{x_{ma}} \cong (S/L)_{x_{ma}}$; here $L = L_1 + \sum_{i=s+1}^{r} J_{\Delta_i}$ with L_1 the determinantal facet ideal of the closed $(m-2)$-dimensional simplicial complex Δ' having the clique decomposition $\Delta' = \Delta'_1 \cup \cdots \cup \Delta'_s$, where $\Delta'_i = \langle F \setminus \{a\} : F \in \mathcal{F}(\Delta_i), a \in F \rangle$ for $1 \le i \le s$. Since $\Delta'_1, \dots, \Delta'_t$ intersect in $\ell - 1 \ge m - t$ vertices, by induction it follows that L_1 is not a prime ideal; this implies (as we shall show) that L is not a prime ideal. Yet this is a contradiction because $(S/L)_{x_{ma}}$ must be a domain.

Since L_1 is not prime, there exist polynomials f, g in S such that $fg \in L_1$ and $f, g \notin L_1$. We claim that $f, g \notin L$. Let us assume, for instance, that $f \in L$. Then we may write $f = \sum_G h_G \gamma_G + \sum_F h_F \mu_F$ for some polynomials $h_G, h_F \in S$, where the first sum is taken over all $G \in \bigcup_{i=1}^{s} \mathcal{F}(\Delta_i')$ and the second over all $F \in \bigcup_{i=s+1}^{r} \mathcal{F}(\Delta_i)$. Then, after mapping the indeterminates x_{mj} to 0 for all $j \neq a$ and the determinants x_{ma} to 1, we obtain $f = \sum_G h_G' \gamma_G$ for some polynomials $h_G' \in S$; hence $f \in L_1$, a contradiction. Therefore, L is not a prime ideal.

It remains to consider the case $t = m$. We may assume that

$$|V(\Delta_1) \cap \cdots \cap V(\Delta_m)| \geq 1.$$

Let $a \in V(\Delta_1) \cap \cdots \cap V(\Delta_m)$. It is clear that $J_\Delta \subset (J_{\Delta'}, x_{1a}, \ldots, x_{ma})$, where $\Delta' = \{F \in \Delta : a \notin F\}$. Since Δ is closed, it follows that Δ' is also closed; moreover, by Corollary 1.3 we have

$$\text{height } J_{\Delta'} = \sum_{i=1}^{m}((n_i - 1) - m + 1) + \sum_{i=m+1}^{r}(n_i - m + 1) = \text{height } J_\Delta - m.$$

Since $x_{1a}, \ldots, x_{ma}$ is obviously a regular sequence on $S/J_{\Delta'}$, it follows that

$$\text{height}(J_{\Delta'}, x_{1a}, \ldots, x_{ma}) = \text{height } J_{\Delta'} + m = \text{height } J_\Delta.$$

Let P be a minimal prime of $(J_{\Delta'}, x_{1a}, \ldots, x_{ma})$ of height equal to $\text{height}(J_\Delta)$. Because J_Δ and P are prime ideals of the same height, we must have $J_\Delta = P$. But P contains the indeterminates $x_{1a}, \ldots, x_{ma}$, which do not belong to J_Δ. We have thus obtained a contradiction, proving the theorem. $\square$

The proofs of primality that follow depend on localization with respect to nonzero divisors. According to the next result, all variables in our situation are nonzero divisors.

LEMMA 2.3. *Let Δ be a closed $(m-1)$-dimensional simplicial complex with the property that any m pairwise distinct cliques of Δ have an empty intersection. Then each of the variables x_{ij} is regular modulo J_Δ.*

Proof. We may assume from the outset that the field K is infinite—given that neither the hypothesis nor the conclusion of the lemma is affected by tensoring with a field extension of K. In order to show that x_{ij} is regular modulo J_Δ, we consider the ideal

$$I = (J_\Delta, x_{1j}, \ldots, x_{mj}).$$

Let Δ' be the simplicial complex whose facets are those of Δ that do not contain j. Observe that Δ' is again closed and that $I = (J_{\Delta'}, x_{1j}, \ldots, x_{mj})$. We use the formula in Corollary 1.3 to compare the height of I with that of J_Δ. If $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ is the clique decomposition of Δ with $n_i = |\Delta_i|$, then $\text{height } J_\Delta = \sum_{i=1}^{r}(n_i - m + 1)$.

We may assume that Δ_i contains the vertex j for $i = 1, \ldots, s$. Our other assumptions imply that $s \leq m - 1$. Note that the clique decomposition of Δ' is $\Delta_1' \cup \cdots \cup \Delta_r'$, where the facets of each Δ_i' are those facets of Δ_i that do not contain j. Therefore, $|\Delta_i'| = |\Delta_i| - 1 = n_i - 1$ for $i = 1, \ldots, s$ and $\Delta_i' = \Delta_i$ for $i > s$. Hence we obtain

$$\begin{aligned}\text{height } I = \text{height } J_{\Delta'} + m &= \sum_{i=1}^{s}(n_i - 1 - m + 1) + \sum_{i=s+1}^{r}(n_i - m + 1) + m \\ &= \text{height } J_\Delta - s + m > \text{height } J_\Delta.\end{aligned}$$

Our considerations show that $I/J_\Delta \subset S/J_\Delta$ has positive height. Since S/J_Δ is Cohen–Macaulay and since K is infinite, it follows that a generic linear combination $a_1x_{1j} + a_2x_{2j} + \cdots + a_mx_{mj}$ of the variables $x_{1j}, \ldots, x_{mj}$ (whose residue classes generate I/J_Δ) is regular modulo J_Δ. Because the preceding linear combination is generic, we may assume that $a_i = 1$.

Now we consider the linear automorphism $\varphi\colon S \to S$ with $\varphi(x_{ik}) = a_1x_{1k} + a_2x_{2k} + \cdots + a_mx_{mk}$ for $k = 1, \ldots, n$ and $\varphi(x_{\ell k}) = x_{\ell k}$ for $\ell \neq i$ and all k. Let X' be the matrix whose entries are the elements $\varphi(x_{\ell k})$ for $\ell = 1, \ldots, m$ and $k = 1, \ldots, n$. Then X' is obtained from X by elementary row operations. It follows that $\varphi(J_\Delta) = J_\Delta$.

Our choice of φ implies that $y_{ij} = \varphi(x_{ij})$ is regular modulo J_Δ. Since $J_\Delta = \varphi(J_\Delta)$ it follows that $x_{ij} = \varphi^{-1}(y_{ij})$ is regular modulo $\varphi^{-1}(J_\Delta) = \varphi^{-1}(\varphi(J_\Delta)) = J_\Delta$, as desired. □

We do not know whether, for a closed simplicial complex Δ, the necessary condition (given in Theorem 2.2) for J_Δ to be a prime ideal is also sufficient. For the moment we can only present a partial converse of this result.

Proposition 2.4. *Let Δ be a simplicial complex with clique decomposition $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$. Assume that all cliques are simplices of dimension $m-1$ and that the following statements hold*:

(1) $|V(\Delta_r) \cap \cdots \cap V(\Delta_{r-s+1})| \leq m - s$ *for* $s = 2, \ldots, r$;
(2) $V(\Delta_{i_1}) \cap \cdots \cap V(\Delta_{i_s}) \subset V(\Delta_r) \cap \cdots \cap V(\Delta_{r-s+1})$ *for all subsets* $\{i_1, \ldots, i_s\} \subset [r]$ *of cardinality* s *with* $2 \leq s \leq r$.

Then J_Δ is a prime ideal.

Proof. Again we proceed by induction on m. The initial step, $m = 2$, is already known [12]. Assume that $|V(\Delta_1) \cap \cdots \cap V(\Delta_r)| = k$. We consider a labeling on the vertices of Δ such that

$$V(\Delta_\ell) = \{a_{\ell 1} < \cdots < a_{\ell, m-k-\ell+1} < b_1 < \cdots < b_k < c_{\ell 1} < \cdots < c_{\ell, \ell-1}\}$$

for all $\ell = 1, \ldots, r$, where the numbers a_{ij} are pairwise distinct. For each $s = 2, \ldots, r$, we choose c_{ij} such that

$$c_{rj} = c_{r-1,j} = \cdots = c_{r-s+1,j}$$

for $j = 1, \ldots, |V(\Delta_r) \cap \cdots \cap V(\Delta_{r-s+1})| - k$.

Then, with respect to this labeling, Δ is closed and so (by Lemma 2.3) x_{mb_1} is a regular element modulo J_Δ. Then by Lemma 2.1 we have $(S/J_\Delta)_{x_{mb_1}} \cong (S/L)_{x_{mb_1}}$, where $L = \sum_{i=1}^r L_i$. Since x_{mb_1} is regular modulo J_Δ, it follows that J_Δ is a prime ideal if and only if $L_{x_{mb_1}}$ is a prime ideal; here L_i is generated by the minor

$$[1 \ldots m-1 | a_{i1} \ldots a_{i,m-k-i+1} b_2 \ldots b_k c_{i1} \ldots c_{i,i-1}].$$

Let Δ' be the $(m-2)$-simplicial complex with the clique decomposition $\Delta'_1 \cup \cdots \cup \Delta'_t$, where $\Delta'_i = \Delta_i \setminus \{b_1\}$. Note that conditions (1) and (2) hold for Δ'. Then, by the inductive hypothesis, $L = J_{\Delta'}$ is a prime ideal. □

EXAMPLE 2.5. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ under the assumption of Theorem 2.4. Then we can describe the vertices of each Δ_i in a nice way as the ith row of a simple matrix. For instance, let $m = 6$, $r = 4$, $|V(\Delta_4) \cap V(\Delta_3)| = 3$, $\left|\bigcap_{i=2}^4 V(\Delta_i)\right| = 3$, and $\left|\bigcap_{i=1}^4 V(\Delta_i)\right| = 2$. Then, by the proof of Proposition 2.4, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & b_1 & b_2 \\ 5 & 6 & 7 & b_1 & b_2 & c_1 \\ 8 & 9 & b_1 & b_2 & c_1 & c_2 \\ 10 & b_1 & b_2 & c_1 & c_3 & c_4 \end{pmatrix}$$

which describes the labels of the $\Delta_1, \ldots, \Delta_4$.

EXAMPLE 2.6. Let $\mathcal{F}(\Delta) = \{\{1,2,3\},\{1,4,5\},\{3,5,6\},\{2,4,6\}\}$. Then one may check with SINGULAR [9] that J_Δ is not a prime ideal. However, the intersection condition of Theorem 2.2 holds for Δ. Hence the converse of Theorem 2.2 requires that Δ be a closed simplicial complex.

This is also an example of a determinantal facet ideal whose initial ideal with respect to the lexicographic order is not squarefree even though J_Δ is a radical ideal.

3. Special Classes of Prime Determinantal Facet Ideals

Let Δ a pure simplicial complex of dimension $m-1 \geq 2$, and let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be its clique decomposition. In this section we pose the following intersection properties on the cliques of Δ:

(i) $|V(\Delta_i) \cap V(\Delta_j)| \leq 1$ for all $i < j$;
(ii) $V(\Delta_i) \cap V(\Delta_j) \cap V(\Delta_k) = \emptyset$ for all $i < j < k$.

Theorem 2.2 implies that: (a) for $m = 3$, conditions (i) and (ii) are satisfied whenever J_Δ is a prime ideal; and (b) for any $m \geq 3$, (i) and (ii) imply the intersection conditions formulated in Theorem 2.2.

In this section we show that, whenever Δ is closed, conditions (i) and (ii) entail the primality of J_Δ under some additional assumptions depending on a graph that we shall define next. For the simplicial complex with properties (i) and (ii), let G_Δ be the simple graph with vertex set $V(G_\Delta) = \{v_1, \ldots, v_r\}$ and edge set

$$E(G_\Delta) = \{\{v_i, v_j\} : V(\Delta_i) \cap V(\Delta_j) \neq \emptyset\}.$$

Hereafter, the phrase "Δ is a simplicial complex with graph G_Δ" will always imply that Δ satisfies the conditions (i) and (ii) (for otherwise G_Δ is not defined).

At present we are able to prove the primality of J_Δ for certain classes of simplicial complexes Δ only under the additional assumption that these complexes are closed. The following lemma provides a necessary condition for a simplicial complex to be closed.

LEMMA 3.1. *Let Δ be a closed simplicial complex with graph G_Δ. Then each vertex v_i of G_Δ has order at most* $\min\{|V(\Delta_i)|, 2\dim(\Delta)\}$.

Proof. We say that a vertex $\ell \in \Delta_i$ *takes the position s* if there is an $(m-1)$-dimensional face $\{a_1 < a_2 < \cdots < a_m\}$ of Δ_i such that $\ell = a_s$. In the clique Δ_i

there are exactly $\min\{|V(\Delta_i)|, 2\dim(\Delta)\}$ vertices that do not take all m positions. The proof now follows from assumption (ii), which implies that each of these vertices can intersect with at most one clique Δ_j (where v_j is a neighbor of v_i). $\square$

Now we are ready to consider the primality of J_Δ for special classes of simplicial complexes.

Theorem 3.2. *Let Δ be simplicial complex such that G_Δ is a tree. Then*

(a) *J_Δ is a prime ideal if Δ is closed, and*
(b) *Δ is closed if and only if each vertex of G_Δ has order at most* $\min\{|V(\Delta_i)|, 2\dim(\Delta)\}$.

Let $\{i_1 < \cdots < i_s\} \subset [m]$ and $\{j_1 < \cdots < j_t\} \subset [n]$. We denote by $X^{j_1 j_2 \dots j_t}_{i_1 \dots i_s}$ the submatrix of X with rows $i_1, \dots, i_s$ and columns $j_1, \dots, j_t$. Observe that Lemma 2.1 implies the well-known fact that if I is generated by all m-minors of the matrix $X^{j_1 \dots j_t}_{1 \dots m}$ then $I_{x_{ij_k}}$ is generated by all $(m-1)$-minors of $X^{j_1 \dots \hat{j}_k \dots j_t}_{1 \dots \hat{i} \dots m}$.

Proof of Theorem 3.2. (a) We may assume that Δ is a connected $(m-1)$-dimensional simplicial complex and that $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$ is the clique decomposition of Δ. The proof is by induction on the number of cliques of Δ (which is the number of vertices of G_Δ). We may assume that v_1 is a vertex of degree 1 in G_Δ and that v_2 is its neighbor. Then Δ_1 intersects with just one clique—namely, Δ_2.

Let $V(\Delta_1) = \{j_1, \dots, j_t\}$ and $V(\Delta_2) = \{\ell_1, \dots, \ell_s\}$ with $m \le t, s$. We may assume that $V(\Delta_1) \cap V(\Delta_2) = \{k\}$, where $k = j_1 = \ell_1$. Since Δ is closed, by Lemma 2.3 we know that x_{mk} is regular modulo J_Δ. It follows from Lemma 2.1 that $(S/J_\Delta)_{x_{mk}} \cong (S/L)_{x_{mk}}$, where $L = L_1 + L_2 + \sum_{i=3}^r J_{\Delta_i}$. Here L_1 is generated by all $(m-1)$-minors of the matrix $X^{j_2 \dots j_t}_{1 \dots m-1}$ and L_2 is generated by all $(m-1)$-minors of the matrix $X^{\ell_2 \dots \ell_s}_{1 \dots m-1}$. The generators of L_1 are polynomials in a set of variables disjoint from those of $L' = L_2 + \sum_{i=3}^r J_{\Delta_i}$. It is known that L_1 is a prime ideal (see [3, Thm. 7.3.1]); therefore, L is a prime ideal if and only if L' is a prime ideal. To see this, observe that $(S/L')_{x_{mk}} \cong (S/J_{\Delta'})_{x_{mk}}$, where Δ' is the closed simplicial complex with clique decomposition $\Delta' = \Delta_2 \cup \cdots \cup \Delta_r$. By the inductive hypothesis, $J_{\Delta'}$ is a prime ideal. Hence $(S/L')_{x_{mk}} \cong (S/J_{\Delta'})_{x_{mk}}$, which implies that $(J_{\Delta'})_{x_{mk}}$ is a prime ideal. Since the generators of L' are polynomials in variables different from x_{mk}, it follows that x_{mk} is regular modulo L'. Consequently, L' is a prime ideal.

(b) According to Lemma 3.1, it suffices to show that Δ is closed if each vertex of G_Δ has order at most $\min\{|V(\Delta_i)|, 2\dim(\Delta)\}$. We prove the assertion by induction on r. As before, we assume that Δ_1 intersects with just one clique: Δ_2. By induction it follows that $\Delta' = \Delta_2 \cup \cdots \cup \Delta_r$ is closed. Our assumption on the order of the vertices of G_Δ implies that Δ_2 has at most $\min\{|V(\Delta_i)|, 2\dim(\Delta)\} - 1$ intersection points in Δ'.

So among the vertices of Δ_2 that are not intersection points in Δ', there is at least one that does not take all m positions; say it misses the kth position. By symmetry we may assume that this vertex is the intersection point with Δ_1. Now we may label Δ_1 such that the vertex in the intersection point does not have position k for any facet of Δ in Δ_1. $\square$

THEOREM 3.3. *Let Δ be a simplicial complex such that G_Δ is a cycle. Then J_Δ is a prime ideal.*

Proof. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ. We consider a labeling on the vertices of Δ such that

$$V(\Delta_1) = \{1, 2, \ldots, a_1\},\ V(\Delta_2) = \{a_1, a_1 + 1, \ldots, a_2\},\ \ldots,$$
$$V(\Delta_{r-1}) = \{a_{r-2}, a_{r-2} + 1, \ldots, a_{r-1}\},\ V(\Delta_r) = \{a_1 - 1, a_{r-1}, a_{r-1} + 1, \ldots, a_r\},$$

where $1 < a_1 < \cdots < a_{r-1} < a_r = n$. Then Δ is closed with respect to the given labeling and, by Lemma 2.3, x_{1a_1} is a regular element modulo J_Δ. It follows from Lemma 2.1 that $(S/J_\Delta)_{x_{1a_1}} \cong (S/L)_{x_{1a_1}}$, where $L = L_1 + L_2 + \sum_{i=3}^{r} J_{\Delta_i}$. Here L_1 is generated by all $(m-1)$-minors of the matrix $X_{2\ldots m}^{1\ldots a_1-1}$ and L_2 is generated by all $(m-1)$-minors of $X_{2\ldots m}^{a_1+1\ldots a_2}$. Hence J_Δ is a prime ideal if $L_{x_{1a_1}}$ is a prime ideal. Since the generators of L are polynomials in variables different from x_{1a_1}, we conclude that x_{1a_1} is regular modulo L. Therefore, J_Δ is a prime ideal if and only if L is a prime ideal.

We first show that the generators of L form a Gröbner basis for L. Toward this end, we observe that the generators of $\sum_{i=3}^{r} J_{\Delta_i} = J_{\Delta_3 \cup \cdots \cup \Delta_r}$ form a Gröbner basis for $\sum_{i=3}^{r} J_{\Delta_i}$ because $\Delta_3 \cup \cdots \cup \Delta_r$ is closed. Also the generators of L_1 form a Gröbner basis for J_{Γ_1}, where Γ_1 is the pure $(m-2)$-dimensional simplicial complex on the vertices $\{1, \ldots, a_1 - 1\}$, and the generators of L_2 form a Gröbner basis for J_{Γ_2}, where Γ_2 is the pure $(m-2)$-dimensional simplicial complex on the vertices $\{a_1 + 1, \ldots, a_2\}$. Finally, we note that the initial ideals of $\sum_{i=3}^{r} J_{\Delta_i}$, L_1, and L_2 are each minimally generated by monomials in pairwise disjoint sets of variables. As a result, the generators of L do indeed form a Gröbner basis.

Next observe that the variable x_{m-1,a_1-1} does not appear in the support of the generators of $\mathrm{in}_<(L)$. In particular, x_{m-1,a_1-1} is regular modulo L. By using Lemma 2.1 we get $(S/L)_{x_{m-1,a_1-1}} \cong \left(S/L_1' + L_2 + L_r + \sum_{i=3}^{r-1} J_{\Delta_i}\right)_{x_{m-1,a_1-1}}$, where L_1' is generated by all $(m-2)$-minors of the matrix $X_{2\ldots m-2,m}^{1\ldots a_1-2}$ and L_r is generated by all $(m-1)$-minors of $X_{1\ldots m-2,m}^{a_{r-1}\ldots a_r}$.

Since the generators of $L' = L_1' + L_2 + L_r + \sum_{i=3}^{r-1} J_{\Delta_i}$ are polynomials in variables different from x_{m-1,a_1-1}, we conclude that x_{m-1,a_1-1} is regular modulo L'. Hence $L_{x_{m-1,a_1-1}}$ is a prime ideal if and only if L' is a prime ideal. Since L_1' is a prime ideal and the generators of L_1' are polynomials in variables different from the variables of the other summands, to prove that L' is prime it suffices to show that $C = L_2 + L_r + \sum_{i=3}^{r-1} J_{\Delta_i}$ is a prime ideal.

We define the pure $(m-1)$-simplicial complex Δ' to be the simplicial complex with clique decomposition $\Delta' = \Delta_2 \cup \cdots \cup \Delta_r$. Because the associated graph of Δ' is a tree, we know from Theorem 3.2 that $J_{\Delta'}$ is a prime ideal. Since $(S/C)_{x_{1a_1}x_{m-1,a_1-1}} \cong (S/J_{\Delta'})_{x_{1a_1}x_{m-1,a_1-1}}$ and since $x_{1a_1}x_{m-1,a_1-1}$ is regular modulo C, the desired conclusion follows. $\square$

Our next result describes the case when each clique of Δ is a simplex.

THEOREM 3.4. *Let Δ be a simplicial complex with graph G_Δ such that each clique of Δ is a simplex. Then the following statements hold.*

(a) *If* Δ *is closed, then* J_Δ *is generated by a regular sequence.*
(b) *Given a graph* G *and an integer* $m \geq |V(G)|$, *there exists a closed simplicial complex* Δ *with* $G_\Delta = G$ *such that each clique of* Δ *is a simplex of dimension* $m-1$.
(c) Δ *is closed if* $\dim \Delta + 1$ *is no less than the number of facets of* Δ.

Proof. (a) Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ. Since each clique is a simplex, it follows that $J_{\Delta_i} = (f_i)$ for all i (where f_i is a suitable m-minor) and that $J_\Delta = (f_1, \dots, f_r)$. Since Δ is closed, the monomials $\mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_r)$ are pairwise relatively prime. This implies that $f_1, \dots, f_r$ is a regular sequence.

(b) We first assume that $m = |V(G)|$ and prove the assertion in this case by induction on the number of vertices of G. The induction beginning is trivial. Now assume that $|G| > 1$, and choose a vertex v of G. Let G' be the induced subgraph on the vertices $V(G) \setminus \{v\}$. By induction, for each $w \in V(G')$ there exists a labeled simplex Δ'_w with $\dim \Delta'_w + 1 = |V(G')| = |V(G)| - 1$ such that the simplicial complex Δ' with clique decomposition $\bigcup_{w \in V(G')} \Delta'_w$ is closed and $G_{\Delta'} = G'$. We define new simplices $\Delta_w = \Delta'_w \cup \{a_w\}$, where the labels a_w are pairwise distinct and are bigger than all labels of Δ'.

Let $w_1, \dots, w_r$ be the neighbors of v in G. Then we let Δ_v be the simplex whose vertices are labeled by the integers $a_{w_1}, \dots, a_{w_r}$ together with $|V(G)| - r$ numbers that are all bigger than all labels used in the construction so far.

Now let $m > |V(G)|$, and let Γ be the closed simplicial complex with $\dim \Gamma = |V(G)| - 1$ that we have just constructed. For each labeled simplex Γ_i of Γ that is of dimension $|V(G)| - 1$, we define the new labeled simplex $\Delta_i = \Gamma_i \cup \{b_{i1}, \dots, b_{is}\}$; here $s = m - |V(G)|$ and the numbers b_{ij} are pairwise distinct and bigger than all labels of Γ. The simplicial complex Δ with facets Δ_i has the desired properties.

(c) Let Δ be a simplicial complex with graph G_Δ such that each clique of Δ is a simplex. Then, up to an isomorphism, Δ is uniquely determined by $\dim \Delta$ and G_Δ. Hence (c) is a simple consequence of (b). $\square$

Corollary 3.5. *Let* Δ *be a simplicial complex with graph* G_Δ *such that each clique of* Δ *is a simplex of dimension* $m-1$. *Suppose that* G_Δ *is the complete graph* K_r. *Then* Δ *is closed if and only if* $m \geq r$.

Proof. Each vertex of K_r has order $r-1$. Hence $m \geq r-1$, for otherwise we could not associate the graph G_Δ to Δ. If $m = r-1$ then Δ has no free vertex, so Δ cannot be closed. Yet if $m \geq r$, the assertion follows from Theorem 3.4. $\square$

References

[1] W. Bruns and A. Conca, *Gröbner bases and determinantal ideals,* Commutative algebra, singularities and computer algebra (Sinaia, 2002), NATO Sci. Ser. II Math. Phys. Chem., 115, pp. 9–66, Kluwer, Dordrecht, 2003.

[2] W. Bruns and J. Herzog, *On the computation of a-invariants,* Manuscripta Math. 77 (1992), 201–213.

[3] ———, *Cohen–Macaulay rings,* Cambridge Stud. Adv. Math., 39, Cambridge Univ. Press, Cambridge, 1993.

[4] W. Bruns and U. Vetter, *Determinantal rings,* Lecture Notes in Math., 1327, Springer-Verlag, Berlin, 1988.

[5] L. Caniglia, J. A. Guccione, and J. J. Guccione, *Ideals of generic minors,* Comm. Algebra 18 (1990), 2633–2640.

[6] A. Conca, *Ladder determinantal rings,* J. Pure Appl. Algebra 98 (1995), 119–134.

[7] A. Conca and J. Herzog, *On the Hilbert function of determinantal rings and their canonical module,* Proc. Amer. Math. Soc. 122 (1994), 677–681.

[8] V. Ene and J. Herzog, *Gröbner bases in commutative algebra,* Grad. Stud. Math., 130, Amer. Math. Soc., Providence, RI, 2011.

[9] G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 2.0: *A computer algebra system for polynomial computations,* Centre for Computer Algebra, University of Kaiserslautern, 2001, ⟨http://www.singular.uni-kl.de⟩.

[10] J. Herzog and T. Hibi, *Monomial ideals,* Grad. Texts in Math., 260, Springer-Verlag, London, 2010.

[11] ———, *Ideals generated by adjacent 2-minors,* preprint, 2010, arXiv:1012.5789v3.

[12] J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, and J. Rauh, *Binomial edge ideals and conditional independence statements,* Adv. in Appl. Math. 45 (2010), 317–333.

[13] J. Herzog and N. V. Trung, *Gröbner bases and multiplicity of determinantal and Pfaffian ideals,* Adv. Math. 96 (1992), 1–37.

[14] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws,* Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., 11, pp. 93–109, North-Holland, Amsterdam, 1987.

[15] M. Hochster and J. A. Eagon, *Cohen–Macaulay rings, invariant theory and the generic perfection of determinantal loci,* Amer. J. Math. 93 (1971), 1020–1058.

[16] S. Hoşten and S. Sullivant, *Ideals of adjacent minors,* J. Algebra 277 (2004), 615–642.

[17] B. Sturmfels, *Gröbner bases and Stanley decompositions of determinantal rings,* Math. Z. 205 (1990), 137–144.

V. Ene
Faculty of Mathematics and Computer Science
Ovidius University
Bd. Mamaia 124
900527 Constanta
Romania
vivian@univ-ovidius.ro

J. Herzog
Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen
45117 Essen
Germany
juergen.herzog@uni-essen.de

T. Hibi
Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka
Osaka 560-0043
Japan
hibi@math.sci.osaka-u.ac.jp

F. Mohammadi
Mathematical Sciences Research Institute
Berkeley, CA 94720
fatemeh.mohammadi716@gmail.com