A SHARP GOOD-λ INEQUALITY WITH AN APPLICATION TO RIESZ TRANSFORMS

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0. Introduction. Let $f \in \mathcal{S}(\mathbf{R}^n)$, the class of rapidly decreasing functions in $\mathbf{R}^n$. For $j = 1, 2, \ldots, n$, define the Riesz transforms as the multiplier operators $(R_j f)\hat{}(\xi) = (i\xi_j/|\xi|)\hat{f}(\xi)$ and set

$$Rf(x) = \left(\sum_{j=1}^{n} |R_j f(x)|^2\right)^{1/2}$$

In [22] Stein proved that $\|Rf\|_p \le C_p \|f\|_p$ for $1 < p < \infty$, with C_p independent of the dimension n. Alternative probabilistic proofs are given in [1], [4], [14], and [16], and in [9] the result is proved by using the method of rotations. Whereas all of these proofs give constants independent of n, none of them give the correct behavior with respect to p (see the remarks following Corollary 2.3). On the other hand, the classical proof [20] which gives constants depending on the dimension does give the right asymptotic behavior with respect to p. That is, C_p is $O(p)$ as $p \to \infty$ and $O(1/(p-1))$ as $p \to 1$. It seems unnatural to us that this should be lost when passing to constants independent of n. The purpose of this note is to correct this deficiency. We do this by proving a sharp good-λ inequality (Lemma 1.2) for vector-valued martingales which itself may be of independent interest.

1. The good-λ inequality. Throughout the paper, we use the following notation. If X_t is an L^p-bounded martingale with $1 < p < \infty$, X will denote the random variable in L^p such that $X_t = E(X \mid \mathfrak{F}_t)$ and $\langle X \rangle_t$ will denote the square function of X_t. By $\langle X \rangle$ we shall mean $\langle X \rangle_\infty$. All our martingales are on the Brownian filtration and hence always continuous.

THEOREM 1.1. *Let $\{X^i\}_{i=1}^m$ be random variables in L^p with $2 \le p < \infty$ and such that $EX^i = 0$ for all i. There are universal constants C_1 and C_2 (independent of p and m) such that*

$$\left\|\left(\sum_{i=1}^{m} |X^i|^2\right)^{1/2}\right\|_p \le C_1\sqrt{p}\left\|\left(\sum_{i=1}^{m} \langle X^i\rangle\right)^{1/2}\right\|_p, \tag{1.1}$$

$$\left\|\left(\sum_{i=1}^{m} \langle X^i\rangle\right)^{1/2}\right\|_p \le C_2\sqrt{p}\left\|\left(\sum_{i=1}^{m} |X^i|^2\right)^{1/2}\right\|_p. \tag{1.2}$$

Part (1.2) of the theorem is well known but we shall prove it here for the convenience of the reader. However, for (1.1) all of the proofs known to the author will give, at best, constants of order p. To prove (1.2) we recall the following lemma.

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LEMMA 1.1 (Garsia [12]). *Let A_t be a positive continuous increasing process with $A_0=0$. If there is a positive random variable Y such that $E(A_\infty - A_T \mid \mathfrak{F}_T) \le E(Y \mid \mathfrak{F}_T)$ for any stopping time T, then $E(A_\infty^p) \le p^p E(Y^p)$, $1<p<\infty$.*

Proof of (1.2). We assume that $p>2$; otherwise the result is trivial. Because $(X_t^i)^2 - \langle X^i \rangle_t$ is a martingale we have, for any stopping time T,

$$E(\langle X^i\rangle - \langle X^i\rangle_T \mid \mathfrak{F}_T) = E((X^i)^2 - (X_T^i)^2 \mid \mathfrak{F}_T) \le E((X^i)^2 \mid \mathfrak{F}_T).$$

Summing both sides and applying Garsia's lemma with $p/2>1$, we obtain (1.2) with $C_2 = \sqrt{1/2}$. □

The first part of the theorem will be a consequence of the following good-λ inequality.

LEMMA 1.2. *Let $\{X^i\}_{i=1}^m$ be as in the statement of the theorem. Set*

$$Y_t = \left(\sum_{i=1}^m |X_t^i|^2\right)^{1/2} \quad \textit{and} \quad Y^* = \sup_t |Y_t|.$$

Let $Z = \sum_{i=1}^m \langle X^i \rangle$ and fix $\epsilon>0$. Then, for all $\lambda>0$,

$$P\{Y^*>2\lambda;\, Z^{1/2} \le \epsilon\lambda\} \le C_1 \exp\left[\frac{-C_2}{\epsilon^2}\right] P\{Y^*>\lambda\},$$

where C_1 and C_2 are universal constants.

The novelty in this lemma is the presence of the exponential square estimate on the right-hand side of the inequality. The original good-λ inequalities of Burkholder and Gundy [5] did not give this sharp estimate. However, for the case of a single martingale ($m=1$) and the case when the martingales are mutually orthogonal with equal area functions, the lemma as we have stated it above was proved by Burkholder in [6]. Our proof follows his argument very closely. From the lemma we easily get (1.1).

Proof of (1.1).

$$\begin{aligned} E\left(\frac{Y^*}{2}\right)^p &= p\int_0^\infty \lambda^{p-1} P\{Y^*>2\lambda\}\, d\lambda \\ &\le p\int_0^\infty \lambda^{p-1} P\{Y^*>2\lambda;\, Z^{1/2}\le\epsilon\lambda\}\, d\lambda + p\int_0^\infty \lambda^{p-1} P\{Z^{1/2}>\epsilon\lambda\}\, d\lambda \\ &\le C_1 \exp\left[\frac{-C_2}{\epsilon^2}\right] p\int_0^\infty \lambda^{p-1} P\{Y^*>\lambda\}\, d\lambda + p\int_0^\infty \lambda^{p-1} P\{Z^{1/2}>\epsilon\lambda\}\, d\lambda \\ &\le C_1 \exp\left[\frac{-C_2}{\epsilon^2}\right] E|Y^*|^p + \frac{1}{\epsilon^p} E(Z^{p/2}), \end{aligned}$$

where the first inequality is clear and we have applied Lemma 1.2 for the second. From here we obtain (using the convention that C is a universal constant)

$$E(Y^*)^p \le \frac{2^p C_1 \exp[C_2/\epsilon^2]}{\epsilon^p(\exp[C_2/\epsilon^2] - 2^p)} E(Z^{p/2}), \tag{1.3}$$

and choosing $\epsilon \sim C/\sqrt{p}$ gives

$$E(Y^*)^p \leq C^p(\sqrt{p})^p E(Z^{p/2}), \tag{1.4}$$

which is the same as (1.1). □

To make the proof of Lemma 1.2 more clear we break it into several lemmas. Besides the random variables Y_t, Y^*, and Z which have already been introduced above, we set $Y=(\sum_{i=1}^m |X^i|^2)^{1/2}$ and $Z_t=\sum_{i=1}^m \langle X^i\rangle_t$.

LEMMA 1.3 (Pipher [19]). *The process*

$$W_t=\exp[\sqrt{1+(Y_t)^2}-\tfrac{1}{2}Z_t] \tag{1.5}$$

is a supermartingale.

The proof of this lemma is an application of the Itô formula with

$$F(x_1,\dots,x_m,y_1,\dots,y_m)=\exp\left[\left(1+\sum_{i=1}^m x_i^2\right)^{1/2}-\frac{1}{2}\sum_{i=1}^m y_i\right].$$

LEMMA 1.4. *Suppose* $\|Z^{1/2}\|_\infty \leq a<\infty$. *Then, for all* $\lambda>0$,

$$P\{Y^*>\lambda\}\leq C_1\exp\left[\frac{-C_2\lambda^2}{a^2}\right]. \tag{1.6}$$

Proof. It follows from Lemma 1.3 that $E(\exp[Y])\leq e\cdot\exp[\frac{1}{2}a^2]$, and repeating this with X^i replaced by $(\lambda/a^2)X^i$ for any $\lambda>0$ we obtain

$$E\left(\exp\left[\frac{\lambda Y}{a^2}\right]\right)\leq e\cdot\exp\left[\frac{\lambda^2}{2a^2}\right], \tag{1.7}$$

which together with Chebyschev's inequality gives

$$P\{Y>\lambda\}\leq e\cdot\exp\left[\frac{-\lambda^2}{2a^2}\right]. \tag{1.8}$$

From Doob's inequality applied to the submartingale Y_t we have that $\|Y^*\|_p^p\leq (p/(p-1))^p\|Y\|_p^p\leq 16\|Y\|_p^p$ for $p\geq 2$. Thus we see, by summing the series for the exponential, that if $E(\exp[C_1Y^2/a^2])\leq C_2$ then the same holds for Y^*, and therefore (1.8) gives the conclusion of the lemma. □

Let T be a stopping time and set $\tilde{\Omega}=\{T<\infty\}$, $\tilde{\mathfrak{F}}_t=\mathfrak{F}_{T+t}$, and $d\tilde{P}=dP/P(\tilde{\Omega})$. Then $\tilde{X}_t^i=X_{T+t}^i-X_t^i$ is a $\tilde{P}$-martingale over $\tilde{\mathfrak{F}}_t$ with $\langle\tilde{X}^i\rangle_t=\langle X^i\rangle_{T+t}-\langle X^i\rangle_T$. Applying the above argument to the $\tilde{X}^i$ (under the assumptions of Lemma 1.4) we obtain

$$P\left\{\sup_t\left(\sum_{i=1}^m |X_{T+t}^i-X_T^i|^2\right)^{1/2}>\lambda;\, T<\infty\right\}\leq C_1\exp\left[\frac{-C_2\lambda^2}{a^2}\right]P\{T<\infty\}. \tag{1.9}$$

LEMMA 1.5. *Suppose* $\|Z^{1/2}\|_\infty\leq a<\infty$. *Then, for all* $\lambda>0$,

$$P\{Y^*>2\lambda\}\leq C_1\exp\left[\frac{-C_2\lambda^2}{a^2}\right]P\{Y^*>\lambda\}.$$

Proof. Let $T=\inf\{t: Y_t>\lambda\}$. The continuity of the paths gives $Y_T=\lambda$ on $\{T<\infty\}=\{Y^*>\lambda\}$. Therefore

$$P\{Y^*>2\lambda\}\le P\left\{\sup_{t>T}(Y_t)>2\lambda;\, T<\infty\right\}\le P\left\{\sup_{t>T}\left(\sum_{i=1}^m |X_t^i-X_T^i|^2\right)^{1/2}>\lambda;\, T<\infty\right\},$$

and applying (1.9) gives the lemma. □

Proof of Lemma 1.2. Let $\tau=\inf\{t: Z_t>(\epsilon\lambda)^2\}$. Then if $\tilde{X}_t=X^i_{t\wedge\tau}$ we h.ve $\|\tilde{Z}^{1/2}\|_\infty\le\epsilon\lambda$ and

$$\begin{aligned}P\{Y^*>2\lambda;\, Z^{1/2}\le\epsilon\lambda\}&=P\{Y^*>2\lambda;\,\tau=\infty\}\\&=P\left\{\sup_{0<t<\tau}|Y_t|>2\lambda;\,\tau=\infty\right\}\le P\{\tilde{Y}^*>2\lambda\}\\&\le C_1\exp\left[\frac{-C_2\lambda^2}{\epsilon^2\lambda^2}\right]P\{\tilde{Y}^*>\lambda\}\le C_1\exp\left[\frac{-C_2}{\epsilon^2}\right]P\{Y^*>\lambda\},\end{aligned}$$

where the next-to-last inequality follows from Lemma 1.5 and the rest are clear. □

We should remark here that the estimate $\exp[-C_2/\epsilon^2]$ is absolutely crucial to obtain the constant $\sqrt{p}$ in (1.1). Even with the extimate $\exp[-C_2/\epsilon]$ one cannot get this behavior in p. Also, asymptotically the constants in Theorem 1.1 are best possible since they are already best possible in the scalar case; see [7].

Let us mention another good-λ inequality that can be obtained by the methods of [6]. We leave the proof to the interested reader.

THEOREM 1.2. *Let B_t be one-dimensional Brownian motion and let τ be a stopping time. For $0\le\alpha<\frac{1}{2}$ define $M_\alpha^*=\sup_{t<\tau}|t^{-\alpha}B_t|$. Then*

$$P\{M_\alpha^*>2\lambda;\,\tau^{1/2-\alpha}\le\epsilon\lambda\}\le C_\alpha\exp\left[\frac{-C'_\alpha}{\epsilon^2}\right]P\{M_\alpha^*>\lambda\},$$

where C_α and C'_α are constants depending on α.

This good-λ inequality can be used to give the right behavior with respect to p in the constants of some of the inequalities proved by Barlow and Yor [2]. For other good-λ inequalities which also use the ideas in [6], see Bass [3] and Davis [8].

2. Martingale transforms and their projections in $\mathbf{R}^n$. Let X_t be an L^2-martingale on the filtration $\mathfrak{F}_t$ of n-dimensional Brownian motion, $n\ge 2$. Such martingales have the representation [10] $X_t=X_0+\int_0^t H_s\cdot dB_s$, where $X_0=EX$ and H_s is a process with values in $\mathbf{R}^n$ adapted to $\mathfrak{F}_s$. If $A(s)$ is an $n\times n$ matrix-valued process adapted to $\mathfrak{F}_s$, we define the martingale transform $(A*X)_t=\int_0^t(A(s)H_s)\cdot dB_s$. The following basic result can be found in [10]:

$$\|A*X\|_p\le C_pM\|X\|_p\quad\text{for } 1<p<\infty.$$

Here C_p depends only on p and $M=\sup_s M(s)$, where $M(s)=\sup\{|A(s)V|: V\in\mathbf{R}^n,\ |V|\le 1\}$. The constant C_p obtained in [10] is, at best, of order $p^{3/2}$ as

$p \to \infty$. In [1] it is shown, using some nontrivial results of Davis [7] on the best constants between $\|X\|_p$ and $\|\langle X\rangle^{1/2}\|_p$, that the constant C_p is $O(p)$ as $p \to \infty$. Here we show that the same conclusion holds for vector-valued martingale transforms.

THEOREM 2.1. *Let $\{A_i(s)\}_{i=1}^{\infty}$ be a sequence of $n \times n$ matrix-valued processes adapted to $\mathfrak{F}_s$. Let $M(s) = (\sup\{\sum_{i=1}^{\infty} |A_i(s)V|^2 : V \in \mathbf{R}^n, |V| \le 1\})^{1/2}$ and set $M = \sup_s M(s)$, which we assume to be finite. There is a constant C_p which depends only on p such that for every $X \in L^p$, $1 < p < \infty$,*

$$\left\| \left(\sum_{i=1}^{\infty} |A_i * X|^2 \right)^{1/2} \right\|_p \le C_p M \|X\|_p. \tag{2.1}$$

Furthermore, $C_p \le C_1 p$ for $p \ge 2$ and $C_p \le C_2/(p-1)$ for $1 < p < 2$, where C_1 and C_2 are absolute constants. Asymptotically these constants are best possible.

Proof. Since the constants in Theorem 1.1 are independent of m, these inequalities actually hold when $m = \infty$. Write

$$X = X_0 + \int_0^{\infty} H_s \cdot dB_s \quad \text{and} \quad A_i * X = \int_0^{\infty} (A_i(s) H_s) \cdot dB_s.$$

With $Y = X - X_0$ we have $\langle Y \rangle = \int_0^{\infty} |H_s|^2\, ds$ and $\langle A_i * X \rangle = \int_0^{\infty} |A_i(s) H_s|^2\, ds$. Also, $E(A_i * X) = 0$ and $E(Y) = 0$. Then for $p \ge 2$, by applying (1.1) and then (1.2) we obtain

$$\begin{aligned} \left\| \left(\sum_{i=1}^{\infty} |A_i * X|^2 \right)^{1/2} \right\|_p &\le C\sqrt{p} \left\| \left(\sum_{i=1}^{\infty} \langle A_i * X \rangle \right)^{1/2} \right\|_p \\ &\le C\sqrt{p} M \|\langle Y \rangle^{1/2}\|_p < C\sqrt{p}\sqrt{p} M \|X - X_0\|_p \le CpM\|X\|_p. \end{aligned}$$

If $1 < p < 2$ our good-λ inequality shows that

$$\left\| \left(\sum_{i=1}^{\infty} |A_i * X|^2 \right)^{1/2} \right\|_p \le C \left\| \left(\sum_{i=1}^{\infty} \langle A_i * X \rangle \right)^{1/2} \right\|_p$$

with C independent of p. From here we have

$$\left\| \left(\sum_{i=1}^{\infty} |A_i * X| \right)^{1/2} \right\|_p \le CM \|\langle Y \rangle^{1/2}\|_p. \tag{2.2}$$

Now apply the classical Burkholder–Gundy inequalities, which show that

$$\|\langle Y \rangle^{1/2}\|_p \le C\|Y^*\|_p \quad \text{for } 1 < p < 2,$$

with C independent of p. This, (2.2), and Doob's maximal inequality give

$$\left\| \left(\sum_{i=1}^{\infty} |A_i * X| \right)^{1/2} \right\|_p \le CM \frac{p}{p-1} \|Y\|_p \le CM \frac{p}{p-1} \|X\|_p,$$

which completes the proof for $1 < p < 2$ and hence, together with the previous case, for all $1 < p < \infty$.

To prove the sharpness part, let B_t be 2-dimensional Brownian motion. Let I_A be the indicator function of a set with, say, $P(A) = \frac{1}{2}$. Let

$$H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It follows easily by time changing to Brownian motion (see [11, p. 29]) that

$$P\{\sup_t |(H*1_A)_t| > \lambda\} \sim C_1 \exp[-C_2\lambda].$$

Thus

$$\|(H*1_A)^*\|_p^p = p\int_0^\infty \lambda^{p-1} P\{H*1_A)^* > \lambda\}\, d\lambda \sim C_1 p \int_0^\infty \lambda^{p-1} \exp[-C_2\lambda]\, d\lambda$$
$$= \frac{C_1}{C_2^p} p \int_0^\infty \lambda^{p-1} e^{-\lambda}\, d\lambda,$$

and Stirling's formula does the rest. Theorem 2.1 is proved. □

We note that we have only used the scalar case of (1.2). The full generality is needed if we wish to prove vector-valued inequalities of the form

$$\left\| \left(\sum_{i=1}^{\infty} |A * X^i|^2 \right)^{1/2} \right\|_p \leq C_p M \left\| \left(\sum_{i=1}^{\infty} |X^i|^2 \right)^{1/2} \right\|_p \tag{2.3}$$

for $1 < p < \infty$ and C_p as above. Here A is just a single matrix.

We now explain the connection between martingale transforms and Riesz transforms. Since this has already been done in so many places ([1], [4], [13], [14]) we shall be brief. Let $Z_t = (X_t, Y_t)$, $-\infty < t < 0$, be the background radiation process of Gundy and Varopoulos [14] in $\mathbf{R}_+^{n+1}$. This is "Brownian motion" which "starts at time $-\infty$ from Lebesgue measure on $\mathbf{R}^n \times \{\infty\}$" and terminates at time $t = 0$ upon hitting the boundary $\mathbf{R}^n$. If $f \in \mathcal{S}(\mathbf{R}^n)$, let u be its Poisson integral to $\mathbf{R}_+^{n+1}$. If $A(x, y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^+$, is an $(n+1)\times(n+1)$ matrix-valued function, we define the martingale transform of f by

$$A * f = \int_{-\infty}^{0} [A(X_s, Y_s)\nabla u(X_s, Y_s)] \cdot dZ_s$$

and the projection operator by the conditional expectation

$$T_A f(x) = E[A * f \mid Z_0 = (x, 0)] = E^x[A * f].$$

THEOREM 2.2. *Let $\{A_i(x, y)\}_{i=1}^\infty$ be a sequence of $(n+1)\times(n+1)$ matrix-valued functions on $\mathbf{R}_+^{n+1}$. Let*

$$M(x, y) = \left(\sup\left\{ \sum_{i=1}^{\infty} |A_i(x, y) V|^2 : V \in \mathbf{R}^{n+1}, |V| \leq 1 \right\} \right)^{1/2}$$

and set $M = \|M(x, y)\|_{L^\infty(\mathbf{R}_+^{n+1})}$, which we assume to be finite. For $f \in \mathcal{S}(\mathbf{R}^n)$ define $Tf(x) = (\sum_{i=1}^\infty |T_{A_i} f(x)|^2)^{1/2}$. Then, for $1 < p < \infty$, there is a constant C_p depending only on p such that $\|Tf\|_p \leq C_p M \|f\|_p$. Furthermore, $C_p \leq C_1 p$ for $p \geq 2$ and $C_p \leq C_2/(p-1)$ for $1 < p < 2$, where C_1 and C_2 are absolute constants.

Theorem 2.2 follows from Theorem 2.1 with the additional observation that the conditional expectation is a contraction in L^p for $1 \leq p \leq \infty$. For full details see [1]. Notice that we have made no smoothness assumptions on the matrices.

Now let A_j, $1 \le j \le n$, be the $(n+1)\times(n+1)$ matrix whose entries are $a^j_{ik}=1$ if $i=1$, $k=j+1$, and zero otherwise. Then ([13], [14]) $T_{A_j}f(x)=\frac{1}{2}R_jf(x)$, and for this sequence of matrices $M\le 1$. Thus we have the following consequence of Theorem 2.2.

COROLLARY 2.3. *$\|Rf\|_p \le C_p\|f\|_p$ for $1<p<\infty$, where C_p depends only on p. Furthermore, $C_p\le C_1 p$ for $p\ge 2$ and $C_p \le C_2/(p-1)$ for $1<p<2$, where C_1 and C_2 are absolute constants. By duality we also have $\|f\|_p\le a_p\|Rf\|_p$, and a_p has the same behavior as C_p.*

Several remarks are in order concerning this corollary. As mentioned in the introduction, the constant independent of the dimension was proved by Stein [22]. Here, however, we should also mention the work of Meyer ([17] and [18]). Their proofs use the fact that the Littlewood–Paley g-function has $\|g(f)\|_p\approx\|f\|_p$ for $1<p<\infty$ with constants independent of dimension. However, at present the best one can say is that $\|f\|_p\le Cp\|g(f)\|_p$ (say, for $p\ge 2$), and therefore the constants obtained by this argument are, at best, $O(p^{3/2})$ as $p\to\infty$. Alternative probabilistic proofs with constants independent of dimension are given in [1] and [4]. In [1] it is shown, using a nontrivial result of Davis [7], that for a single Riesz transform we do have the correct behavior in p. This argument does not work for the full vector. On the other hand, Duoandikoetxea and Rubio de Francia [9] used the method of rotations to obtain constants independent of dimension and of order p as $p\to\infty$. However, their argument only gives $O(1/(p-1)^{3/2})$ as $p\to 1$. The above proof corrects the deficiencies in the previous probabilistic arguments and improves the constant obtained by Duoandikoetxea and Rubio de Francia, giving the correct behavior of this constant for the full range of p.

If we consider matrices of the form $A=a(y)I$, with $a(y)$ a scalar function on $(0,\infty)$ and I the $(n+1)\times(n+1)$ identity matrix, we obtain the operators of Laplace transform type studied in [17], [21], and [23]. More precisely, if $a(y)\in L^\infty(0,\infty)$ then $(T_Af)^\wedge(\xi)=m_A(\xi)\hat{f}(\xi)$, with

$$m_A(\xi)=16\pi^2|\xi|^2\int_0^\infty a(y)e^{-4\pi y|\xi|}y\,dy.$$

COROLLARY 2.4. *Let $\{a_i(y)\}_{i=1}^\infty$ be a sequence of real-valued functions in $(0,\infty)$. Let $a(y)=(\sum_{i=1}^\infty|a_i(y)|^2)^{1/2}$ and $M=\|a(y)\|_{L^\infty(0,\infty)}$, which we assume to be finite. For $f\in\mathcal{S}(\mathbf{R}^n)$ define $Tf(x)=(\sum_{i=1}^\infty|T_{A_i}f(x)|^2)^{1/2}$, where $A_i=a_iI$. Then, for $1<p<\infty$, there is a constant C_p depending only on p and having the same asymptotic behavior as the constant in Theorem 2.2, such that $\|Tf\|_p\le C_pM\|f\|_p$.*

The corollary follows from Theorem 2.2. The constants we obtain here are better than those obtained in [17], [21], and [23], which are, at best, of order $p^{3/2}$ as $p\to\infty$. The corollary applies to the semigroups studied in [23].

Very recently Gundy [15] has shown that the same probabilistic machinery used to study the Riesz transforms in $\mathbf{R}^n$ can be used to study the Riesz transforms of the Orstein–Uhlenbeck semigroup (a truly infinite-dimensional situation). He uses this to prove the boundedness of these operators in L^p, $1<p<\infty$, a result first proved by Meyer [18] using Littlewood–Paley theory. A careful reading of

Gundy's proof reveals that the constant he obtains is of order p^2 as $p \to \infty$. The good-λ inequality in this paper improves his constant to order $p^{3/2}$. As far as we know, the question of whether the Riesz transforms of the Ornstein–Uhlenbeck semigroup have constants of same order as the constants for the Riesz transforms in $\mathbf{R}^n$ remains open.

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