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Geometry of Brill–Noether Loci on Prym Varieties

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1. Introduction

Given a smooth curve X, it is well known that the Brill–Noether loci $W_d^r X$ contain much interesting information about the curve X and its polarized Jacobian (JX, Θ_X). Given a smooth curve C and an étale double cover $\pi : \tilde{C} \to C$, one can analogously define Brill–Noether loci V^r for the Prym variety (P, Θ) (see Section 2). Several fundamental results on these loci have been known for some time: the expected dimension is $g(C) - 1 - \binom{r+1}{2}$, the loci are nonempty if the expected dimension is nonnegative [Ber, Thm. 1.4], and they are connected if the expected dimension is positive [D3, Exm. 6.2]. If C is general in the moduli space of curves, then all the Brill–Noether loci are smooth and have the expected dimension [W2, Thm. 1.11]. Whereas the Brill–Noether locus $V^1 \subset P^+$ is the canonically defined theta-divisor and has received the attention of many authors, the study of higher Brill–Noether loci (and the information they contain about the étale cover $\pi : \tilde{C} \to C$) is a more recent development. Casalaina-Martin, Lahoz, and Viviani [CaLV] show that V^2 is set-theoretically the theta-dual (cf. Definition 2.1) of the Abel–Prym curve. Lahoz and Naranjo [LN] refine this statement and prove a Torelli theorem: the Brill–Noether locus V^2 determines the covering $\tilde{C} \to C$. That finding motivates a more detailed study of the geometry of V^2. Our first result is as follows.

1.1. Theorem. *Let C be a smooth curve of genus $g(C) \geq 6$, and let $\pi : \tilde{C} \to C$ be an étale double cover such that the Prym variety (P, Θ) is an irreducible principally polarized abelian variety.*

(a) *Suppose that C is hyperelliptic. Then V^2 is irreducible of dimension $g(C) - 3$.*
(b) *Suppose that C is not hyperelliptic. Then V^2 is a reduced Cohen–Macaulay scheme of dimension $g(C) - 4$. If the singular locus V^2_{sing} has an irreducible component of dimension at least $g(C) - 5$, then C is a plane quintic, trigonal, or bielliptic.*

The condition on the irreducibility is always satisfied unless C is hyperelliptic and $\tilde{C}$ is not. In that case, (P, Θ) is isomorphic to a product of Jacobians [M2].

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In the hyperelliptic case (cf. Proposition 4.2), the statement is a straightforward extension of [CaLV]. In the non-hyperelliptic case, it is based on the following observation: if the singular locus of V^2 is large, then the singularities are exceptional in the sense of [B3]. This provides a link with certain Brill–Noether loci on JC.

An immediate consequence of the theorem is that V^2 is irreducible unless C is a plane quintic, trigonal, or bielliptic (Corollary 3.6). The case of trigonal curves is very simple: (P, Θ) is isomorphic to a Jacobian JX and V^2 splits into two copies of $W^0_{g(C)-4}X$. For a plane quintic, V^2 is reducible if and only if (P, Θ) is isomorphic to the intermediate Jacobian of a cubic threefold; in this case, V^2 splits into two copies of the Fano surface F. Note that the Fano surface F and the Brill–Noether loci $W^0_d X$ are expected to be the only subvarieties of principally polarized abelian varieties having the minimal cohomology class $\left[\frac{\Theta^k}{k!}\right]$ [D2]. By [dCPr], the cohomology class of V^2 is $\left[2\frac{\Theta^{g(C)-4}}{(g(C)-4)!}\right]$; therefore, a reducible V^2 provides an important test for this conjecture. Our second result is the following theorem.

1.2. THEOREM. *Let C be a smooth non-hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi : \tilde{C} \to C$ be an étale double cover. Denote by (P, Θ) the polarized Prym variety. The Brill–Noether locus V^2 is reducible if and only if at least one of the following statements holds*:

(a) *C is trigonal*;
(b) *C is a plane quintic and (P, Θ) an intermediate Jacobian of a cubic threefold*;
(c) *C is bielliptic and the covering $\pi : \tilde{C} \to C$ belongs to the family $\mathcal{R}_{\mathcal{B}_{g(C),g(C_1)}}$ with $g(C_1) \geq 2$ (cf. Remark 5.11). Then V^2 has two or three irreducible components, but none of them has minimal cohomology class.*

If C is bielliptic of genus $g(C) \geq 8$, then the Prym variety is not a Jacobian of a curve [S]. Moreover, these Prym varieties form $\left\lfloor\frac{g(C)-1}{2}\right\rfloor$ distinct subvarieties of $\mathcal{A}_{g(C)-1}$ [D1]. For exactly one of these families, the general member has the property that the cohomology class of *any* subvariety is a multiple of the minimal class $\frac{\Theta^k}{k!}$. The proof of Theorem 1.2 shows that the Brill–Noether locus V^2 is irreducible if and only if the Prym variety belongs to this family! This is the first evidence for Debarre's conjecture that is not derived from low-dimensional cases or considerations on Jacobians and intermediate Jacobians (cf. [D2, Hö2, R]).

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2. Notation

Most of our arguments are valid for an arbitrary algebraically closed field of characteristic $\neq 2$. However, we work over $\mathbb{C}$ so that we can apply [ACGH] and [D3], which are crucial for Theorem 1.1 and its consequences. For standard definitions in algebraic geometry we refer to [Ha] and for Brill–Noether theory to [ACGH].

Given a smooth curve C, we denote by $\operatorname{Pic} C$ its Picard scheme and by

$$\operatorname{Pic} C = \bigcup_{d \in \mathbb{Z}} \operatorname{Pic}^d C$$

the decomposition into its irreducible components. We will identify the Jacobian JC and the degree-0 component $\operatorname{Pic}^0 C$ of the Picard scheme. In order to simplify the notation we denote by $L \in \operatorname{Pic} C$ the point corresponding to a given line bundle L on C. We will abuse terminology somewhat and say that a line bundle is *effective* if it has a global section.

For $\varphi\colon X \to Y$ a finite cover between smooth curves and D a divisor on X, we denote the norm by $\operatorname{Nm}\varphi(D)$. In the same way, $\operatorname{Nm}\varphi\colon \operatorname{Pic} X \to \operatorname{Pic} Y$ denotes the norm map. If $\mathcal{F}$ is a coherent sheaf on X (in general, $\mathcal{F}$ will be the locally free sheaf corresponding to some divisor), then we denote by $\varphi_* \mathcal{F}$ the push-forward as a sheaf.

Let C be a smooth curve of genus $g(C)$ and let $\pi : \tilde{C} \to C$ be an étale double cover. We have $(\operatorname{Nm}\pi)^{-1}(K_C) = P^+ \cup P^-$, where $P^- \simeq P^+ \simeq P$ are defined by

$$P^- := \{L \in (\operatorname{Nm}\pi)^{-1}(K_C) \mid \dim|L| \equiv 0 \bmod 2\},$$

$$P^+ := \{L \in (\operatorname{Nm}\pi)^{-1}(K_C) \mid \dim|L| \equiv 1 \bmod 2\}.$$

For $r \geq 0$ we set

$$W^r_{2g(C)-2}\tilde{C} := \{L \in \operatorname{Pic}^{2g(C)-2} \tilde{C} \mid \dim|L| \geq r\}.$$

The Brill–Noether loci of the Prym variety [W2] are defined as the scheme-theoretical intersections

$$V^r := \begin{cases} W^r_{2g(C)-2}\tilde{C} \cap P^- & \text{if } r \text{ is even,} \\ W^r_{2g(C)-2}\tilde{C} \cap P^+ & \text{if } r \text{ is odd.} \end{cases}$$

The notion of theta-dual was introduced by Pareschi and Popa in their work on Fourier–Mukai transforms (see [PPo2] for a survey).

2.1. Definition. Let (A, Θ) be a principally polarized abelian variety, and let $X \subset A$ be any closed subset. Then the *theta-dual* $T(X)$ of X is the maximal subset $Z \subset A$ such that $A - Z \subset \Theta$.

Note that $T(X)$ has a natural scheme structure [PPo2].

3. The Singular Locus of V^2

Throughout this section we denote by C a smooth non-hyperelliptic curve of genus $g(C)$ and by $\pi : \tilde{C} \to C$ an étale double cover. The following lemma will be used repeatedly.

3.1. Lemma. *Let $L \in V^r$ be a line bundle such that* $\dim|L| = r$. *If the Zariski tangent space $T_L V^r$ satisfies*

$$\dim T_L V^r > g(C) - 2r,$$

then there exist

(a) *a line bundle M on C such that* $\dim|M| \geq 1$ *and*
(b) *an effective line bundle F on $\tilde{C}$ such that*

$$L \simeq \pi^* M \otimes F.$$

3.2. REMARK. For $r = 1$, the scheme $V^1 = W^1_{2g(C)-2}\tilde{C} \cap P^+$ identifies with the canonical polarization Θ. The theta-divisor has dimension $g(C) - 2$, so the condition

$$\dim T_L V^1 > g(C) - 2$$

is equivalent to V^1 being singular in L. Thus, for $r = 1$ we obtain the well-known statement that if a point $L \in \Theta$ with $\dim|L| = 1$ is in Θ_{sing} then the singularity is exceptional (in the sense of Beauville [B3]).

Proof of Lemma 3.1. We consider the Prym–Petri map introduced by Welters [W2, 1.8]:

$$\beta \colon \wedge^2 H^0(\tilde{C}, L) \to H^0(\tilde{C}, K_{\tilde{C}})^-, \quad s_i \wedge s_j \mapsto s_i \sigma^* s_j - s_j \sigma^* s_i;$$

here $\sigma \colon \tilde{C} \to \tilde{C}$ is the involution induced by the double cover. Note that $H^0(\tilde{C}, K_{\tilde{C}})^-$ is identified with the tangent space of the Prym variety; in particular, it has dimension $g(C) - 1$. By [W2, Prop. 1.9], the Zariski tangent space of V^r at the point L is equal to the orthogonal of the image of β. Thus, if $\dim T_L V^r > g(C) - 2r$ then $\operatorname{rk} \beta < 2r - 1$. Because $\wedge^2 H^0(\tilde{C}, L)$ has dimension $\frac{r(r+1)}{2}$, that statement is equivalent to

$$\dim \ker \beta > \frac{r(r+1)}{2} - 2r - 1. \tag{1}$$

The locus of decomposable 2-forms in $\wedge^2 H^0(\tilde{C}, L)$ is the affine cone over the Plücker embedding of $G(2, H^0(\tilde{C}, L))$ in $\mathbb{P}(\wedge^2 H^0(\tilde{C}, L))$, so it has dimension $2r - 1$. Thus, by (1) there is a nonzero decomposable vector $s_i \wedge s_j$ in $\ker \beta$. This means that $s_i \sigma^* s_j - s_j \sigma^* s_i = 0$ and so s_j / s_i defines a rational function h on C. We conclude by taking $M = \mathcal{O}_C((h)_0)$ and F the maximal common divisor between $(s_i)_0$ and $(s_j)_0$. By construction, F is effective and $\dim|M| \geq 1$. $\square$

By [CaLV, Thm. 2.2; IPau, Lemma 2.1], every irreducible component of the Brill–Noether locus V^2 has dimension at most $g(C) - 4$ provided C is not hyperelliptic. The following estimate is a generalization of their statement to arbitrary r.

3.3. LEMMA. *We have*

$$\dim V^r \leq g(C) - 2 - r \quad \forall r \geq 2.$$

Proof. Denote by $|K_C| \subset C^{(2g(C)-2)}$ the set of effective canonical divisors and by $\operatorname{Nm} \pi : \tilde{C}^{(2g(C)-2)} \to C^{(2g(C)-2)}$ the norm map. Since the canonical linear system $|K_C|$ defines an embedding, it follows from [B3, Sec. 2, Cor.] that $\operatorname{Nm} \pi^{-1}(|K_C|)$

has exactly two irreducible components, Λ_0 and Λ_1, and that both are normal varieties of dimension $g(C) - 1$. Let

$$i : \tilde{C}^{(2g(C)-2)} \to \operatorname{Pic}^{2g(C)-2} \tilde{C}, \quad D \mapsto \mathcal{O}_{\tilde{C}}(D)$$

be the Abel–Jacobi map; then, up to renumbering,

$$\varphi(\Lambda_0) = P^- \quad \text{and} \quad \varphi(\Lambda_1) = \Theta \subset P^+.$$

Recall that for all $L \in \operatorname{Pic} \tilde{C}$ we have the set-theoretic equality $i^{-1}(L) = |L|$. In particular, we see that

$$\dim i^{-1}(V^r) \geq \dim V^r + r \tag{2}$$

for every $r \geq 0$.

Suppose now that r is even (the odd case is analogous and is left to the reader). For a general point $L \in P^-$ one has $\dim|L| = 0$. Thus, for $r \geq 2$,

$$i^{-1}(V^r) \subsetneq \Lambda_1;$$

hence $i^{-1}(V^r)$ has dimension at most $g(C) - 2$. We conclude by using (2). □

3.4. Remark. In the proof we used non-hyperelliptic C only to show that Λ_0 and Λ_1 are irreducible. Since inequality (2) is valid without this property, we obtain

$$\dim V^r \leq g(C) - 1 - r \quad \forall r \geq 2.$$

We will see in Section 4.A that this estimate is optimal.

We can now use Marten's theorem to give an estimate of the dimension of the singular locus V^2_{sing}.

3.5. Proposition. *Suppose that $g(C) \geq 6$ and V^2_{sing} has an irreducible component S of dimension at least $g(C) - 5$. Then there exist*

(a) *a $d \in \{3, 4\}$ such that*

$$\dim W^1_d C = d - 3$$

and

(b) *an irreducible component $W \subset W^1_d C$ of maximal dimension such that, for every $M \in W$,*

$$\dim|K_C \otimes M^{\otimes -2}| = g - d - 2.$$

For every L in S we have

$$L \simeq \pi^* M \otimes F$$

for some $M \in W$ and some effective line bundle F on $\tilde{C}$. In particular, S is of dimension $g(C) - 5$.

Proof. Let $L \in S$ be a generic point; then, by Lemma 3.3, $\dim|L| = 2$. Since V^2 is singular in L, it follows that

$$\dim T_L V^2 > g(C) - 4.$$

Hence by Lemma 3.1 there exist a line bundle $M \in W^1_d C$ for some $d \leq g(C) - 1$ and an effective line bundle F on $\tilde{C}$ such that

$$L \simeq \pi^* M \otimes F.$$

The family of such pairs (M, F) is a finite cover of the set of pairs (M, B) for which $M \in W^1_d C$ for some $d \leq g(C) - 1$ and B is an effective divisor of degree $2g(C) - 2 - 2d \geq 0$ on C such that $B \in |K_C \otimes M^{\otimes -2}|$.

By hypothesis, the parameter space T of the pairs (M, B) has dimension at least $g(C) - 5$. Note that if $\deg M = g(C) - 1$ then $K_C \otimes M^{\otimes -2} \simeq \mathcal{O}_C$. Thus M is a theta-characteristic and the space of pairs (M, B) is finite—a contradiction to $g(C) - 5 > 0$. Because C is not hyperelliptic, $3 \leq \deg M < g(C) - 1$. Moreover, by Clifford's theorem we have

$$\dim |H^0(C, K_C \otimes M^{\otimes -2})| \leq g(C) - 1 - d - 1. \tag{3}$$

Thus the variety W parameterizing the line bundles M has dimension at least $d - 3$. By construction we have $W \subset W^1_d$; by Marten's theorem [ACGH, IV, Thm. 5.1],

$$\dim W \leq \dim W^1_d C \leq d - 3. \tag{4}$$

Therefore, T and S each have dimension at most $g(C) - 5$. Since (by hypothesis) S has dimension at least $g(C) - 5$, it follows that (3) and (4) are equalities—at least for $M \in W$ generic. By upper semicontinuity and Clifford's theorem, we obtain equality for every $M \in W$.

The last remaining point is to show that this situation can occur only for $d \in \{3, 4\}$. We have already established the existence of a finite map

$$W \to W^{g-d-2}_{2g(C)-2-2d} C, \quad M \mapsto K_C \otimes M^{\otimes -2}.$$

If $2g(C) - 2 - 2d \leq g(C) - 1$ then, by Marten's theorem, $\dim W^{g(C)-d-2}_{2g(C)-2-2d} C \leq 1$. Because $\dim W = d - 3$, we see that $d \leq 4$. Now if $2g(C) - 2 - 2d \geq g(C)$, we use the isomorphism

$$W^{g(C)-d-2}_{2g(C)-2-2d} C \to W^{d-1}_{2d} C, \quad K_C \otimes M^{\otimes -2} \mapsto M^{\otimes 2}$$

together with Marten's theorem to show that $\dim W^{g(C)-d-2}_{2g(C)-2-2d} C \leq 1$; hence, again we obtain $d \leq 4$. □

Proof of Theorem 1.1. The hyperelliptic case is settled in Proposition 4.2, so we suppose that C is not hyperelliptic.

By [D3, Exm. 6.2.1], the Brill–Noether-locus V^2 is a determinantal variety. Since for non-hyperelliptic C it has the expected dimension, V^2 is Cohen–Macaulay. Since $\dim V^2_{\text{sing}} \leq g(C) - 5$ by Proposition 3.5, it follows that all the irreducible components of V^2 are generically reduced. Recall that a generically reduced Cohen–Macaulay scheme is itself reduced. If $\dim V^2_{\text{sing}} \geq g(C) - 5$ then, by Proposition 3.5, $\dim W^1_d C = d - 3$ for $d = 3$ or 4. Thus the second statement follows from Mumford's refinement of Marten's theorem [ACGH, IV, Thm. 5.2]. □

REMARK. Lahoz and Naranjo [LN] use completely different methods to show that V^2 is reduced and Cohen–Macaulay.

3.6. Corollary. *Let C be a smooth non-hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi : \tilde{C} \to C$ be an étale double cover. If V^2 is reducible, then C is a plane quintic, trigonal, or bielliptic.*

Remark. Teixidor i Bigas [T] uses the Martens–Mumford theorem to determine when the singular locus of a Jacobian of a curve is reducible.

Proof of Corollary 3.6. By a theorem of Debarre [D3, Exm. 6.2.1], the locus V^2 is $(g(C) - 5)$-connected. In other words, if V^2 is not irreducible then there exist two irreducible components $Z_1, Z_2 \subset V^2$ such that $Z_1 \cap Z_2$ has dimension at least $g(C) - 5$ in one point [D3, p. 287]. So if V^2 is reducible, its singular locus has dimension at least $g(C) - 5$. Now conclude using Theorem 1.1. □

4. Examples

4.A. Hyperelliptic Curves

Let C be a smooth hyperelliptic curve of genus $g(C)$. Let $\pi : \tilde{C} \to C$ be an étale double cover such that the Prym variety (P, Θ) is an irreducible principally polarized abelian variety (i.e., $\tilde{C}$ is also a hyperelliptic curve). Let $\sigma : \tilde{C} \to \tilde{C}$ be the involution induced by π.

Recall from [BiLa, Chap. 12, Sec. 5] that in this case, for a fixed $p_0 \in C$, the Abel–Prym map

$$\alpha : \tilde{C} \to P, \quad p \mapsto \sigma(p) - p + \sigma(p_0) - p_0$$

is two-to-one onto its image C' (which is a smooth curve) and the Prym variety (P, Θ) is isomorphic to $(J(C'), \Theta_{C'})$.

In [CaLV, Lemma 2.1] the authors show that, for C not hyperelliptic, V^2 is a translate of the theta-dual of the Abel–Prym embedded curve $\tilde{C} \subset P$. In fact, their argument works also for C hyperelliptic if one replaces $\tilde{C} \subset P$ by $\alpha(\tilde{C}) = C' \subset P$. Thus we have the following statement.

4.1. Lemma. *The Brill–Noether locus V^2 is a translate of the theta-dual $T(C')$.*

Since the Prym variety (P, Θ) is isomorphic to $(J(C'), \Theta_{C'})$, it follows that the theta-dual of C' is a translate of $W^0_{g(C)-3}C'$. In particular, V^2 is irreducible of dimension $g(C) - 3$.

4.2. Proposition. *Let C be a smooth hyperelliptic curve of genus $g(C) \geq 6$, and let $\pi : \tilde{C} \to C$ be an étale double cover such that the Prym variety (P, Θ) is an irreducible principally polarized abelian variety. Then V^2 is irreducible of dimension $g(C) - 3$ and, set-theoretically, it is a translate $W^0_{g(C)-3}C'$.*

For any point $L \in V^2$ we have

$$L \simeq \pi^* H \otimes F,$$

where H is the unique g^1_2 on C and F is an effective line bundle on $\tilde{C}$.

Proof. By Remark 3.4 we have a proper inclusion $V^4 \subsetneq V^2$, so a general $L \in V^2$ satisfies $\dim|L| = 2$. By Lemma 3.1 there exists a line bundle $M \in W^1_d C$ for some $d \leq g(C) - 1$ and an effective line bundle F on $\tilde{C}$ such that

$$L \simeq \pi^* M \otimes F.$$

We can now argue as in the proof of Proposition 3.5 to obtain the statement. We need only observe that the inequality

$$\dim|H^0(C, K_C \otimes M^{\otimes -2})| \leq g(C) - 1 - d - 1$$

is also valid on a hyperelliptic curve unless M is a multiple of the g^1_2. $\square$

4.B. *Plane Quintics*

Let $C \subset \mathbb{P}^2$ be a smooth plane quintic and let $\pi : \tilde{C} \to C$ be an étale double cover. We denote by H the restriction of the hyperplane divisor to C and by $\eta \in \operatorname{Pic}^0 C$ the 2-torsion line bundle inducing π. Let $\sigma : \tilde{C} \to \tilde{C}$ be the involution induced by π.

4.3. EXAMPLE. Suppose that $h^0(C, \mathcal{O}_C(H) \otimes \eta)$ is odd—that is, suppose the Prym variety P^- is isomorphic to the intermediate Jacobian $J(X)$ of a cubic threefold X [ClG]. Let us fix such an isomorphism of principally polarized abelian varieties $J(X) \xrightarrow{\sim} P^-$. The Fano variety F parameterizing lines on the threefold X is a smooth surface that has a natural embedding in the intermediate Jacobian $J(X)$. By [ClG], the surface $F \subset P$ has minimal cohomology class $\left[\frac{\Theta^3}{3!}\right]$. Moreover, it follows from [Höl] and [PPo1] that the theta-dual satisfies $T(F) = -F$. It is well known that $\tilde{C} \subset F$ (up to translation), so

$$-F = T(F) \subset V^2 = T(\tilde{C}).$$

Since the condition $\dim|L| \geq 2$ is invariant under isomorphism, the Brill–Noether locus V^2 is stable under the map $x \mapsto -x$. Thus $-F \subset V^2$ implies that $F \subset V^2$. Since the cohomology class of V^2 is $\left[2\frac{\Theta^3}{3!}\right]$, we see that (up to translation) V^2 is a union of F and $-F$. In particular, V^2 is reducible and its singular locus is the intersection of the two irreducible components. Since V^2 is Cohen–Macaulay, the singular locus has pure dimension 1.

We will now prove the converse of this example.

4.4. PROPOSITION. *The Brill–Noether locus V^2 is reducible if and only if $h^0(C, \mathcal{O}_C(H) \otimes \eta)$ is odd—in other words, iff the Prym variety is isomorphic to the intermediate Jacobian of a cubic threefold. In this case, the singular locus V^2_{sing} is a translate of $\tilde{C}$.*

Proof. Suppose that V^2_{sing} has a component S of dimension 1. Since C is not trigonal, we know from Proposition 3.5 that S corresponds to a 1-dimensional component $W \subset W^1_4 C$ such that, for every $[M] \in W$,

$$|K_C \otimes M^{\otimes -2}| \neq \emptyset.$$

By adjunction we have $K_C \simeq \mathcal{O}_C(2H)$, and from [B2, Sec. 2, (iii)] it follows that $M \simeq \mathcal{O}_C(H - p)$, where $p \in C$ is a point. Hence $K_C \otimes M^{\otimes -2} \simeq \mathcal{O}_C(2p)$ and a general point $L \in S$ is of the form

$$L \simeq \pi^* \mathcal{O}_C(H - p) \otimes \mathcal{O}_{\tilde{C}}(q_1 + q_2),$$

where q_1, q_2 are points in $\tilde{C}$. Since $\operatorname{Nm} \pi(L) \simeq \mathcal{O}_C(2H)$ and C is not hyperelliptic, we obtain that $q_i \in \pi^{-1}(p)$. Then we can write

$$L \simeq \pi^* H \quad \text{or} \quad L \simeq \pi^* \mathcal{O}_C(H) \otimes \mathcal{O}_{\tilde{C}}(q - \sigma(q)) \ \text{ for some } q \in \tilde{C}.$$

Because L varies in a 1-dimensional family, we can exclude the first case. By Mumford's description of a Prym variety whose theta-divisor has a singular locus of dimension $g(C) - 5$, we know that $h^0(C, \mathcal{O}_C(H) \otimes \eta)$ is even if and only if $h^0(\tilde{C}, \pi^* \mathcal{O}_C(H) \otimes \mathcal{O}_{\tilde{C}}(q - \sigma(q)))$ is even [M2, p. 347]. Since $V^2 \subset P^-$, this shows the statement.

The description of the general points $L \in S$ shows that V^2_{sing} has a unique 1-dimensional component and that V^2_{sing} is the translate by $\pi^* \mathcal{O}_C(H)$ of the Abel–Prym embedded $\tilde{C} \subset P$. $\square$

4.C. Trigonal Curves

Let C be a trigonal curve of genus $g(C) \geq 6$. Let $\pi : \tilde{C} \to C$ be an étale double cover and (P, Θ) the corresponding Prym variety. By a theorem of Recillas [Re], the Prym variety is isomorphic as a principally polarized abelian variety to the polarized Jacobian (JX, Θ_X) of a tetragonal curve X of genus $g(C) - 1$. By Recillas's construction [BiLa, Chap. 12.7] we also know how to recover the double cover $\pi : \tilde{C} \to C$ from the curve X. Namely, let $s : X^{(2)} \times X^{(2)} \to X^{(4)}$ be the sum map; then

$$\tilde{C} \simeq p_1(s^{-1}(\mathbb{P}^1)),$$

where $\mathbb{P}^1 \subset X^{(4)}$ is the linear system giving the tetragonal structure and p_1 is the projection onto the first factor. In particular, we see that

$$\tilde{C} \subset X^{(2)} \simeq W^0_2 X.$$

Therefore, up to choosing an isomorphism $(P, \Theta) \simeq (JX, \Theta_X)$ (and appropriate translates),

$$T(W^0_2 X) \subset T(\tilde{C}) \simeq V^2.$$

By [PPo1, Exm. 4.5], the theta-dual of $W^0_2 X$ is $-W^0_{g(C)-4} X$. As in the case of the intermediate Jacobian described in Example 4.3, we see that (up to translation)

$$V^2 = -W^0_{g(C)-4} X \cup W^0_{g(C)-4} X;$$

moreover, the singular locus of V^2 is the union of $\pm (W^0_{g(C)-4} X)_{\text{sing}}$, which has dimension at most $g(C) - 6$, and the intersection of the two irreducible components, which has dimension $g(C) - 5$.

5. Prym Varieties of Bielliptic Curves, I

5.A. Special Subvarieties

We recall some well-known facts about special subvarieties that we will use in the next section.

Let $\varphi\colon X \to Y$ be a double cover (which may be étale or ramified) of smooth curves. We suppose that $g(Y)$ is at least 1 and denote by $\operatorname{Nm}\varphi\colon \operatorname{Pic} X \to \operatorname{Pic} Y$ the norm morphism. Let M be a globally generated line bundle of degree $d \geq 2$ on Y. Denote by $\mathbb{P}^r \subset Y^{(d)}$, where $r := \dim|M|$, the set of effective divisors in the linear system $|M|$. If $\operatorname{Nm}\varphi\colon X^{(d)} \to Y^{(d)}$ is the norm map, then $\Lambda := \operatorname{Nm}\varphi^{-1}(\mathbb{P}^r)$ is a reduced Cohen–Macaulay scheme of pure dimension r and the map $\Lambda \to |M|$ is étale of degree 2^d over the locus of smooth divisors in $|M|$ that do not meet the branch locus of φ.

If φ is étale then Λ has exactly two connected components, Λ_0 and Λ_1 [W1]. If φ is ramified, the scheme Λ is connected [N, Prop. 14.1]. Let

$$i_Y\colon Y^{(d)} \to \operatorname{Pic}^d Y, \quad D \mapsto \mathcal{O}_Y(D)$$

and

$$i_X\colon X^{(d)} \to \operatorname{Pic}^d X, \quad D \mapsto \mathcal{O}_X(D)$$

be the Abel–Jacobi maps; then we have the commutative diagram

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[d] & X^{(d)} \arrow[r, "i_X"] \arrow[d, "\operatorname{Nm}\varphi"'] & \operatorname{Pic}^d X \arrow[d, "\operatorname{Nm}\varphi"] \\ \mathbb{P}^r \arrow[r, hook] & Y^{(d)} \arrow[r, "i_Y"'] & \operatorname{Pic}^d Y. \end{tikzcd}$$

The fibre of $i_X(X^{(d)}) \to i_Y(Y^{(d)})$ over the point M—and thus the intersection of $i_X(X^{(d)})$ with $\operatorname{Nm}\varphi^{-1}(M)$—is equal (at least set-theoretically) to $i_X(\Lambda)$.

Fix now a connected component $S \subset \Lambda$. Then we call $V := i_X(S)$ *a special subvariety* associated to M. (In general it is not true that S is irreducible; in particular, the special subvariety may not be a variety. Note also that in general it should be obvious which covering we consider, and otherwise we say that V is a φ-special subvariety associated to M.) It is clear that

$$\dim V = r - \dim|\mathcal{O}_X(D)|, \tag{5}$$

where $D \in S$ is a general point.

The following technical definition will be crucial in the next section.

5.1. DEFINITION. Let $\varphi\colon X \to Y$ be a double cover of smooth curves. An effective divisor $D \subset X$ is *not simple* if there exists a point $y \in Y$ such that $\varphi^* y \subset D$, and it is *simple* if this is not the case.

Note that if an effective divisor $D \subset X$ is not simple then $\operatorname{Nm}\varphi(D)$ is not reduced. Hence, if Y is an elliptic curve and M a line bundle of degree $d \geq 2$ on Y, then a general divisor $D \in X^{(d)}$ such that $\operatorname{Nm}\varphi(D) \in |M|$ is simple: the linear system $|M|$ is base point free, so a general element is reduced.

5.2. LEMMA. *Let $\varphi: X \to Y$ be a ramified double cover of smooth curves such that Y is an elliptic curve. Denote by δ_φ the line bundle of degree $g(X) - 1$ defining the cyclic cover φ. Let $M \not\simeq \delta_\varphi$ be a line bundle of degree $2 \le d \le g(X) - 1$ on Y. Then the following statements hold*:

(a) *Λ is smooth and irreducible*;
(b) *a general divisor $D \in \Lambda$ is simple and satisfies* $\dim|\mathcal{O}_X(D)| = 0$.

In particular, there exists a unique special subvariety associated to M and it is irreducible of dimension $d - 1$.

Proof. We start by showing part (b). By the foregoing, D is simple and so, according to [M1, p. 338], we have an exact sequence

$$0 \to \mathcal{O}_Y \to \varphi_*\mathcal{O}_X(D) \to \mathcal{O}_Y(\operatorname{Nm}\varphi(D)) \otimes \delta_\varphi^* \to 0.$$

Since $\deg D \le \deg \delta_\varphi$ and $\mathcal{O}_Y(\operatorname{Nm}\varphi(D)) \simeq M \not\simeq \delta_\varphi$, we have

$$h^0(Y, \mathcal{O}_Y(\operatorname{Nm}\varphi(D)) \otimes \delta_\varphi^*) = 0.$$

Therefore, $1 = h^0(Y, \mathcal{O}_Y) = h^0(Y, \varphi_*\mathcal{O}_X(D))$.

For the proof of part (a) we note first that, since Λ is connected, it is sufficient to show the smoothness. Let $D \in \Lambda$ be any divisor. Then we have a unique decomposition

$$D = \varphi^*A + R + B,$$

where A is an effective divisor on Y; the divisor R is effective, with support contained in the ramification locus of φ; and B is effective, simple, and has support disjoint from the ramification locus of φ. Since Y is an elliptic curve, we have

$$h^0(Y, M \otimes \mathcal{O}_Y(-A - \operatorname{Nm}\varphi(R))) = h^0(Y, M) - \deg(A + \operatorname{Nm}\varphi(R))$$

unless $\deg M = \deg(A + \varphi_* R)$ and $M \otimes \mathcal{O}_Y(-A - \operatorname{Nm}\varphi(R))$ is not trivial. Because $\deg M = \deg D$, this last case could occur only when $A = 0$ and $B = 0$; hence $D = R$. Yet by construction we have $M \simeq \mathcal{O}_Y(\operatorname{Nm}\varphi(D)) = \mathcal{O}_Y(\operatorname{Nm}\varphi(R))$, so $M \otimes \mathcal{O}_Y(-A - \operatorname{Nm}\varphi(R))$ is trivial. By [N, Prop. 14.3] this shows the smoothness of Λ. The statement on the dimension follows by part (b) and equation (5). □

5.B. The Irreducible Components of V^2

In this section C will be a smooth curve of genus $g(C) \ge 6$ that is bielliptic; in other words, we have a double cover $p: C \to E$ onto an elliptic curve E. As usual, $\pi: \tilde{C} \to C$ will be an étale double cover. In this section we suppose that the covering $p \circ \pi: \tilde{C} \to E$ is Galois. Then one sees easily that the Galois group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Using the Galois action on $\tilde{C}$ yields the commutative diagram

$$\begin{tikzcd} & \tilde{C} \arrow[dl, "\pi"'] \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ C \arrow[dr, "p"'] & C_1 \arrow[d, "p_1"] & C_2 . \arrow[dl, "p_2"] \\ & E & \end{tikzcd}$$

(The presentation here follows [D1, Chap. 5], to which we refer for details.) It is straightforward to see that

$$g(C_1) + g(C_2) = g(C) + 1,$$

and we will assume without loss of generality that $1 \leq g(C_1) \leq g(C_2) \leq g(C)$. Denote by Δ the branch locus of p and by δ the line bundle inducing the cyclic cover p. Then $2\delta \simeq \Delta$ and, by the Hurwitz formula, $\deg K_C = \deg \Delta$; hence

$$\deg \delta = g(C) - 1.$$

The cyclic covers p_1 and p_2 are analogously given by line bundles δ_1 and δ_2 such that $\deg \delta_1 = g(C_1) - 1$ and $\deg \delta_2 = g(C_2) - 1$.

For any $a \in \mathbb{Z}$ we define closed subsets $Z_a \subset \operatorname{Pic} C_1 \times \operatorname{Pic} C_2$ by

$$\bigl\{(L_1, L_2) \mid L_1 \in W^0_{g(C_1)-1+a} C_1, \\ L_2 \in W^0_{g(C_2)-1-a} C_2,\ \operatorname{Nm} p_1(L_1) \otimes \operatorname{Nm} p_2(L_2) \simeq \delta\bigr\}.$$

We note that the sets Z_a are empty unless $1 - g(C_1) \leq a \leq g(C_2) - 1$. Pulling back to $\tilde{C}$ we obtain natural maps

$$(\pi_1^*, \pi_2^*) \colon Z_a \to \operatorname{Pic} \tilde{C}, \quad (L_1, L_2) \mapsto \pi_1^* L_1 \otimes \pi_2^* L_2,$$

and by [D1, p. 230] the image $(\pi_1^*, \pi_2^*)(Z_a)$ is in P^- if and only if a is odd. Moreover, we can argue as in [D1, Prop. 5.2.1] to show that

$$V^2 \subset (\pi_1^*, \pi_2^*)\Bigl(\bigcup_{a \text{ odd}} Z_a\Bigr). \tag{6}$$

5.3. LEMMA. *For a odd, the sets Z_a are empty or*

$$\dim Z_a = g(C) - 1 - a. \tag{7}$$

Furthermore, Z_a is irreducible unless $g(C_1) = 1$ and $a \geq g(C_2) - 2$.

Proof. We divide the proof into two cases as follows.

Case 1: $g(C_1) > 1$. We prove the statement for positive a (the argument is analogous for negative a). The projection onto the second factor gives a surjective map $Z_a \to W^0_{g(C_2)-1-a}C_2$, and the fibres of this map are parameterized by effective line bundles L_1 with fixed norm. Because $a \geq 1$, the line bundles L_1 are of degree at least $g(C_1)$ and so are automatically effective. Thus the fibres identify to fibres of the norm map $\operatorname{Nm} p_1 \colon \operatorname{Pic} C_1 \to \operatorname{Pic} E$. Since the double covering p_1 is ramified, it follows that the $(\operatorname{Nm} p_1)$-fibres are irreducible of dimension $g(C_1)-1$; hence Z_a is irreducible of the expected dimension.

Case 2: $g(C_1) = 1$. The sets Z_a are empty for a negative, so suppose that a is positive. Arguing as in the first case, we obtain the statement on the dimension. In order to see that Z_a is irreducible for $a \leq g(C_2)-3$, we consider the surjective map induced by the projection onto the first factor $Z_a \to \operatorname{Pic}^{g(C_1)-1+a} C_1$. The fibre over a line bundle L_1 is the union of the p_2-special subvarieties associated to $\delta \otimes \operatorname{Nm} p_1 L_1^*$. Since $2 \leq \deg \delta \otimes \operatorname{Nm} p_1(L_1^*) \leq g(C_2) - 2$, it follows from Lemma 5.2 that the unique special subvariety is irreducible and so the fibres are irreducible. □

Since all the irreducible components of V^2 have dimension $g(C) - 4$, by (6) and (7) we have

$$V^2 \subset (\pi_1^*, \pi_2^*)\Bigg(\bigcup_{a \text{ odd}, |a| \leq 3} Z_a \Bigg). \tag{8}$$

If $(L_1, L_2) \in Z_{\pm 3}$ then, by the Riemann–Roch theorem, it follows that $\dim|L_1| \geq 2$ and $\dim|L_2| \geq 2$; therefore,

$$(\pi_1^*, \pi_2^*)(Z_{\pm 3}) \subset V^2.$$

For the sets $Z_{\pm 1}$ this cannot be true, since equation (7) shows that they have dimension $g(C) - 2$. We introduce the following smaller loci:

$$W_1 := \{(L_1, L_2) \in Z_1 \mid L_1 \in W^1_{g(C_1)} C_1\};$$

$$W_{-1} := \{(L_1, L_2) \in Z_{-1} \mid L_2 \in W^1_{g(C_2)} C_2\}.$$

Note that if $g(C_1) = 1$ then $W_1 = \emptyset$: there is no g^1_1 on a nonrational curve. Because $\dim W^1_{g(C_1)} C_1 = g(C_1) - 2$ (resp., $\dim W^1_{g(C_2)} C_1 = g(C_2) - 2$), one may easily deduce (from the proof of Lemma 5.2) that the sets $W_{\pm 1}$ are either empty or irreducible of dimension $g(C) - 4$.

By the same lemma we see that, for fixed L_1 (resp. L_2) and general L_2 (resp. L_1) such that $(L_1, L_2) \in W_1$ (resp. $(L_1, L_2) \in W_{-1}$), the linear system $|L_1|$ (resp. $|L_2|$) contains a unique effective divisor and this divisor is simple.

Observe that if $(L_1, L_2) \in W_{\pm 1}$ then $\dim|(\pi_1^*, \pi_2^*)(L_1, L_2)| \geq 1$. Since these sets map into the component P^-, we obtain

$$(\pi_1^*, \pi_2^*)(W_{\pm 1}) \subset V^2.$$

5.4. Proposition. *We have*

$$V^2 = (\pi_1^*, \pi_2^*)(Z_{-3} \cup W_{-1} \cup W_1 \cup Z_3).$$

The proof requires some technical preparation.

5.5. DEFINITION. Let $\varphi\colon X \to Y$ be a double cover of smooth curves, and let L be a line bundle on X such that $\dim|L| \geq 1$. Then the line bundle L is simple if every divisor in $D \in |L|$ is simple in the sense of Definition 5.1.

5.6. LEMMA [D1, Cor. 5.2.8]. *In our situation, let $L_1 \in \operatorname{Pic} C_1$ and $L_2 \in \operatorname{Pic} C_2$ be effective line bundles such that $L \simeq \pi_1^* L_1 \otimes \pi_2^* L_2$. If L_1 is p_1-simple, then*

$$h^0(\tilde{C}, L) \leq 2h^0(C_2, L_2) + g(C_2) - 1 - \deg L_2.$$

Analogously, if L_2 is p_2-simple then

$$h^0(\tilde{C}, L) \leq 2h^0(C_1, L_1) + g(C_1) - 1 - \deg L_1.$$

Proof of Proposition 5.4. Let $L \in V^2$ be an arbitrary line bundle. By the inclusion (8) we need only show that if $L \in (\pi_1^*, \pi_2^*)(Z_{\pm 1})$ and $L \notin (\pi_1^*, \pi_2^*)(Z_{-3} \cup Z_3)$ then $L \in (\pi_1^*, \pi_2^*)(W_{\pm 1})$. We will suppose that $L \in (\pi_1^*, \pi_2^*)(Z_1)$; the other case is analogous and is left to the reader. Because $L \in (\pi_1^*, \pi_2^*)(Z_1)$, we can write

$$L \simeq \pi_1^* L_1 \otimes \pi_2^* L_2$$

with L_1 effective of degree $g(C_1)$ and L_2 effective of degree $g(C_1) - 2$. If L_2 is not simple then L is in $(\pi_1^*, \pi_2^*)(Z_3)$, which we have already excluded. Hence L_2 is simple and so, by Lemma 5.6,

$$3 \leq h^0(\tilde{C}, L) \leq 2h^0(C_1, L_1) + g(C_1) - 1 - g(C_1).$$

Therefore, $\dim|L_1| \geq 1$ and $L \in (\pi_1^*, \pi_2^*)(W_1)$. □

5.7. COROLLARY. *If $g(C_1) = 1$ then*

$$V^2 = (\pi_1^*, \pi_2^*)(Z_3).$$

In particular, V^2 is irreducible.

Proof. Since the sets Z_{-3}, W_{-1}, and W_1 are empty for $g(C_1) = 1$, the first statement is immediate from Proposition 5.4. Since $g(C_1) = 1$ implies that $g(C_2) = g(C)$ and $g(C) \geq 6$ by hypothesis, it follows from Lemma 5.3 that Z_3 is irreducible. □

We now focus on the case $g(C_1) \geq 2$. Proposition 5.4 reduces the study of V^2 to understanding the sets $W_{\pm 1}$, $Z_{\pm 3}$ and their images in P^-. We start with the following observation.

5.8. LEMMA. *For $g(C_1) \geq 2$,*

$$(\pi_1^*, \pi_2^*)(W_1) = (\pi_1^*, \pi_2^*)(W_{-1}).$$

Proof. We claim that the following holds: If $L_1 \in W^1_{g(C_1)} C_1$ is a general point then

(a) L_1 is not simple and
(b) there exists a point $x \in E$ such that

$$L_1 \simeq p_1^* \mathcal{O}_E(x) \otimes \mathcal{O}_{C_1}(D_1),$$

with D_1 an effective divisor such that $\mathcal{O}_E(\operatorname{Nm} p_1(D_1) + x) \simeq \delta_1$.

Assuming this for the time being, let us show how to conclude. If $L \in (\pi_1^*, \pi_2^*)(W_1)$ is a general point, then $L \simeq \pi_1^* L_1 \otimes \pi_2^* L_2$ with $L_1 \in W^1_{g(C_1)} C_1$ a general point and L_2 a p_2-simple line bundle. Thus, by the claim we can write

$$L \simeq \pi_1^* \mathcal{O}_{C_1}(D_1) \otimes \pi_2^*(L_2 \otimes p_2^* \mathcal{O}_E(x)).$$

Since $\mathcal{O}_E(\mathrm{Nm}\, p_1(D_1) + x) \simeq \delta_1$ and $\delta \simeq \delta_1 \otimes \delta_2$, a short computation shows that $\mathrm{Nm}\, p_2(L_2) \otimes \mathcal{O}_E(x) \simeq \delta_2$. Moreover L_2 is p_2-simple and so, by [D1, Prop. 5.2.7], $\dim|L_2 \otimes p_2^* \mathcal{O}_E(x)| \geq 1$. Hence L is in $(\pi_1^*, \pi_2^*)(W_{-1})$. This shows one inclusion; the proof of the other is analogous.

Proof of the claim. Set

$$S := \{(x, D_1) \in E \times C_1^{(g(C_1)-2)} \mid x + \mathrm{Nm}\, p_1(D_1) \in |\delta_1|\}.$$

(For $g(C_1) = 2$, the symmetric product $C_1^{(g(C_1)-2)}$ is a point; it corresponds to the zero divisor on C_1.) Observe that the projection $p_2\colon S \to C_1^{(g(C_1)-2)}$ on the second factor is an isomorphism, so S is not uniruled. For $(x, D_1) \in S$ general, the divisor D_1 is p_1-simple by Lemma 5.2 and so, by [M1, p. 338], we have the exact sequence

$$0 \to \mathcal{O}_E(x) \to (p_1)_* \mathcal{O}_{C_1}(p_1^* x + D_1) \to \mathcal{O}_E(x + \mathrm{Nm}\, p_1(D_1)) \otimes \delta_1^* \to 0.$$

By construction we have $\mathcal{O}_E(x + \mathrm{Nm}\, p_1(D_1)) \otimes \delta_1^* \simeq \mathcal{O}_E$. Thus $H^1(E, \mathcal{O}_E(x)) = 0$ implies that $h^0(C_1, \mathcal{O}_{C_1}(p_1^* x + D_1)) = 2$. Hence the image of

$$\tau\colon S \to \mathrm{Pic}\, C_1, \quad (x, D_1) \mapsto \mathcal{O}_{C_1}(p_1^* x + D_1)$$

is contained in $W^1_{g(C_1)} C_1$. Because S is not uniruled, the general fibre of $S \to \tau(S)$ has dimension 0. By Riemann–Roch, the residual map $W^1_{g(C_1)} C_1 \to W^0_{g(C_1)-2} C_1$ is an isomorphism and so $W^1_{g(C_1)} C_1$ is irreducible of dimension $g(C_1) - 2$. Hence τ is surjective on $W^1_{g(C_1)}(C_1)$. $\square$

Suppose that $g(C_1) \geq 2$. Let (JC_1, Θ_{C_1}) and (JC_2, Θ_{C_2}) be the Jacobians of the curves C_1 and C_2 with their natural principal polarizations. Since p_1 and p_2 are ramified, the pull-backs $\pi_1^*\colon JE \to JC_1$ and $\pi_2^*\colon JE \to JC_2$ are injective and the restricted polarizations $B_1 := \Theta_{C_1}|_{JE}$ and $B_2 := \Theta_{C_2}|_{JE}$ are of type (2) [M1, Chap. 3]. We define

$$P_1 := \ker(\mathrm{Nm}\, p_1\colon JC_1 \to JE), \qquad P_2 := \ker(\mathrm{Nm}\, p_2\colon JC_2 \to JE).$$

We set $A_1 := \Theta_{C_1}|_{P_1}$ and $A_2 := \Theta_{C_2}|_{P_2}$; then the polarizations A_1 and A_2 are of type $(1, \ldots, 1, 2)$ [BiLa, Cor. 12.1.5].

If $p_j^* \times i_{P_j}\colon JE \times P_j \to JC_j$ denotes the natural isogeny, then $(p_j^* \times i_{P_j})^* \Theta_{C_j} \equiv B_j \boxtimes A_j$. Thus if $\alpha_j\colon JC_j \to JE \times \widehat{P_j}$ is the dual map then

$$\Theta_{C_j}^{\otimes 2} \equiv \alpha_j^*(\widehat{B_j} \boxtimes \widehat{A_j}) \tag{9}$$

[BiLa, Prop. 14.4.4], where $\widehat{B_j}$ and $\widehat{A_j}$ are the dual polarizations. We remark that $\widehat{A_j}$ has type $(1, 2, \ldots, 2)$.

By [D1, Prop. 5.5.1], the pull-back maps P_1 and P_2 into the Prym variety P and we obtain an isogeny $(\pi_1^*, \pi_2^*)|_{P_1 \times P_2} \colon P_1 \times P_2 \to P$ such that

$$(\pi_1^*, \pi_2^*)|_{P_1 \times P_2}^* \Theta \equiv A_1 \boxtimes A_2.$$

In particular, if $g \colon P \to \widehat{P_1 \times P_2}$ denotes the dual map then

$$\Theta^{\otimes 2} \equiv g^*(\widehat{A_1} \boxtimes \widehat{A_2}). \tag{10}$$

5.9. PROPOSITION. *If $g(C_1) \geq 3$ then the cohomology classes of $(\pi_1^*, \pi_2^*)(Z_{-3})$, $(\pi_1^*, \pi_2^*)(W_1)$, and $(\pi_1^*, \pi_2^*)(Z_3)$ are not minimal. Moreover, their cohomology classes are distinct and so they are distinct irreducible components of V^2.*

If $g(C_1) = 2$ then the same holds for $(\pi_1^*, \pi_2^*)(W_1)$ and $(\pi_1^*, \pi_2^*)(Z_3)$.

Proof. In order to simplify the notation, we denote the pull-back of the polarizations $\widehat{A_1}$ and $\widehat{A_2}$ to $\widehat{P_1 \times P_2}$ by the same letter.

We start by observing that is sufficient to show that $[(\pi_1^*, \pi_2^*)(Z_{-3})]$ (resp. $[(\pi_1^*, \pi_2^*)(Z_{-3})]$) is a nonnegative multiple of $g^*\widehat{A_1}^3$ ($g^*\widehat{A_2}^3$). Indeed, once we have shown this property, we can use that

$$[(\pi_1^*, \pi_2^*)(Z_{-3})] + [(\pi_1^*, \pi_2^*)(Z_3)] + [(\pi_1^*, \pi_2^*)(W_1)] = [V^2] = \frac{\Theta^3}{3!}$$

and the identity (10) to compute that

$$\begin{aligned}&[(\pi_1^*, \pi_2^*)(W_1)] \\ &\quad = \frac{1}{3!\,2^3}[(1-a_1)g^*\widehat{A_1}^3 + 3g^*\widehat{A_1}^2\widehat{A_2} + 3g^*\widehat{A_1}\widehat{A_2}^2 + (1-a_2)g^*\widehat{A_2}^3],\end{aligned}$$

where $a_1, a_2 \geq 0$ correspond to the cohomology class of $Z_{\pm 3}$. It is clear that none of these classes is (a multiple of) a minimal cohomology class. If $g(C_1) \geq 4$ then all the classes are nonzero and distinct, in which case the images of $Z_{\pm 3}$ and W_1 are distinct irreducible components of V^2. If $2 \leq g(C_1) \leq 3$ then the set Z_{-3} is empty (and the corresponding class zero), so we obtain only two irreducible components.

Computation of the cohomology class of $(\pi_1^*, \pi_2^*)(Z_{\pm 3})$. We will prove the claim for Z_3; the proof for Z_{-3} is analogous. We have the commutative diagram

$$\begin{tikzcd}
P \arrow[r, hook, "i_P"] & J\tilde{C} \arrow[r, "\simeq"] & \widehat{J\tilde{C}} \arrow[r, "\widehat{i_P}"] \arrow[d, "\widehat{(\pi_1^*, \pi_2^*)}"'] & \widehat{P} \simeq P \arrow[d, "g"'] \\
& JC_1 \times JC_2 \arrow[u, "(\pi_1^*, \pi_2^*)"] \arrow[r, "\simeq"] & \widehat{JC_1 \times JC_2} \arrow[r, "q"] & \widehat{P_1 \times P_2};
\end{tikzcd}$$

therefore, if $X \subset JC_1 \times JC_2$ is a subvariety such that $(\pi_1^*, \pi_2^*)(X) \subset P$, then its cohomology class is determined (up to a multiple) by the class of $q(X)$ in $\widehat{P_1 \times P_2}$.

We choose a translate of Z_3 that is in $JC_1 \times JC_2$ and denote it by the same letter. We want to understand the geometry of $q(Z_3)$. Since the norm maps $\mathrm{Nm}\, p_j$

are dual to the pull-backs p_j^* [M1, Chap. 1], the map q fits into an exact sequence of abelian varieties

$$0 \to JE \times JE \xrightarrow{p_1^* \times p_2^*} JC_1 \times JC_2 \xrightarrow{q} \widehat{P_1 \times P_2} \to 0. \tag{11}$$

Recall from the proof of Lemma 5.3 that Z_3 is a fibre space over $W^0_{g(C_1)-4}$ such that, for given $L_2 \in W^0_{g(C_2)-4}$, the fibre identifies to the fibre of $\mathrm{Nm}\, p_1 \colon \mathrm{Pic}^{g(C_1)+2} C_1 \to \mathrm{Pic}^{g(C_1)+2} E$ over $\delta \otimes \mathrm{Nm}\, p_2(L_2^*)$. Thus Z_3 identifies to a fibre product

$$\mathrm{Pic}^{g(C_1)+2} C_1 \times_{JE} W^0_{g(C_2)-4}.$$

Together with the exact sequence (11) this shows that

$$q(Z_3) = \widehat{P_1} \times q_2(W^0_{g(C_2)-4}),$$

where $q_2 \colon JC_2 \to P_2$ is the restriction of q to JC_2.

Thus we are left to compute the cohomology class of $q_2(W^0_{g(C_2)-4})$. Note first that q_2 is the composition of the isogeny $\alpha_2 \colon JC_2 \to JE \times \widehat{P_2}$ with the projection on $\widehat{P_2}$. Since the polarization $\widehat{B_2}$ is numerically equivalent to a multiple of $e \times \widehat{P_2} \subset JE \times \widehat{P_2}$ and since the cohomology class of $W^0_{g(C_2)-4}$ is $\frac{\Theta^4_{C_2}}{4!}$, it follows from the identity (9) that the cohomology class of $q_2(W^0_{g(C_2)-4})$ is a multiple of $\widehat{A_2}^3$. □

5.10. Remark. With some additional effort one can prove the following statement. If $g(C_1) \geq 2$, then the following equalities in $H^6(P, \mathbb{Z})$ hold:

$$[(\pi_1^*, \pi_2^*)(Z_{-3})] = \frac{1}{4!} g^* p^*_{\widehat{P_1}} \widehat{A_1}^3; \tag{12}$$

$$[(\pi_1^*, \pi_2^*)(Z_3)] = \frac{1}{4!} g^* p^*_{\widehat{P_2}} \widehat{A_2}^3; \tag{13}$$

$$[(\pi_1^*, \pi_2^*)(W_1)] = \frac{1}{8} g^* (p^*_{\widehat{P_1}} \widehat{A_1}^2 p^*_{\widehat{P_2}} \widehat{A_2} + p^*_{\widehat{P_1}} \widehat{A_1} p^*_{\widehat{P_2}} \widehat{A_2}^2). \tag{14}$$

The polarization $\widehat{A_j}$ is of type $(1, 2, \ldots, 2)$ and so, by [BiLa, Thm. 4.10.4], $\frac{1}{4!}\widehat{A_j}^3$ is a "minimal" cohomology class for $(P_j, \widehat{A_j})$; in other words, it is in $H^6(P_j, \mathbb{Z})$ and is not divisible.

5.11. Remark. Let $\mathcal{R}_{g(C)}$ be the moduli space of pairs (C, π), where C is a smooth projective curve of genus $g(C)$ and $\pi : \tilde{C} \to C$ is an étale double cover. We denote by

$$\mathrm{Pr} \colon \mathcal{R}_{g(C)} \to \mathcal{A}_{g(C)-1}$$

the Prym map associating to (C, π) the principally polarized Prym variety (P, Θ).

Let $\mathcal{B}_{g(C)}$ be the moduli space of bielliptic curves of genus $g(C) \geq 6$, and let $\mathcal{R}_{\mathcal{B}_{g(C)}} \subset \mathcal{R}_{g(C)}$ be the moduli space of étale double covers over them. Let $\mathcal{R}_{\mathcal{B}_{g(C), g(C_1)}}$ be those étale double covers such that $\tilde{C} \to C \to E$ has Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the curve C_1 has genus $g(C_1)$.

By [D1, Thm. 4.1(i)], the closure of $\mathrm{Pr}(\mathcal{R}_{\mathcal{B}_{g(C),1}})$ in $\mathcal{A}_{g(C)-1}$ contains the locus of Jacobians of hyperelliptic curves of genus $g(C)-1$. A general hyperelliptic Jacobian has the property that the cohomology class of every subvariety is an integral multiple of the minimal class [Bis]. Hence the same property holds for a general element in $\mathrm{Pr}(\mathcal{R}_{\mathcal{B}_{g(C),1}})$. So if V^2 were reducible then the irreducible components would have minimal cohomology class.

6. Prym Varieties of Bielliptic Curves, II

6.A. Tetragonal Construction and V^2

Denote by C an irreducible nodal curve of arithmetic genus $p_a(C) \geq 6$ and by $\pi: \tilde{C} \to C$ a Beauville admissible cover. By [B1], the corresponding Prym variety (P, Θ) is a principally polarized abelian variety. Suppose that C is a tetragonal curve—that is, suppose there exists a finite morphism $f: C \to \mathbb{P}^1$ of degree 4. We set $H := f^*\mathcal{O}_{\mathbb{P}^1}(1)$. By Donagi's tetragonal construction ([Do]; see also [BiLa, Chap. 12.8]), the corresponding special subvarieties give Beauville admissible covers $\tilde{C}' \to C'$ and $\tilde{C}'' \to C''$ such that C' and C'' are tetragonal and the Prym varieties are isomorphic to (P, Θ).

Consider now the residual line bundle $K_C \otimes H^*$. By Riemann–Roch, the linear series $|K_C \otimes H^*|$ is a $g^{p_a(C)-4}_{2p_a(C)-6}$ to which we can apply the construction of special subvarieties (cf. Section 5.A). If $S \subset \Lambda$ is a connected component then, by [B3, Thm. 1 and Rem. 4], the cohomology class of $V := i_{\tilde{C}}(S)$ is $\left[2\frac{\Theta^3}{3!}\right]$. Denote by (P^+, Θ^+) the canonically polarized Prym variety; thus,

$$\Theta^+ = \{L \in (\mathrm{Nm}\,\pi)^{-1}(K_C) \mid |L| \neq \emptyset,\ \dim|L| \equiv 0 \bmod 2\}.$$

Up to exchanging $\tilde{C}'$ and $\tilde{C}''$, we can suppose that the image of the natural map

$$V \times \tilde{C}' \to J\tilde{C}$$

is contained in P^+. By construction, the image is then contained in Θ^+; hence a translate of $-V$ is contained in the theta-dual $T(\tilde{C}')$. Since $T(\tilde{C}')$ equals the Brill–Noether locus $(V^2)'$ of the covering $\tilde{C}' \to C'$, it has cohomology class $\left[2\frac{\Theta^3}{3!}\right]$ and the inclusion is a (set-theoretical) equality.

The preceding argument shows that the special subvariety V is isomorphic to the Brill–Noether locus $(V_2)'$ of a tetragonally related covering. The following technical lemma shows that this special subvariety is irreducible unless we are in a very special situation.

6.1. LEMMA. *Let C be an irreducible nodal curve of arithmetic genus $p_a(C) \geq 6$, and let $\pi: \tilde{C} \to C$ be a Beauville admissible cover. Suppose that C is a tetragonal curve but is not hyperelliptic, trigonal, or a plane quintic. Assume that the normalization $\nu: T \to C$ is a hyperelliptic curve, and denote by $h: T \to \mathbb{P}^1$ the hyperelliptic covering.*

Denote by $f: C \to \mathbb{P}^1$ the morphism of degree 4 and set $H := f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Suppose that the base locus of the linear system $|K_C \otimes H^*|$ does not contain any points of C_{sing}, and suppose that the special subvariety S corresponding to $|K_C \otimes H^*|$ is reducible. *Then the following claims hold.*

(a) *There exist points p, q in the smooth locus C_{sm} such that we have a two-to-one cover*

$$\varphi_{|H\otimes\mathcal{O}_C(p+q)|}\colon C \to \bar{C} \subset \mathbb{P}^2$$

onto a singular plane cubic $\bar{C}$. This morphism factors through the hyperelliptic covering; *that is, we have a commutative diagram*

$$\begin{tikzcd} T \arrow[r, "\nu"] \arrow[d, "h"'] & C \arrow[d, "\varphi_{|H\otimes\mathcal{O}_C(p+q)|}"] \\ \mathbb{P}^1 \arrow[r, dotted, "\bar{\nu}"] & \bar{C}. \end{tikzcd}$$

(b) *If $x_1, x_2 \in T$ such that $\nu(x_1) = \nu(x_2)$, then $h(x_1) = h(x_2)$ unless $\bar{C}$ is nodal and $h(x_1)$ and $h(x_2)$ are mapped onto the unique node.*

Proof. Since C is not a plane quintic, the linear system $|K_C \otimes H^*|$ is base point free. By [B3, Sec. 2, Cor.] applied to the pull-back of the linear system to T, we know that S is irreducible if the linear system $|K_C \otimes H^*|$ induces a map $f\colon C \to \mathbb{P}^{p_a(C)-4}$ that is birational onto its image $f(C)$. Suppose now that this is not the case; then, for every generic point $p \in C$, there exists another generic point $q \in C$ such that

$$h^0(C, K_C \otimes H^* \otimes \mathcal{O}_C(-p-q)) = h^0(C, K_C \otimes H^* \otimes \mathcal{O}_C(-p)).$$

By Riemann–Roch, this equality implies that the linear system $|H \otimes \mathcal{O}_C(p+q)|$ is a base point free g^2_6. Because C is not hyperelliptic, we obtain in this way a 1-dimensional subset $W \subset W^2_6 C$. Consider now the morphism $\varphi_{|H\otimes\mathcal{O}_C(p+q)|}\colon C \to \bar{C} \subset \mathbb{P}^2$. Because C is irreducible and not trigonal, the curve $\bar{C}$ is an irreducible cubic or sextic curve.

Since H and $H \otimes \mathcal{O}_C(p+q)$ are base point free, it is easy to see that $|\nu^*(H \otimes \mathcal{O}_C(p+q))|$ is a g^3_6. Thus we have $\nu^*(H \otimes \mathcal{O}_C(p+q)) \simeq h^*\mathcal{O}_{\mathbb{P}^1}(3)$ and a factorization $\bar{\nu}\colon \mathbb{P}^1 \to \bar{C}$ such that $\bar{\nu} \circ h = \varphi_{|H\otimes\mathcal{O}_C(p+q)|} \circ \nu$.

In particular, $\varphi_{|H\otimes\mathcal{O}_C(p+q)|}$ is not birational onto its image and $\bar{C}$ is a singular cubic. A look at the lemma's commutative diagram shows that if $x_1, x_2 \in T$ such that $\nu(x_1) = \nu(x_2)$, then $h(x_1) = h(x_2)$ unless $\bar{C}$ is nodal and $h(x_1)$ and $h(x_2)$ are mapped onto the unique node. □

REMARK. For the sake of completeness we also consider the case where, in Lemma 6.1, the normalization T is not hyperelliptic. In this case the pull-backs $\nu^*(H \otimes \mathcal{O}_C(p+q))$ define a 1-dimensional subset $\tilde{W} \subset W^2_6 T$. It follows from [ACGH, p. 198] that T is bielliptic, and if $h\colon T \to E$ is a two-to-one map onto an elliptic curve E then $\nu^*(H \otimes \mathcal{O}_C(p+q)) \simeq h^*L$, where $L \in \operatorname{Pic}^3 E$. As before, we have a factorization $\bar{\nu}\colon E \to C'$, which is easily seen to be an isomorphism. In particular, C is obtained from T by identifying points that are in a h-fibre.

6.B. The Irreducible Components of V^2

Let C' be a smooth curve of genus $g(C) \geq 6$ that is bielliptic; in other words, we have a double cover $p'\colon C' \to E$ onto an elliptic curve E. As usual, $\pi'\colon \tilde{C}' \to C'$

will be an étale double cover. We suppose that the morphism $p' \circ \pi' \colon \tilde{C}' \to E$ is not Galois (in the terminology of [D1; N], the covering belongs to the family $\mathcal{R}'_{\mathcal{B}_{g(C)}} \subset \mathcal{R}_{g(C)}$; cf. Remark 5.11).

If we apply the tetragonal construction to a general g^1_4 on C, the result is a Beauville admissible cover $\pi : \tilde{C} \to C$ such that the normalization $\nu \colon T \to C$ is a smooth hyperelliptic curve T of genus $g(C) - 2$. Denote by $h \colon T \to \mathbb{P}^1$ the hyperelliptic structure. Then ν identifies two pairs of points, x_1, x_2 and y_1, y_2, such that $h(x_1), h(x_2), h(y_1), h(y_2)$ are four distinct points in $\mathbb{P}^1$ (this follows from the "figure locale" in [D1, 7.2.4]).

By [N, Chap. 15], a tetragonal structure on C can be constructed as follows. There exists a unique double cover $j \colon \mathbb{P}^1 \to \mathbb{P}^1$ sending each pair $h(x_1), h(x_2)$ and $h(y_1), h(y_2)$ onto a single point. The four-to-one covering $j \circ h \colon T \to \mathbb{P}^1$ factors through the normalization ν, so we have a four-to-one cover $f \colon C \to \mathbb{P}^1$. After applying the tetragonal construction to $H := f^*\mathcal{O}_{\mathbb{P}^1}(1)$, we recover the original étale double cover $\pi' \colon \tilde{C}' \to C'$. We have already seen in Section 6.A that the Brill–Noether locus V^2 associated to π' is isomorphic to a special subvariety associated to $|K_C \otimes H^*|$.

Now, by considering the exact sequence

$$0 \to \nu_*(K_T \otimes \nu^* H^*) \to K_C \otimes H^* \to \mathbb{C}_{\nu(x_1)} \oplus \mathbb{C}_{\nu(y_1)} \to 0,$$

one sees easily that the linear system $|K_C \otimes H^*|$ is base point free yet does not separate the singular points $\nu(x_1)$ and $\nu(y_1)$. Since the points $h(x_1), h(x_2), h(y_1), h(y_2)$ are distinct, it follows from Lemma 6.1 that the special subvarieties are irreducible. Our final proposition summarizes these considerations.

6.2. PROPOSITION. *Let C' be a smooth curve of genus $g(C') \geq 6$ that is bielliptic (i.e., we have a double cover $p' \colon C' \to E$ onto an elliptic curve E). Let $\pi' \colon \tilde{C}' \to C'$ be an étale double cover such that the cover $\tilde{C}' \to E$ is not Galois. Then V^2 is irreducible.*

7. Proof of Theorem 1.2

If V^2 is reducible then, by Corollary 3.6, C is trigonal, a plane quintic, or bielliptic. The first two cases are settled in Sections 4.B and 4.C, respectively. If C is bielliptic, we distinguish two cases depending on whether or not the four-to-one cover $\tilde{C} \to C \to E$ is Galois. In the Galois case we conclude by Corollary 5.7 and Proposition 5.9; otherwise, we use Proposition 6.2.

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