Spectra of Slant Toeplitz Operators with Continuous Symbols

MARK C. HO

1. Introduction

Let $\varphi(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}$ be a bounded function on the unit circle $\mathbf{T}$, where c_n is the nth Fourier coefficient of φ. The *slant Toeplitz operator* A_φ with symbol φ is an operator on $L^2(\mathbf{T})$ whose representing matrix with respect to the usual basis $\{e^{in\theta} : n \in \mathbf{Z}\}$ is:

$$\begin{pmatrix} \ddots & & & & & & \\ \cdots & c_{-2} & c_{-3} & \cdots & & & \\ \cdots & c_0 & c_{-1} & c_{-2} & \cdots & & \\ & \cdots & c_1 & c_0 & c_{-1} & \cdots & \\ & & \cdots & c_2 & c_1 & c_0 & \cdots \\ & & & \cdots & c_3 & c_2 & \cdots \\ & & & & & & \ddots \end{pmatrix}.$$

Note the double shift between rows instead of the single shift, as in the doubly infinite Toeplitz matrix which represents a multiplication operator on L^2, and note that one can obtain a matrix like this by eliminating every other row of a doubly infinite Toeplitz matrix.

Currently, the most general result about the spectrum of A_φ is that if φ is invertible then the spectrum of A_φ contains a closed disc, and the interior of that disc consists of eigenvalues with infinite multiplicity. In this paper, we will show that the spectrum of A_φ is a closed disc if φ is continuous. Furthermore, we will show that if φ is a continuous function that does not vanish on $\mathbf{T}$, then the interior of its spectrum consists of eigenvalues with infinite multiplicity.

2. Definitions and Notation

In this section, we will introduce some notation and definitions that will be used throughout the paper. We will also assume, throughout this section, that φ is continuous. Let us first define the following operator on $C(\mathbf{T})$:

Received March 1, 1996. Revision received July 22, 1996.

This paper is part of the author's doctoral thesis, written under the direction of Professor Carl C. Cowen at Purdue University.

Michigan Math. J. 44 (1997).

$$(L_\varphi f)(\theta) := \varphi\left(\frac{\theta}{2}\right) f\left(\frac{\theta}{2}\right) + \varphi\left(\frac{\theta}{2}+\pi\right) f\left(\frac{\theta}{2}+\pi\right)$$

for any f in $C(\mathbf{T})$. It's easy to see that, for any f in $C(\mathbf{T})$, the nth Fourier coefficient of $L_\varphi f$ is given by

$$\begin{aligned}\langle L_\varphi f, e^{in\theta}\rangle &= \int_0^{2\pi} (L_\varphi f)(\theta) e^{-in\theta}\, \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \left[\varphi\left(\frac{\theta}{2}\right) f\left(\frac{\theta}{2}\right) + \varphi\left(\frac{\theta}{2}+\pi\right) f\left(\frac{\theta}{2}+\pi\right)\right] e^{-in\theta}\, \frac{d\theta}{2\pi} \\ &= 2\int_0^{\pi} (\varphi(\zeta) f(\zeta) + \varphi(\zeta+\pi) f(\zeta+\pi)) e^{-i2n\zeta}\, \frac{d\zeta}{2\pi} \\ &= 2\int_0^{2\pi} \varphi(\zeta) f(\zeta) e^{-i2n\zeta}\, \frac{d\zeta}{2\pi}.\end{aligned}$$

On the other hand, since $A_\varphi = A_1 M_\varphi$ (where M_φ is the multiplication operator and A_1 is easily seen to be the adjoint of the composition operator induced by the map $\tau(e^{i\theta}) = e^{i2\theta}$ on $\mathbf{T}$; see [8]), the nth Fourier coefficient of $A_\varphi f$ is

$$\begin{aligned}\langle A_\varphi f, e^{in\theta}\rangle = \langle f, A_\varphi^* e^{in\theta}\rangle &= \langle f, \bar{\varphi} e^{i2n\theta}\rangle = \langle \varphi f, e^{i2n\theta}\rangle \\ &= \int_0^{2\pi} \varphi(\theta) f(\theta) e^{-i2n\theta}\, \frac{d\theta}{2\pi}.\end{aligned}$$

Therefore, $L_\varphi f = 2A_\varphi f$ for all f in $C(\mathbf{T})$.

REMARK. Let X be a compact metric space with a differential structure, and let $T\colon X \to X$ be an n-to-1 covering map. For an $\alpha > 0$ and a g in $C^\infty(X)$, consider the following operator on $C^\alpha(X)$:

$$\mathcal{L}_g f(x) := \sum_{y \in T^{-1}(x)} g(y) f(y)$$

for all f in $C^\alpha(X)$. Operators of this type are sometimes called the Ruelle–Perron–Frobenius operators, and were studied extensively by mathematicians in dynamical systems in the past few decades (see e.g. [1; 10; 12; 13]). The spectra of these operators under the $C^\alpha(X)$ topology ($\alpha > 0$) generally consist of a closed disc centered at the origin and some isolated eigenvalues outside that disc. Obviously, in our case $L_\varphi = \mathcal{L}_\varphi$ with $X = \mathbf{T}$ and $T = \tau$.

Next, let us consider the following set of functions on $\mathbf{T}$:

$$E := \left\{ f \in C(\mathbf{T}) : f \sim \sum_n a_n e^{in\theta} \text{ and } \sum_n |a_n|^2 e^{2|n|} < \infty \right\}.$$

It is not difficult to see that E is a Hilbert space with the inner product given by

$$\langle f, g\rangle_E = \sum_n a_n \bar{b}_n e^{2|n|}$$

for $f \sim \sum_n a_n e^{in\theta}$ and $g \sim \sum_n b_n e^{in\theta}$ in E. Moreover, given any f in E, one can extend f to an analytic function F on the annulus $\{z : e^{-1} < |z| < e\}$. Now consider φ in $C(\mathbf{T})$ whose Fourier coefficients c_n decay exponentially, that is, the coefficients satisfy

$$|c_n| \leq Ce^{-\gamma|n|} \tag{1}$$

for some C and $\gamma > 0$. The following properties are consequences of the work of Cohen and Daubechies in [2].

(i) L_φ is a compact operator on E if $\frac{1}{2} < \gamma < 1$.

(ii) If φ does not vanish on $\mathbf{T}$ and $\frac{1}{2} < \gamma < 1$, then r_e, the spectral radius of $L_{|\varphi|^2}$ on E, is an eigenvalue of algebraic multiplicity 1, and the corresponding eigenfunction is strictly positive. Also, $r_e > 0$.

Evidently, (i) and (ii) hold for any nonvanishing trigonometric polynomial φ on $\mathbf{T}$. We sketch a proof for (i) and (ii) as follows.

For $\frac{1}{2} < \gamma < 1$, the boundedness of L_φ follows easily from the estimate

$$\begin{aligned}
\|L_\varphi f\|_E^2 &= 4\sum_n \left|\sum_m c_{2n-m}a_m\right|^2 e^{2|n|} \\
&\leq 4\|f\|_E^2 \sum_{n,m} |c_{2n-m}|^2 e^{-2|m|}e^{2|n|} \\
&\leq 4C^2\|f\|_E^2 \sum_{n,m} e^{-2(1-\gamma)|m|}e^{-2(2\gamma-1)|n|},
\end{aligned}$$

and the compactness of L_φ follows from the fact that L_φ is trace class on E, since for any n and m we have

$$|\langle L_\varphi e_m, e_n\rangle_E| = 2|c_{2n-m}|e^{|n|}e^{-|m|} \leq 2Ce^{-(1-\gamma)|m|}e^{-(2\gamma-1)|n|},$$

where $\{e_n = e^{-|n|}e^{in\theta} : n \in \mathbf{Z}\}$ forms an orthonormal basis of E. Notice that both inequalities depend on the fact that $|2n - m| \geq 2|n| - |m|$ for all n and m.

For (ii), the space $E_0 = \{f \in E : f(\theta) \in \mathbf{R}, \theta \in [\theta, 2\pi]\}$ over $\mathbf{R}$ is an ordered real Banach space with the usual order, and it is easy to check that $L_{|\varphi|^2}$ sends E_0^+ into E_0^+ (where $E_0^+ = \{f \in E_0 : f \geq 0\}$). Hence, by the Krein–Rutman theorem (which can be regarded basically as a version of the Perron–Frobenius theorem for the infinite-dimensional case), r_e, the spectral radius of $L_{|\varphi|^2}$, is an eigenvalue of $L_{|\varphi|^2}$ (since, according to the terminology used in [14], E is the "complexification" of E_0). Furthermore, we can find a Φ in E_0^+ such that $L_{|\varphi|^2}\Phi = r_e\Phi$. Interested readers can find a detailed explanation for this argument in the appendix of [14].

To show that Φ is strictly positive, we use the following identity:

$$(L_\varphi^n f)(\theta) = \sum_{m=-2^{n-1}+1}^{2^{n-1}} \left[\prod_{k=1}^{n} \varphi(2^{-k}(\theta + 2m\pi))\right] f(2^{-n}(\theta + 2m\pi))$$

for all f in $C(\mathbf{T})$ and θ in $[0, 2\pi]$. Now, since the set $\{e^{2^{-n}(\theta+2m\pi)} : n, m \in \mathbf{Z}\}$ is a countable dense set in $\mathbf{T}$ for any θ, and since Φ is nonzero and continuous, there must exist N, M for each θ such that $\Phi\big(2^{-N}(\theta+2M\pi)\big) > 0$. From the above identity this means that

$$r_e^N\Phi(\theta) = (L_{|\varphi|^2}^N\Phi)(\theta) > 0,$$

since by assumption φ does not vanish on $\mathbf{T}$. But then this implies that $\Phi > 0$ on $\mathbf{T}$. Finally, the algebraic multiplicity of r_e can be derived from [14, 3.2, p. 270].

3. Lower Bound for A_φ^{*n}

Let us now return to slant Toeplitz operators. For any f in $L^2(\mathbf{T})$, we have

$$\|A_\varphi^{*n}f\|^2 = \langle A_\varphi^{*n}f, A_\varphi^{*n}f\rangle = \langle A_\varphi^n A_\varphi^{*n}f, f\rangle = \langle\psi_n f, f\rangle = \int_0^{2\pi}\psi_n|f|^2\,\frac{d\theta}{2\pi},$$

where $\psi_n = A_{|\varphi|^2}^n(1) \geq 0$ (see [8]). This implies that

$$\|A_\varphi^{*n}f\| \geq (\operatorname{essinf}\{\psi_n\})^{1/2}\|f\|_2$$

and hence $C_n = (\operatorname{essinf}\{\psi_n\})^{1/2}$ is a lower bound for A_φ^{*n}. In this section we will show that $C_n^{1/n} \to r(A_\varphi)$, where $r(A_\varphi)$ is the spectral radius of A_φ on $L^2(\mathbf{T})$. To achieve this, we need the following lemma.

LEMMA 3.1. *Recall that r_e is the spectral radius of $L_{|\varphi|^2}$ on E. Let Φ in E be a strictly positive eigenfunction of $L_{|\varphi|^2}$ associated with r_e. Then there exists a constant $\kappa \geq 0$ such that $(2r_e^{-1})^n\psi_n \to \kappa\Phi$ uniformly if φ is a trigonometric polynomial that does not vanish on the unit circle.*

Proof. The argument of the proof is borrowed from the proof of Lemma 4.7 in [2]. Let us first define a continuous function

$$g(\theta) := \frac{1}{r_e\Phi(2\theta)}\Phi(\theta)|\varphi(\theta)|^2, \quad \theta \in [0, 2\pi].$$

It is obvious that g is differentiable and strictly positive. Furthermore, for any θ in $[0, 2\pi]$ we have

$$\begin{aligned}
g(\theta) + g(\theta+\pi) &= \frac{1}{r_e\Phi(2\theta)}\Phi(\theta)|\varphi(\theta)|^2 + \frac{1}{r_e\Phi(2\theta)}\Phi(\theta+\pi)|\varphi(\theta+\pi)|^2 \\
&= \frac{1}{r_e\Phi(2\theta)}\big(\Phi(\theta)|\varphi(\theta)|^2 + \Phi(\theta+\pi)|\varphi(\theta+\pi)|^2\big) \\
&= \frac{r_e\Phi(2\theta)}{r_e\Phi(2\theta)} \quad (\text{since } L_{|\varphi|^2}(\Phi) = r_e\Phi) \\
&= 1.
\end{aligned}$$

It is then a consequence of a result in [10] that, for any f in $C(\mathbf{T})$, the sequence $L_g^n f$ converges uniformly to a constant.

Next we will show by induction that

$$L_{|\varphi|^2}^n f = r_e^n\Phi L_g^n(f\Phi^{-1})$$

for all f in $C(\mathbf{T})$ and all $n \geq 1$. Let us check this for $n = 1$. For any f in $C(\mathbf{T})$ and θ in $[0, 2\pi]$, we have

$$\begin{aligned}(L_g(f\Phi^{-1}))(\theta) &= g\left(\frac{\theta}{2}\right) f\left(\frac{\theta}{2}\right) \Phi\left(\frac{\theta}{2}\right)^{-1} \\ &\quad + g\left(\frac{\theta}{2}+\pi\right) f\left(\frac{\theta}{2}+\pi\right) \Phi\left(\frac{\theta}{2}+\pi\right)^{-1} \\ &= \frac{1}{r_e\Phi(\theta)}\left[\left|\varphi\left(\frac{\theta}{2}\right)\right|^2 f\left(\frac{\theta}{2}\right) + \left|\varphi\left(\frac{\theta}{2}+\pi\right)\right|^2 f\left(\frac{\theta}{2}+\pi\right)\right] \\ &= \frac{1}{r_e\Phi(\theta)}(L_{|\varphi|^2} f)(\theta).\end{aligned}$$

Now assume $L^k_{|\varphi|^2} f = r_e^k \Phi L_g^k (f\Phi^{-1})$ for $k < n-1$ and all f in $C(\mathbf{T})$. Then

$$\begin{aligned}L^n_{|\varphi|^2} f &= L_{|\varphi|^2}(L^{n-1}_{|\varphi|^2} f) \\ &= r_e^{n-1} L_{|\varphi|^2}\big(\Phi L_g^{n-1}(f\Phi^{-1})\big) \\ &= r_e^n \Phi L_g\big(\Phi L_g^{n-1}(f\Phi^{-1})\Phi^{-1}\big) \\ &= r_e^n \Phi L_g^n (f\Phi^{-1});\end{aligned}$$

hence the induction is complete.

Finally, let us prove the lemma. Let $\kappa = \lim_{n\to\infty} L_g^n(\Phi^{-1})$. Since $L_{|\varphi|^2}{=}2A_{|\varphi|^2}$ on $C(\mathbf{T})$, we have

$$(2r_e^{-1})^n \psi_n = (2r_e^{-1})^n A^n_{|\varphi|^2}(1) = (r_e^{-1})^n L^n_{|\varphi|^2}(1) = \Phi L_g^n(\Phi^{-1}) \to \kappa\Phi$$

uniformly on $\mathbf{T}$. It is also clear that $\kappa \geq 0$, since $L_g^n(\Phi^{-1}) \geq 0$ for all n. □

We will now present our main result in this section as a result of Lemma 3.1.

PROPOSITION 3.2. *If φ is a trigonometric polynomial with no zeros on the unit circle, then*

$$C_n^{1/n} = (\operatorname{essinf}\{\psi_n\})^{1/2n} \to r(A_\varphi).$$

Proof. Since Φ is strictly positive and continuous, the sequence $\psi_n^{1/n}$ converges uniformly to $\frac{1}{2}r_e > 0$ (if $\kappa > 0$) or 0 (if $\kappa = 0$), by Lemma 3.1. But since $\|\psi_n\|_\infty^{1/n} \to r(A_\varphi)^2$ and $r(A_\varphi) \geq r(A_{\varphi^{-1}})^{-1} > 0$ (see [8]), $\psi_n^{1/n} \to r(A_\varphi)^2$ uniformly. This completes the proof. □

Note that we have also proved the following.

COROLLARY 3.3. *If φ is a nonvanishing trigonometric polynomial, then*

$$r_e = 2r(A_\varphi)^2.$$

4. Spectrum of A_φ

We are now ready to prove the main theorem. (Theorem 4.1 also appears in [11], a fact that was brought to the author's attention in May 1996. The work here, however, was done independently and its proof uses different methods.)

THEOREM 4.1. *If φ is a nonvanishing trigonometric polynomial on the unit circle, then $\sigma(A_\varphi)$ and $\sigma_e(A_\varphi)$, the spectrum and the essential spectrum of A_φ, are the closed disc*

$$\{\lambda : |\lambda| \leq r(A_\varphi)\}.$$

Furthermore, if $|\lambda| < r(A_\varphi)$ then $\dim\ker(A_\varphi - \lambda) = \infty$ *and* $\dim\ker(A_\varphi^* - \lambda) = 0$.

Proof. The idea of the proof can be found in [5] (or see [6, Thm. 7.44]). From Proposition 3.2 we have

$$C_n^{1/n} = (\operatorname{essinf}\{\psi_n\})^{1/2n} \to r(A_\varphi).$$

Thus, if $|\lambda| = r < r(A_\varphi)$ then we can choose large n such that $|\lambda|^n = r^n < C_n$ and $C_n > 0$.

On the other hand, from the definition of C_n we have

$$C_n \|f\|_2 \leq \|A_\varphi^{*n} f\|_2$$

for all f in $L^2(\mathbf{T})$. In particular, this means that A_φ^{*n} is one-to-one and the range $\mathcal{R}$ of A_φ^{*n} is closed. Let P be the projection from $L^2(\mathbf{T})$ onto $\mathcal{R}$, and let S be the inverse of A_φ^{*n} as a map from $\mathcal{R}$ onto $L^2(\mathbf{T})$—that is, $SA_\varphi^{*n} = I$ and $A_\varphi^{*n} S = I_{\mathcal{R}}$. This implies that

$$\|SP\| \leq \|S\| \leq C_n^{-1} < r^{-n} = |\lambda|^{-n}.$$

Therefore, the series

$$S_\lambda = \sum_{k=0}^{\infty} \bar{\lambda}^{nk} (SP)^{k+1}$$

converges absolutely. Furthermore, it is easy to see that $S_\lambda(A_\varphi^{*n} - \bar{\lambda}^n) = I$. Hence S_λ is a left inverse of $A_\varphi^{*n} - \bar{\lambda}^n$ (hence S_λ^* is a right inverse of $A_\varphi^n - \lambda^n$).

Claim: $A_\varphi^n - \lambda^n$ is semi-Fredholm but not Fredholm.

First, we would like to point out that the range of $A_\varphi^n - \lambda^n$ is closed since $A_\varphi^n - \lambda^n$ is right invertible. Second, since $\mathcal{R}^\perp = \ker(A_\varphi^n) \supseteq \ker(A_\varphi) = \{\varphi^{-1} e^{i(2k-1)6} : k \in \mathbf{Z}\}$ is infinite-dimensional (see [9]), $\ker(P) = \mathcal{R}^\perp$ is infinite-dimensional. This means that $\dim\ker(S_\lambda) \geq \dim\ker(P) = \infty$. Suppose that $A_\varphi^n - \lambda^n$ is Fredholm; then $A_\varphi^{*n} - \bar{\lambda}^n$ is also Fredholm. But since S_λ is the left inverse of $A_\varphi^{*n} - \bar{\lambda}^n$, this would imply that S_λ is Fredholm, which is impossible (note we have actually shown that $\dim\ker(A_\varphi^n - \lambda^n) = \infty$).

The claim above implies that $\sigma_e(A_\varphi^n)$, the essential spectrum of A_φ^n, includes the circle $\{\zeta : |\zeta| = r^n\}$. Now consider the identity

$$\prod_{k=1}^{n} (A_\varphi - \lambda e^{i2\pi k/n}) = A_\varphi^n - \lambda^n,$$

and define

$$\tilde{S}_\lambda = \left[\prod_{k=1}^{n-1}(A_\varphi - \lambda e^{i2\pi k/n})\right] S_\lambda^*.$$

A direct computation shows that $(A_\varphi - \lambda)\tilde{S}_\lambda = I$. Since $\sigma_e(A_\varphi^n)$ includes the circle of radius r^n, the spectral mapping theorem for the Calkin algebra implies that $\sigma_e(A_\varphi)$ intersects the circle of radius r. Suppose that there is a λ_0 on the boundary of $\sigma_e(A_\varphi)$ with $|\lambda_0| = r$. Then, as $\lambda \to \lambda_0$ from the essential resolvent of A_φ with $|\lambda| = r$, the essential norm of the essential inverse of $A_\varphi - \lambda$ approaches infinity. If $A_\varphi - \lambda$ is essentially invertible then $\tilde{S}_\lambda$, its right inverse, must be its essential inverse. But the estimate

$$\|\tilde{S}_\lambda\|_e \leq \|\tilde{S}_\lambda\| \leq \|S\|(1 - r^n\|S\|)^{-1}(\|A_\varphi\| + r)^{n-1}$$

for all $|\lambda| = r$ shows that the boundary of $\sigma_e(A_\varphi)$ cannot intersect the circle of radius r, which means that $\sigma_e(A_\varphi)$ contains this circle. Therefore, $A_\varphi - \lambda$ is right invertible but not Fredholm for all λ with $|\lambda| = r$, and the proof is complete since r is arbitrary. □

We can extend Theorem 4.1 to the case when φ is a nonvanishing continuous function on **T** as follows.

THEOREM 4.2. *If φ is a continuous function with no zero on the unit circle, then both the spectrum and the essential spectrum of A_φ are the closed disc*

$$\{\lambda : |\lambda| \leq r(A_\varphi)\}$$

Moreover, $\dim\ker(A_\varphi - \lambda) = \infty$ *and* $\dim\ker(A_\varphi^* - \bar{\lambda}) = 0$ *if* $|\lambda| < r(A_\varphi)$.

Proof. The key in proving Theorem 4.1 is to show that

$$C_n^{1/n} = (\operatorname{essinf}\{\psi_n\})^{1/2n} \to r(A_\varphi).$$

Given $\rho > 1$, it is not difficult to see that there exists an $\varepsilon > 0$ such that $|\psi| < \rho|\varphi|$ a.e. whenever $\|\psi - \varphi\|_\infty < \varepsilon$. Now let p be a nonvanishing trigonometric polynomial such that $\|p - \varphi\|_\infty < \varepsilon$. This implies that $|p| < \rho|\varphi|$, and therefore $\tilde{\psi}_n = A_{|p|^2}^n(1) \leq \rho^n A_{|\varphi|^2}^n(1) = \rho^n \psi_n$. Thus, by applying Proposition 3.2 to p and setting $\tilde{C}_n = (\operatorname{essinf}\{\tilde{\psi}_n\})^{1/2}$, we obtain

$$r(A_p) = \lim_{n\to\infty} \tilde{C}_n^{1/n} \leq \sqrt{\rho}\, \liminf_{n\to\infty} C_n^{1/n}.$$

Let us choose a sequence of nonvanishing trigonometric polynomials p_n such that $p_n \to \varphi$ uniformly on **T**. We can find large N so that $\|p_n - \varphi\|_\infty < \varepsilon$ if $n > N$. This means that

$$r(A_{p_n}) \leq \sqrt{\rho}\, \liminf_{n\to\infty} C_n^{1/n}$$

if $n > N$. By [9, Prop. 5.1], we have

$$r(A_\varphi) = \lim_{n\to\infty} r(A_{p_n}) \leq \sqrt{\rho}\, \liminf_{n\to\infty} C_n^{1/n}.$$

Since ρ is arbitrary, we have

$$r(A_\varphi) \le \liminf_{n\to\infty} C_n^{1/n},$$

and the theorem follows again from the fact that $\|\psi_n\|_\infty^{1/n} \to r(A_\varphi)^2$. ❑

Our next theorem is an immediate consequence of Theorem 4.2.

THEOREM 4.3. *If φ is a continuous function on the unit circle, then the spectrum of A_φ is the closed disc*

$$\{\lambda : |\lambda| \le r(A_\varphi)\}.$$

Proof. We will use an elementary result from operator theory:

$$r(A) \le \liminf_{n\to\infty} r(A_n)$$

if $\|A_n - A\| \to 0$. Since the set of nonvanishing continuous functions is dense in $C(\mathbf{T})$, we may choose a sequence φ_n of nonvanishing continuous functions on $\mathbf{T}$ such that $\|\varphi_n - \varphi\|_\infty \to 0$. This means that $\|A_{\varphi_n} - A_\varphi\| \to 0$.

Now let $r < \liminf_{n\to\infty} r(A_{\varphi_n})$. Choose a subsequence $r(A_{\varphi_{n_j}})$ of $r(A_{\varphi_n})$ such that

$$r(A_{\varphi_{n_j}}) \searrow \liminf_{n\to\infty} r(A_n).$$

In particular, $r(A_{\varphi_{n_j}}) > r$ for all j. Therefore, by Theorem 4.2, there exist f_j with $\|f_j\|_2 = 1$ in $L^2(\mathbf{T})$ for every j such that

$$A_{\varphi_{n_j}} f_j = \lambda f_j$$

if $|\lambda| = r$. But this means that

$$\|(A_\varphi - \lambda)f_j\| \le \|(A_{\varphi_{n_j}} - A_\varphi)f_j\| + \|(A_{\varphi_{n_j}} - \lambda)f_j\| = \|(A_{\varphi_{n_j}} - A_\varphi)f_j\| \to 0,$$

which implies that $\{\lambda : |\lambda| = r\}$ is contained in the approximate point spectrum of A_φ, and this completes the proof. ❑

Next we show that the spectral radius $r(A_\varphi)$ is nonzero for a sufficiently large class of continuous symbols on $\mathbf{T}$.

PROPOSITION 4.4. *If φ is a nonzero trigonometric polynomial on $\mathbf{T}$, then $r(A_\varphi)$ is nonzero, that is, $\sigma(A_\varphi) \ne \{0\}$.*

Proof. Without loss of generality, let $\varphi(\theta) = \sum_{n=0}^{N} a_n e^{in\theta}$ with $a_0 \ne 0$ (since clearly $r(A_\varphi) = r(A_{\lambda\varphi})$ if $|\lambda| = 1$). In [8] it is shown that the subspace H of $L^2(\mathbf{T})$ spanned by $\{e^{in\theta} : |n| \le N\}$ is invariant under A_φ. Let $A = A_\varphi|_H$. Now assume that $r(A_\varphi) = 0$; this means that $r(A) = 0$, so $\det(xI - A) = x^m$ for some integer $m > 0$. By the Cayley–Hamilton theorem, $A^m = 0$ (i.e., A is nilpotent). However, we have

$$\langle A^k 1, 1\rangle = a_0^k \ne 0$$

for any integer $k > 0$ (since $a_0 \ne 0$), and this leads to a contradiction. This completes the proof. ❑

5. Final Thoughts

Keane [10] has actually shown that the sequence $L_g^n f$ converges uniformly to a constant if g is in $C^1(\mathbf{T})$, $g(\theta) + g(\theta + \pi) = 1$ for all θ in $[0, 2\pi]$, and one of the following conditions is satisfied:

(i) g has only one zero on $\mathbf{T}$;
(ii) g has finitely many zeros and none of the zeros wanders into a periodic orbit of the mapping $\tau(e^{i\theta}) = e^{2i\theta}$; or
(iii) all zeros of g lie in $[\frac{\pi}{2}, \frac{3\pi}{2})$ or $(\frac{\pi}{2}, \frac{3\pi}{2}]$

(see [10, p. 313]). This means that the conclusion of Theorem 4.2 will hold for a larger class of the symbols φ (in fact, we believe that Theorem 4.2 is still valid for the class of Hölder-continuous φ on $\mathbf{T}$). The reason for our insistence on nonvanishing symbols is just for the sake of simplifying the statements. In fact, judging from the kind of results we have in Section 4, we believe that the spectral properties of A_φ depend more on the zero set of φ than on the smoothness of φ. Hence it is natural to pose the following questions: What is the weakest requirement for φ under which the conclusion of Theorem 4.2 will hold? Does there exist a φ in $L^\infty(\mathbf{T})$ such that $\sigma(A_\varphi)$ is not a closed disc? (As we have already shown in [8], $\sigma(A_\varphi)$ contains a closed disc centered at 0.)

This work also seems to have pointed out some interesting relations between the spectral properties of A_φ and the topologies of the spaces on which A_φ are acting. First, we would like to present the following theorem, which is a special case of a remarkable result due to Ruelle [13].

THEOREM 5.1. *Let φ be in $C(\mathbf{T})$, whose Fourier coefficients c_n decay exponentially (see Section 2 for definition). Given $\alpha > 0$, let r_α and ρ_α be the spectral radii of A_φ and $A_{|\varphi|}$ on $C^\alpha(\mathbf{T})$, respectively. Let ρ_0 be the spectral radius of $A_{|\varphi|}$ on $C(\mathbf{T})$. Then, for any $\alpha > 0$, we have $r_\alpha \le \rho_0$ and $\rho_\alpha = \rho_0$. The spectrum of A_φ on $C^\alpha(\mathbf{T})$ consists of a closed disc centered at 0 whose interior consists of eigenvalues with infinite multiplicity and some isolated eigenvalues with finite multiplicity in the region $\{\lambda : |\lambda| > 2^{-\alpha}\rho_0\}$.*

By letting $\alpha \searrow 0$, one can show that $\sigma_0(A_{|\varphi|})$, the spectrum of $A_{|\varphi|}$ on $C(\mathbf{T})$, is a closed disc centered at 0 whose interior consists of eigenvalues with infinite multiplicity. Moreover, the spectral radius ρ of $A_{|\varphi|}$ on $C(\mathbf{T})$ is also an eigenvalue (no longer isolated). This observation may lead one to believe that Theorem 4.1 can be derived from Theorem 5.1 with some modification. But this turns out to be not true: one can show that, for example, there is no eigenvalue on the circle of the spectral radius of $A_{|e^{i\theta}+\alpha|}$ on $L^2(\mathbf{T})$ if $\alpha \neq 0$ (see [9]). On the other hand, by Theorem 4.1 and Theorem 5.1 we have

$$\cdots \subseteq \sigma_\alpha(A_\varphi) \subseteq \cdots \subseteq \sigma_0(A_\varphi) \subseteq \sigma(A_\varphi)$$

(where $\sigma_\alpha(A_\varphi)$ is the spectrum of A_φ on $C^\alpha(A_\varphi)$) if φ is a trigonometric polynomial with no zeros on $\mathbf{T}$. The foregoing discussion suggests that the spectral radius of $A_{|\varphi|}$ is no longer the constant ρ_0 if the topology on the space is weaker than the uniform topology. In fact, we believe that ρ_p, the spectral radius of $A_{|\varphi|}$ on $L^p(\mathbf{T})$, is strictly increasing as $p \searrow 1$ unless φ is unimodular on $\mathbf{T}$.

One of the reasons that the operator $\mathcal{L}_g$ (see Section 2) is called the Ruelle–Perron–Frobenius operator is because, when $g \geq 0$, it sends $C^\alpha(\mathbf{T})^+$ into $C^\alpha(\mathbf{T})^+$ for all $\alpha \geq 0$. As a consequence, the spectral radius of $\mathcal{L}_g$ on $C^\alpha(\mathbf{T})$ is an eigenvalue with a corresponding eigenvector in $C^\alpha(\mathbf{T})^+$, if g satisfies certain conditions (i.e., if g is a strictly positive C^∞ function). This conclusion, however, does not hold in $L^2(\mathbf{T})$, as we have seen in the example $A_{|e^{i\theta}+\alpha|}$ for $\alpha \neq 0$ (although these operators still map $L^2(\mathbf{T})^+$ into $L^2(\mathbf{T})^+$). We believe that the main reason for this is because—whereas $C^\alpha(\mathbf{T})^+$ has nonempty interior in $C^\alpha(\mathbf{T})$ for all $\alpha \geq 0$—the interior of $L^2(\mathbf{T})^+$ is empty. This raises some interesting questions: Does there exist a probability measure μ on $\mathbf{T}$ such that the spectral radius of A_φ (φ a strictly positive C^∞ function, for instance) on $L^2(\mu)$ is still an eigenvalue; and if so, how do we characterize such μ?

References

[1] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Math., 470, Springer, New York, 1975.

[2] A. Cohen and I. Daubechies, *A new technique to estimate the regularity of refinable functions,* preprint, 1994.

[3] J. B. Conway, *A course in functional analysis,* 2nd ed., Springer, New York, 1990.

[4] ———, *The theory of subnormal operators,* Math. Surveys Monographs, 36, Amer. Math. Soc., Providence, RI, 1991.

[5] C. C. Cowen, *Composition operators on H^2,* J. Operator Theory 9 (1983), 77–106.

[6] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions,* CRC Press, Boca Raton, FL, 1995.

[7] R. G. Douglas, *Banach algebra techniques in operator theory,* Academic Press, New York, 1972.

[8] M. C. Ho, *Properties of slant Toeplitz operators,* Indiana Univ. Math. J. (to appear).

[9] ———, *Adjoints of slant Toeplitz operators,* preprint, 1995.

[10] M. Keane, *Strongly mixing g-measure,* Invent. Math. 16 (1972), 309–324.

[11] Y. D. Latushkin, *On integro-functional operators with a shift which is not one-to-one,* Math. USSR-Izv. 19 (1982), 479–493.

[12] D. Ruelle, *Zeta functions for expanding map and Anosov flows,* Invent. Math. 34 (1976), 231–242.

[13] ———, *An extension of the theory of Fredholm determinants,* Inst. Hautes Études Sci. Publ. Math. 72 (1990), 175–193.

[14] H. H. Schaefer, *Topological vector spaces* (Macmillan Series in Advanced Mathematics and Theoretical Physics), Macmillan, New York, 1966.

Department of Mathematics
Purdue University
West Lafayette, IN 47907-1395

mho@math.purdue.edu