ANALYTIC MULTIPLIERS OF BERGMAN SPACES

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Basics and introduction. Let W be a nonempty region in the complex plane and let $L^p(W)$ be the usual Lebesgue p-space of complex functions with domain W, relative to the Lebesgue two-dimensional area measure dm. For $0<p\leq\infty$, let the Bergman p-space be defined by $L_a^p(W)=L^p(W)\cap H(W)$, where $H(W)$ is the space of analytic functions on W. For $f\in L_a^p(W)$ let

$$\|f\|_p=\left(\int_W |f|^p\,dm\right)^{1/p} \quad \text{if } 0<p<\infty$$

$$=\sup_{z\in W}|f(z)| \quad \text{if } p=\infty.$$

The class $L_a^\infty(W)$ of bounded analytic functions on W is usually denoted by $H^\infty(W)$. Let $0<p\leq\infty$ and let $\{f_n\}$ be a Cauchy sequence in $L_a^p(W)$. Then by using a theorem of Hardy and Littlewood ([8], Chapter 3, Lemma 3.7), one deduces the existence of f in $H(W)$ such that $f_n\to f$ uniformly on compact sets. It follows that if $p\geq 1$ then $L_a^p(W)$ is a Banach space, and that if $0<p<1$ then $L_a^p(W)$ is an F-space.

$L_a^2(W)$ is a Hilbert space, with the inner product $\langle f,g\rangle=\int_W f\bar{g}\,dm$. For each $w\in W$ there exists a unique k_w in $L_a^2(W)$ such that $f(w)=\int_W f\bar{k}_w\,dm$ for each f in $L_a^2(W)$. This k_w is called the reproducing kernel associated with w. Let D denote the unit disc. When $W=D$, we have

$$k_w(z)=\frac{1}{\pi}\cdot\frac{1}{(1-\bar{w}z)^2}$$

for $z\in D$ and $w\in D$. Let P be the orthogonal projection from $L^2(W)$ onto $L_a^2(W)$, so that

$$P(f)(w)=\int_W f\bar{k}_w\,dm.$$

Taking this as the definition of $P(f)$ for each f in $L^p(D)$, Zaharjuta and Judovic [16] (also see [4]) proved that P projects $L^p(D)$ onto $L_a^p(D)$ continuously for $1<p<\infty$. An immediate consequence would be that the dual of $L_a^p(D)$ can be identified with $L_a^q(D)$, where $1<p<\infty$ and $1/p+1/q=1$.

The map P does not project $L^1(D)$ to $L_a^1(D)$ continuously. However $L^1(D)$ can be continuously projected onto $L_a^1(D)$ ([3]). In fact, it is not hard to see that

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Received September 22, 1983. Revision received April 26, 1984.

This paper constitutes a portion of the author's doctoral dissertation submitted to Michigan State University, written under the supervision of Professor Sheldon Axler. The author wishes to thank the referee for many useful comments.

Michigan Math. J. 31 (1984).

$$R(f)(w)=\frac{2}{\pi}\int_D \frac{(1-|z|^2)}{(1-w\bar{z})^3} f(z)\,dm(z) \quad \text{for } f\in L^1(D),\ w\in D,$$

continuously projects $L^1(D)$ onto $L_a^1(D)$. Consequently, if v is a harmonic function in $L^1(D)$, then its harmonic conjugate also belongs to $L^1(D)$, a fact used in Proposition 5.1.

Let $f\in L_a^p(D)$ and let $\{z_n\}_{n\geq 1}$ be any subset of the zeros of f. In general $\sum_1^\infty (1-|z_n|)$ does not converge, but Horowitz ([10], Corollary 6.8) proved that $\sum_1^\infty (1-|z_n|)^2$ does converge, from which it follows that

$$H(z)=\prod_1^\infty \left(\frac{z-z_n}{1-\bar{z}_n z}\right)\left(2-\frac{z-z_n}{1-\bar{z}_n z}\right)$$

converges uniformly on compact subsets of D, and so H is analytic. Horowitz further proved that f/H is in $L_a^p(D)$ and that

$$\left(\int_D |f/H|^p\,dm\right)^{1/p}\leq c\left(\int_D |f|^p\,dm\right)^{1/p},$$

where c is a constant that depends only on p.

Another key theorem used in the paper is the following factorization theorem of Horowitz. (See [10], Theorem 3.)

THEOREM. *Let*

$$p, p_1, \ldots, p_n>0 \quad \textit{with} \quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}.$$

If $f\in L_a^p(D)$, *then there exists* $f_1\ldots f_n$ *such that each* $f_i\in L_a^{p_i}(D)$, $f=\prod_{i=1}^n f_i$, *and*

$$\sum_{i=1}^n \|f_i\|_{p_i}^{p_i}\leq c\|f\|_p^p,$$

where the constant c *depends only on* $p, p_1, \ldots, p_n$ *and* n.

The integral $\int_W f\,dm$ will be usually written as just $\int f$. Also c will denote a constant, not necessarily the same each time, and n will always denote a positive integer.

1. Let W be a plane region and let v be a function on W. We say that v is a multiplier of $L_a^p(W)$ to $L^q(W)$ if $vL_a^p(W)\subset L^q(W)$. The multiplication operator $M_v: L_a^p(W)\to L^q(W)$ is defined by $M_v(f)=vf$ for $f\in L_a^p(W)$.

Lemma 3.7 of Chapter 3 in [8] shows that the linear functionals of evaluation at a point are continuous and thus an application of the Closed Graph Theorem shows that M_v is bounded.

2. This section is devoted to some examples.

EXAMPLE 1. Let D denote the unit disc. It is known that there are unbounded multipliers from $L_a^2(D)$ to $L^2(D)$. For example, let

$$S=\{re^{i\theta}\mid 0\leq r<1,\ |\theta|<(1-r)^2\}$$

and let

$$\varphi(z) = \frac{1}{(1-|z|)^{1/2}} \chi_S(z).$$

We will prove that $\varphi L_a^2(D) \subset L^2(D)$. Pick $f(z) = \sum_{n=0}^{\infty} a_n z^n$ from $L_a^2(D)$. Then

$$\begin{aligned}
\int_D |\varphi|^2 |f|^2\, dm &\leq \int_0^1 \frac{1}{(1-r)} \left(\int_{|\theta|<(1-r)^2} \sum_{n,m\geq 0} |a_n|\,|a_m| r^{n+m}\, d\theta \right) r\, dr \\
&\leq 2 \sum_{n,m\geq 0} |a_n|\,|a_m| \int_0^1 (1-r) r^{n+m}\, dr \\
&= 2 \sum_{n,m\geq 0} \frac{|a_n|\,|a_m|}{(n+m+2)(n+m+1)} \\
&\leq 2 \sum_{n,m\geq 0} \frac{[|a_n|/\sqrt{2(n+1)}][|a_m|/\sqrt{2(m+1)}]}{(n+m+1)} \\
&\leq 2\pi \sum_{n\geq 0} \frac{|a_n|^2}{2(n+1)} \\
&= \|f\|_2^2. \qquad \square
\end{aligned}$$

For the last inequality see, for example, [6, p. 48]. Thus $M_\varphi : L_a^2(D) \mapsto L^2(D)$ is continuous. In fact one can verify, through a computation quite similar to the one above, that M_φ is Hilbert-Schmidt and therefore compact.

The following example was pointed out to me by Sheldon Axler. Let $\psi = \chi_S$. Clearly $M_{\sqrt{1-|z|}} : L^2(D) \to L^2(D)$ is bounded. Therefore $T_\psi : L_a^2(D) \to L_a^2(D)$ defined by $T_\psi = PM_{\sqrt{1-|z|}} M_\psi \mid L_a^2(D)$ is compact, where P is the orthogonal projection from $L^2(D)$ onto $L_a^2(D)$. On the other hand the cluster set of ψ on ∂D is $\{0,1\}$. Compare this with Proposition 5 in [13].

EXAMPLE 2. Let $D' = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ and let $v(z) = \log|z|$ for $z \in D'$. We claim that $vL_a^2(D') \subset L^2(D')$. Since $L_a^2(D') = L_a^2(D)$ [1], given f in $L_a^2(D')$, for some $\{a_n\}$,

$$f(re^{i\theta}) = \sum_0^\infty a_n r^n e^{ni\theta} \quad \text{and} \quad \sum_0^\infty \frac{|a_n|^2}{n+1} < \infty.$$

Also, pick M such that $|\log r|^2 r \leq M$ for all r in $(0,1)$. Then,

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} |\log r f(re^{i\theta})|^2 r\, dr\, d\theta &= 2\pi \int_0^1 \sum_{n=0}^\infty (\log r)^2 |a_n|^2 r^{2n+1}\, dr \\
&\leq M 2\pi \int_0^1 \sum_0^\infty |a_n|^2 r^{2n}\, dr \leq M 2\pi \sum_0^\infty \frac{|a_n|^2}{(n+1)} < \infty,
\end{aligned}$$

whence $vL_a^2(D') \subset L^2(D)$.

Moreover it is not hard to show that $T_v : L_a^2(D') \to L_a^2(D')$, defined by $T_v = PM_v \mid L_a^2(D')$ is compact.

Thus we have an example of an unbounded harmonic function which multiplies $L_a^2(W)$ to $L^2(W)$ for a suitable region W. However in the case of analytic multipliers of $L_a^p(W)$ to $L_a^p(W)$ we do get the expected result. See [7, Lemma II] for a more general result.

3.

PROPOSITION 3.1. *Let W be a region of finite area, $p>0$, and suppose $fL_a^p(W)\subset L_a^p(W)$. Then $f\in L_a^\infty(W)$.*

Proof. We may assume that $p<\infty$. By the continuity of $M_f\colon L_a^p(W)\to L_a^p(W)$ we have

$$\int_W |f|^p|g|^p\,dm\le c\int_W |g|^p\,dm \quad \text{for all } g\in L_a^p(W).$$

Note that for every positive integer n, we have $f^n\in L_a^p(W)$ and so

$$\int_W |f|^{np}\,dm\le c\int_W |f|^{(n-1)p}\,dm.$$

Thus,

$$\int_W |f|^{pn}\,dm\le c^{n-1}\int_W |f|^p\,dm,$$

so

$$\left(\int_W |f|^{pn}\,dm\right)^{1/n}\le c^{(n-1)/n}\left(\int_W |f|^p\,dm\right)^{1/n}.$$

Letting $n\to\infty$ completes the proof with $\|f\|_\infty\le c^{1/p}$. □

Note that W may be unbounded. When the region W is the disc we will show that the harmonic multipliers of $L_a^p(W)$ to $L^p(W)$ must be bounded.

PROPOSITION 3.2. *Let v be harmonic, $0<p<\infty$, and suppose $vL_a^p(D)\subset L^p(D)$. Then $v\in L^\infty(D)$.*

Proof. We have that

$$\left(\int_D |f|^p|v|^p\,dm\right)^{1/p}\le c\left(\int_D |f|^p\,dm\right)^{1/p} \quad \text{for all } f\in L_a^p(D).$$

Hence $|v|^p\,dm$ is a Carleson measure on the disc [11]. Thus, given α in D,

$$\int_{D_\alpha} |v|^p\,dm\Big/ m(D_\alpha)\le c,$$

where D_α is a hyperbolic disc of radius 1/2 with center α. Therefore by a theorem of Hardy and Littlewood ([8], Chapter 3, Lemma 3.7) we have that v is bounded. One notes that the above proof would have gone through had v been subharmonic (instead of being harmonic) provided $p\ge 1$. □

4. In order to classify the harmonic multipliers of $L_a^2(D')$ to $L^2(D')$ we need the following lemma which surely must be known; nevertheless a proof is included.

LEMMA 4.0. *Let u be a real-valued harmonic function on $D' = \{z \mid 0 < |z| < 1\}$. Then there exists α in $\mathbf{R}$ such that $u(z) - \alpha \log|z|$ is the real part of an analytic function on D'.*

Proof. Let

$$D_1 = D - \{z \in \mathbf{C} \mid z \le 0\}$$
$$D_2 = D - \{z \in \mathbf{C} \mid z \ge 0\}$$
$$R_1 = \{z \in D \mid \operatorname{Im} z > 0\}$$
$$R_2 = \{z \in D \mid \operatorname{Im} z < 0\}.$$

Now since D_1 and D_2 are simply connected, for $i = 1, 2$, there exist analytic maps h_i on D_i such that $\operatorname{Re} h_i = u$ on D_i. Hence there exist constants c_i such that

$$\operatorname{Im} h_1 - \operatorname{Im} h_2 = c_1 \text{ on } R_1$$

and

$$\operatorname{Im} h_1 - \operatorname{Im} h_2 = c_2 \text{ on } R_2.$$

Pick b on $\{z \in D \mid z \le 0\}$ and let γ be a smooth curve that lies in D' and joins b to b while passing once around the origin. Let us prove that $\int_\gamma (\partial u/\partial n)$ is independent of γ. Pick $b_i \in \gamma \cap R_i$, $i = 1, 2$. Then

$$\begin{aligned}\int_\gamma \frac{\partial u}{\partial n} &= \int_{b_2 \text{ to } b_1 \text{ on } \gamma} \frac{\partial u}{\partial n} + \int_{b_1 \text{ to } b_2 \text{ on } \gamma} \frac{\partial u}{\partial n} \\ &= \operatorname{Im} h_1(b_2) - \operatorname{Im} h_1(b_1) + \operatorname{Im} h_2(b_1) - \operatorname{Im} h_2(b_2) \\ &= c_2 - c_1,\end{aligned}$$

and thus the asserted independence is proven.

Incidently, note that we also proved that $\int_\gamma (\partial u/\partial n) = 0$ provided that γ does not enclose the origin.

Hence, if γ is a curve that lies in D' and if either γ does not enclose the origin or goes only once around it, there exists a in $\mathbf{R}$ such that

$$\int_\gamma \frac{\partial u}{\partial n} - a \int_\gamma \frac{\partial}{\partial n} \log|z| = 0.$$

[Notice that $\int_\gamma (\partial/\partial n) \log|z| = 2\pi$ or 0]. Pick $z \in D'$. Let γ be a simple curve that joins b to z. Then there exists a single valued function $f: D' \to C$ such that

$$\operatorname{Re} f(z) = u(z) - a \log|z| \quad \text{and} \quad \operatorname{Im} f(z) = -\int_\gamma \frac{\partial}{\partial n} (\operatorname{Re} f).$$

Let us show that f is analytic on D'. Since $u(z) - a\log|z|$ is harmonic on D_2, there exists an analytic function h_2 on D_2 such that $\operatorname{Re} h_2(z) = u(z) - a \log|z|$. Let γ be a curve lying in D_2 which joins b to z. Then

$$\operatorname{Im} h_2(z) - \operatorname{Im} h_2(b) = -\int_\gamma \frac{\partial}{\partial n} (u(z) - a \log|z|).$$

Since the definition of f does not depend on γ, $f = h_2 + \operatorname{Im} h_2(b)$ on D_2 and thus f is analytic on D_2. Similarly, f is analytic on $D - \{iy \mid y \geq 0\}$ and thus the analyticity of f on D' is established. □

PROPOSITION 4.1. *Let $D' = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$. Suppose v is a real valued, harmonic function and $vL_a^2(D') \subset L^2(D')$. Then there exists α such that $v(z) - \alpha \log|z| \in L^\infty(D')$.*

Proof. In view of Lemma 4.0 and since $\log|z| L_a^2(D') \subset L^2(D')$ [§2, Example 2], we may assume that v is the real part of an analytic function f on D' and prove that $v \in L^\infty(D)$. The Laurent expansion of f gives

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{ni\theta} + \sum_{n=1}^{\infty} \frac{b_n}{r^n} e^{-ni\theta}.$$

Hence,

$$2v = f + \bar{f} = \sum_{n=1}^{\infty} \left(a_n r^n + \frac{\bar{b}_n}{r^n}\right) e^{ni\theta} + \sum_{n=1}^{\infty} \left(\bar{a}_n r^n + \frac{b_n}{r^n}\right) e^{-ni\theta} + a_0 + \bar{a}_0.$$

But $v \in L^2(D')$, so

$$\int_0^1 \sum_{n=1}^{\infty} \left| a_n r^n + \frac{\bar{b}_n}{r^n} \right|^2 r\,dr < \infty,$$

and so for every n,

$$\int_0^1 \left| a_n r^n + \frac{\bar{b}_n}{r^n} \right|^2 r\,dr < \infty.$$

Hence for all $n \geq 1$, we have $b_n = 0$ which shows that v extends to be a harmonic function on D. By Proposition 1 of [1] $L_a^2(D') = L_a^2(D)$, whence $vL_a^2(D) \subset L^2(D)$, and now the proposition follows from Proposition 3.2. □

5. Suppose v is a harmonic function on D. If $v \in L^1(D)$, then unlike the case of the circle its harmonic conjugate is also in $L^1(D)$. (See the introduction and also Theorem 1 of [15].) We will use this fact below.

PROPOSITION 5.1. *Let $1 \leq p \leq \infty$, let v be harmonic on D, and suppose $vL_a^p(D) \subset L^1(D)$. Then $v \in L^r(D)$ where $1/p + 1/r = 1$.*

Proof. By Proposition 3.1 and the above remark we may assume that $1 < p$, and to avoid a triviality, let $p < \infty$. We have

$$\int |v|\,|f| \leq c\left(\int |f|^p\right)^{1/p} \quad \text{for all } f \in L_a^p(D).$$

Hence, we may assume that v is real-valued, and it also follows that

$$f \to \int vf \quad \text{for all } f \in L_a^p(D)$$

is a continuous linear functional on $L_a^p(D)$. Thus by [16], for some $g \in L_a^r(D)$ we have $2\int vf = \int \bar{g} f$ for all $f \in L_a^p(D)$.

But, as remarked at the beginning of the section, $v = \operatorname{Re} h$ for some $h \in L_a^1(D)$. Thus,

$$\int hf + \int \bar{h}f = \int \bar{g}f \quad \text{for all } f \in H^\infty(D).$$

Let

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } |z|<1$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{for } |z|<1.$$

Put $f = z^m$ for $m \geq 1$ to get $a_m/(m+1) = b_m/(m+1)$ for all $m \geq 1$. It follows that $g - g(0) = h - h(0)$, whence $h \in L_a^r(D)$. Thus $v \in L^r(D)$. □

PROPOSITION 5.2. *Suppose v is harmonic, $1 \leq q < p$, and $vL_a^p(D) \subset L^q(D)$. Then $v \in L^r(D)$ where $1/p + 1/r = 1/q$.*

Proof. We may assume that $q > 1$. Pick q', r' such that

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Then by Theorem 3 of [9], $L_a^p(D)L_a^{q'}(D) = L_a^{r'}(D)$. But then $vL_a^p(D)L_a^{q'}(D) \subset L^q(D)L_a^{q'}(D)$. Hence, $vL_a^{r'}(D) \subset L^1(D)$. Now invoke Proposition 5.1 to complete the proof. □

PROPOSITION 5.3. *Let v be a harmonic function on D, $q > p > 0$, and suppose $vL_a^p(D) \subset L^q(D)$. Then $v \equiv 0$.*

An unpublished result of Sheldon Axler says that for regions of finite areas, if $q > p$ and $fL_q^p(W) \subset L_a^q(W)$, then $f \equiv 0$. The proof for the case of the disc is quite simple.

Proof. The continuity of $M_v : L_a^p(D) \to L^q(D)$ implies the existence of c such that

$$\left(\int_D |v|^q |g|^q \, dm\right)^{1/q} \leq c\left(\int_D |g|^p \, dm\right)^{1/p} \quad \text{for all } g \in L_a^p(D).$$

Hence, by Theorem 2.2 of [11], for each a in D

$$\int_{D_\alpha} |v|^p \, dm \leq c(1-|\alpha|^2)^{2q/p},$$

where D_α is a hyperbolic disc of radius 1/2 with center α, and c is some constant. Now there exist constants c_1 and c_2 such that

$$c_1 \leq \frac{m(D_\alpha)}{(1-|\alpha|^2)^2} \leq c_2$$

(See [12], p. 4.) Moreover

$$c|v|^p(a) \le \frac{\int_{D_\alpha} |v|^p}{m(D_\alpha)}$$

(see [8], Chapter 3, Lemma 3.7). Hence

$$|v|^p(\alpha) \le c(1-|\alpha|^2)^{2(q-p)/p}.$$

Therefore $v(\alpha) \to 0$ as $|\alpha| \to 1$, and so $v \equiv 0$ on D. □

6. We now want to show that if $0<q<p\le 1$ and if $fL_a^p(D) \subset L_a^q(D)$, then $f \in L_a^r(D)$, where $1/p+1/r=1/q$. The proof is harder than for the previous cases, and will be accomplished through a series of lemmas. Be reminded that c will denote a constant, perhaps not the same in each occurrence and n will always denote a positive integer.

LEMMA 6.1. *Let $0<q<p$ and suppose $fL_a^p(D) \subset L_a^q(D)$. Then*

$$|f|^{1/n} L_a^{np}(D) \subset L^{nq}(D) \quad \textit{for every } n.$$

Proof.

$$\left(\int |f|^q |g|^q\right)^{1/q} \le c\left(\int |g|^p\right)^{1/p} \quad \text{for every } g \in L_a^p(D).$$

Let $n>0$ be an integer. Then given $g \in L_a^{np}(D)$, we have $g^n \in L_a^p(D)$, so

$$\left(\int |f|^q |g|^{nq}\right)^{1/nq} \le c^{1/n}\left(\int |g|^{np}\right)^{1/np};$$

that is, $|f|^{1/n} L_a^{np}(D) \subset L^{nq}(D)$. □

PROPOSITION 6.2. *Suppose f has finitely many zeros, $0<q<p$, and $fL_a^p(D) \subset L_a^q(D)$. Then $f \in L_a^r(D)$, where $1/p+1/r=1/q$.*

Proof. Let b be the Blaschke product formed by all the zeros of f. Then since b is a finite Blaschke product

$$M_b : L_a^q(D) \to L_a^q(D),$$

the multiplication operator by b has closed range ([13], Proposition 22) and so there exists $c>0$ such that for each g in $L_a^q(D)$ for which g/b is analytic

$$\left(\int \left|\frac{g}{b}\right|^q\right)^{1/q} \le c\left(\int |g|^q\right)^{1/q}.$$

Hence

$$\left(\int \left|f\frac{g}{b}\right|^q\right)^{1/q} \le c\left(\int |gh|^q\right)^{1/q} \quad \text{for every } g \in L_a^p(D);$$

but then

$$\left(\int |f|^q |g|^q\right)^{1/q} \le c\left(\int |g|^p\right)^{1/p},$$

so

$$\left(\int \left|f\frac{g}{b}\right|^q\right)^{1/q} \le c\left(\int |g|^p\right)^{1/p} \quad \text{for every } g \in L_a^p(D),$$

whence

$$(f/b)L_a^p(D) \subset L_a^q(D).$$

Now pick n such that $np, nq > 1$. By the previous lemma and since f/b has no zeros,

$$\left(\frac{f}{b}\right)^{1/n} L_a^{np}(D) \subset L_a^{nq}(D).$$

Now the proposition follows from Proposition 5.2. □

LEMMA 6.3. *Let $f \in L_a^r(D)$ and suppose*

$$\left(\int |fg|^{r/n}\right)^{n/r} \le c\left(\int |g|^{r/(n-1)}\right)^{(n-1)/r} \quad \textit{for all } g \in L^{r/(n-1)}(D)$$

for some $n > 1$. Then $\|f\|_r \le c$. Here c denotes the same constant.

Proof. Since $f \in L_a^r(D)$, we have $f^{n-1} \in L_a^{r/(n-1)}(D)$. Whence,

$$\left(\int |f|^{r/n}|f|^{(n-1)r/n}\right)^{n/r} \le c\left(\int |f|^r\right)^{(n-1)/r}$$

or

$$\left(\int |f|^r\right)^{n/r} \le c\left(\int |f|^r\right)^{(n-1)/r};$$

that is,

$$\left(\int |f|^r\right)^{1/r} \le c.$$ □

LEMMA 6.4. *Suppose $fL_a^{r/(n-1)}(D) \subset L_a^{r/n}(D)$ for some $r > 0$ and $n > 1$. Then $f \in L_a^r(D)$.*

Proof. Let $|z_1| \le |z_2| \le \cdots$ be the zeros of f. Let

$$b_k(z) = \left(\frac{z - z_k}{1 - \bar{z}_k z}\right)\left(2 - \frac{z - z_k}{1 - \bar{z}_k z}\right)$$

and

$$B_k = \prod_{j \ge k} b_j.$$

By Horowitz ([10], §7.9) there exists c such that given g in $L_a^{r/n}(D)$ with g/B_k is analytic,

$$\left(\int \left|\frac{g}{B_k}\right|^{r/n}\right)^{n/r} \le c\left(\int |g|^{r/n}\right)^{n/r}.$$

The constant c does not depend on k. Whence for some constants independent of k,

$$(*)\qquad \left(\int \left|\frac{fg}{B_k}\right|^{r/n}\right)^{n/r} \le c\left(\int |fg|^{r/n}\right)^{n/r},$$

so

$$\left(\int \left|\frac{fg}{B_k}\right|^{r/n}\right)^{n/r} \le c\left(\int |g|^{r/(n-1)}\right)^{(n-1)/r} \quad \text{for every } g \in L_a^{r/(n-1)}(D).$$

So

$$\frac{f}{B_k} L_a^{r/(n-1)}(D) \subset L_a^{n/r}(D).$$

But then f/B_k has only finitely many zeros, and

$$\left(\frac{r}{n-1}\right)^{-1} + r^{-1} = \left(\frac{r}{n}\right)^{-1}.$$

Hence by the corollary to Lemma 6.1, $f/B_k \in L_a^r(D)$. Now by Lemma 6.2 $\|f/B_k\|_r \le c$, where c is the constant occurring in (*). Apply Fatou's Lemma to deduce that $\|f\|_r \le c$. □

LEMMA 6.5. *Let $\varphi L_a^p(D) \subset L^q(D)$ for some nonnegative function φ. Then for all $n > 0$ we have $\varphi^n L_a^{p/n}(D) \subset L^{q/n}(D)$.*

Proof. Fix g in $L_a^{p/n}(D)$. Then by Theorem 1 of [9] there exist c_1 and $g_1, \dots, g_n$ in $L_a^p(D)$ such that

$$g = \prod_{i=1}^n g_i \quad \text{and} \quad \sum_{i=1}^n \|g_i\|^p \le c_1 \|g\|_{p/n}^{p/n}.$$

Here c_1 is independent of g, and depends only on p and n. Since $M_\varphi : L_a^p(D) \to L^q(D)$ is continuous, for all i we have $\int |f|^q |g_i|^q \le c^q (\int |g_i|^p)^{q/p}$, so

$$\int |f|^q \left(\sum_{i=1}^n |g_i|^q\right) \le c^q \sum_{i=1}^n \left(\int |g_i|^p\right)^{q/p}.$$

Now the arithmetic-geometric inequality gives

$$\sum_{i=1}^n |g_i|^q \ge n|g|^{q/n},$$

and so

$$n\int |f|^q |g|^{q/n} \le c^q n \sum_{i=1}^n \left(\frac{\int |g_i|^p}{n}\right)^{q/p}$$

$$\le c^q \frac{n}{n^{q/p}} \left(\sum_{i=1}^n \int |g_i|^p\right)^{q/p}.$$

Thus

$$\left(\int |f|^q |g|^{q/n}\right)^{n/q} \le \frac{c^n c_1^n}{n^{n/p}} \|g\|_{p/n}.$$ □

LEMMA 6.6. *Let $0 < q < p$ and suppose $f L_a^p(D) \subset L_a^q(D)$. Then*

$$fL_a^{r/(n-1)}(D) \subset L_a^{r/n}(D)$$

for some integer $n>1$, *where* $1/r=1/q-1/p$.

Proof. By Lemma 6.1, for all $n \geq 1$,

$$|f|^{1/n} L_a^{np}(D) \subset L^{nq}(D).$$

Pick $n>1$ so that $np, nq>1$ and such that there exists positive R with

$$\frac{1}{nq}+\frac{1}{R}=\frac{1}{r}.$$

Then

$$|f|^{1/n} L_a^{np}(D) L_a^R(D) \subset L^{nq}(D) L^R(D) \subset L^r(D).$$

Now

$$\frac{1}{np}+\frac{1}{R}=-\frac{1}{nr}+\frac{1}{nq}+\frac{1}{R}$$
$$=-\frac{1}{nr}+\frac{1}{r}=\frac{1}{r/[1-(1/n)]}.$$

Hence, by Theorem 3 of [9],

$$L_a^{np}(D) L_a^R(D)=L_a^{r/[1-(1/n)]}(D),$$

so

$$|f|^{1/n} L_a^{r/[1-(1/n)]}(D) \subset L^r(D).$$

Thus by Lemma 6.4,

$$|f| L_a^{r/(n-1)}(D) \subset L^{r/n}(D),$$

and so

$$fL_a^{r/(n-1)}(D) \subset L_a^{r/n}(D). \qquad \square$$

THEOREM 6.7. *Let* $\xi>0$ *be a rational number,* $1>p>q>0$, *and suppose* $|f|^{\xi} L_a^p(D) \subset L^q(D)$ *for some analytic function* f. *Then* $f \in L_a^{r\xi}(D)$, *where* $1/p+1/r=1/q$.

Proof. Apply Lemma 6.5 for the obvious integer and then Lemma 6.6 and Lemma 6.5. $\square$

7. We now return to the question of determining the subharmonic multipliers of $L_a^2(D)$ to $L^1(D)$. Must they be in $L^2(D)$? A partial solution to this question is provided by the following proposition.

PROPOSITION 7.1. *Let* $v>0$ *be subharmonic on* D *and* $vL_a^2(D) \subset L^1(D)$. *Let* $M(v,\rho)=(1/2\pi)\int_0^{2\pi} v(\rho e^{i\theta})\,d\theta$. *Then*

$$M(v,\rho)\sqrt{1-\rho}=O(1).$$

Proof. Since $M_v: L_a^2(D) \to L^1(D)$ is continuous, we have

$$\int_D |v|\,|z^n|\,dm \le c\left(\int_D |z^{2n}|\,dm\right)^{1/2} \quad \text{for all } n \ge 0.$$

So

$$\int_0^1 r^{n+1}\left(\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta\right)dr \le c\left(\int_0^1 r^{2n+1}\,dr\right)^{1/2}.$$

Since v is subharmonic, $M(v, r)$ increases with r. Hence,

$$M(v,\rho)\frac{1-\rho^{n+2}}{n+2} \le c\frac{1}{\sqrt{2n+2}} \quad \text{for } 0<\rho<1,$$

hence $M(v,\rho) \le c\sqrt{n}/(1-\rho^n)$ for all $n \ge 1$ and for all ρ in $(0,1)$. Put $n=[1/(1-\rho)]$, the greatest integer less than or equal to $1/(1-\rho)$. Then $1-\rho^n$ is bounded away from zero as $\rho \to 1$. Thus, $M(v,\rho) \le c/\sqrt{1-\rho}$.

As a corollary we have that $v \in L^p_a(D)$ for all p in $(0, 2)$. One may ask "does the subharmonic function $z \to 1/\sqrt{1-|z|}$ for $z \in D$ multiply $L^2_a(D)$ to $L^1(D)$?" This was answered by Dr. Daniel H. Luecking, whose proof is given below.

PROPOSITION 7.2. *The subharmonic function* $1/\sqrt{1-|z|}$ *on* D *does not multiply* $L^2_a(D)$ *to* $L^1(D)$.

Proof. First suppose $(1/\sqrt{1-|z|})L^2_a(D) \subset L^1(D)$. Then the operator

$$S(f)(w) = \int_D \frac{f(z)(1-|z|^2)^{1/2}}{(1-\bar{z}w)^3}\,dm(z) \quad \text{for } w \in D,$$

maps $L^2_a(D)$ to $L^1_a(D)$ (see the introduction). By computing what S does to coefficients (using $\int_0^1 (1-r)^{1/2} r^{2n+1}\,dr \sim n^{-3/2}$), it is easy to verify that S maps $L^2_a(D)$ onto $L^{2,1}_a(D)$, where

$$L^{2,1}_a(D) = \left\{ f \in H(D) \mid \int |f(z)|^2(1-|z|)\,dm(z) < \infty \right\}.$$

Hence $L^{2,1}_a(D) \subseteq L^1_a(D)$, which is false since the derivative of an H^∞ function is always in $L^{2,1}_a(D)$ but not necessarily in $L^1_a(D)$ ([14], Theorem II).

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