ON SYSTEMS OF FUNCTIONS RESEMBLING THE WALSH SYSTEM

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In memory of David L. Williams

Certain systems of orthogonal functions have been shown to have properties very similar to those of the Walsh functions. Let $\{\phi_n\}$, $n=0,1,2,\ldots$ be a system of real functions on $[0,1]$; set $\Psi_0 \equiv 1$, and for $n=2^{k_1}+2^{k_2}+\cdots+2^{k_s}$, $0 \le k_1 < k_2 < \cdots < k_s$ set $\Psi_n = \phi_{k_1}\cdot\phi_{k_2}\cdot\ldots\cdot\phi_{k_s}$. If $\{\Psi_n\}$ is an orthogonal system on $[0,1]$, it has been called a *W-system* by Alexits [1, pp. 185–196] after the Walsh system, $\{w_n\}$, which is formed from the Rademacher functions, $\{r_n\}$, in this way. (The definition given by Alexits differs from ours in form because we use Paley's definition of the Rademacher functions, $r_n(x) = \operatorname{sign}\sin 2^{n+1}\pi x$. Alexits sets $r_0 \equiv 1$ and; accordingly, sets $\phi_0 \equiv 1$, an assumption we do not make.) A usual assumption is

$$(*) \qquad \int \Psi_n^2 = K \quad \text{for} \quad n \ge n_0$$

and many results have appeared which may be summed up simply by saying that the properties of series $\sum c_n \Psi_n$ are much the same as those of Walsh series if we make the additional requirement

$$(**) \qquad |\phi_n(x)| \le 1 \text{ a.e.} \quad \text{for every } n.$$

We will consider only systems satisfying $(*)$ and $(**)$ and we will refer to them simply as *W-systems.*

In our research announcement [5] we have pointed out the reason for the strong parallels between *W*-systems and the Walsh functions: *A W-system is the Walsh system in disguise.*

If we assume for the moment that we can restrict our attention to *W*-systems for which $|\Psi_n(x)| = 1$ a.e. for all n, we can state our principal result.

THEOREM. *If* $\{\Psi_n\}$ *is a W-system, then there is a measurable function y mapping* $[0,1]$ *into itself such that* $m(y^{-1}(E)) = m(E)$ *for every measurable* $E \subset [0,1]$ *and* $w_n \circ y(x) = \Psi_n(x)$ *a.e. for every n.*

The W-system $\{\Psi_n\}$ *is complete if and only if there is a metric automorphism* η *of* $[0,1]$ *such that* $\eta(x) = y(x)$ *a.e.*

Here by a metric automorphism of a set E is meant a 1–1 mapping of E onto E such that the mapping and its inverse are measurable and measure preserving.

We see then that if $g \in L^p$, $p \ge 1$, $g \sim \sum c_n w_n$, then $g \circ y \in L^p$ and $g \circ y \sim \sum c_n \Psi_n$. If $\{\Psi_n\}$ is complete and $f \in L^p$, $p \ge 1$, $f \sim \sum c_n \Psi_n$, then $f \circ \eta^{-1} \in L^p$ and $f \circ \eta^{-1} \sim \sum c_n w_n$.

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Examples of results which can be obtained from these considerations are to be found in [5].

The Rademacher functions are independent random variables taking on the values ± 1 and having expected value zero. Thus the products of the Rademacher functions form an orthonormal system. Our theorem implies the converse of this statement: *If the products of φ_n's form an orthonormal system, and $|\varphi_n| \leq 1$, then $\{\varphi_n\}$ is a system of independent random variables taking on values ± 1.*

Suppose $\{r_{n_i}\}$, $i=0,1,2,\ldots$ is a rearrangement of the Rademacher system. This induces a rearrangement $\{w_{m_n}\}$ of the Walsh system given by the relation

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k}, \quad 0 \leq i_1 < i_2 < \cdots < i_k$$

implies $m_n = 2^{n_{i_1}} + 2^{n_{i_2}} + \cdots + 2^{n_{i_k}}$ and $m_0 = 0$.

We have called such a rearrangement *coherent* [5]. Alexits [1, pp. 188–189] has briefly considered such rearrangements of W-systems. If we consider $\{w_{m_n}\}$, $n=0,1,2,\ldots$ as a complete W-system, there is a metric automorphism η of $[0,1]$ such that $w_{m_n}(x) = w_n \circ \eta(x)$ a.e. for every n. We may then conclude:

If $\{w_{m_n}\}$ is a coherent rearrangement of the Walsh system, then the almost everywhere convergence and summability behaviors of the series $\sum c_n w_n$ and $\sum c_n w_{m_n}$ are the same.

We now prove a lemma which allows us to assume, without loss of generality, that $|\phi_n(x)| = 1$ a.e. for every n.

LEMMA 1. (a) *To any system $\{\phi_n\}$ on* $[0,1]$ *with $\int |\phi_i|^2 = \int |\phi_i \phi_j|^2 = K$ for $i \neq j$ and* $|\phi_n(x)| \leq 1$ a.e. *for all n, there corresponds a measurable set $E \subset [0,1]$, $m(E) = K$, such that, for every n, $|\phi_n| \equiv 1$ on E and $\phi_n = 0$* a.e. *on E^c.*

(b) *A W-system $\{\Psi_n\}$ lives on a set E of measure K in the sense that*

(i) *$|\Psi_n| \equiv 1$ on E for every n,*

(ii) *$m(\{\Psi_n \neq 0\} \cap E^c) > 0$ for only finitely many n,*

(iii) *$\{\Psi_n\}$ is orthogonal relative to E.*

Proof. (a) Let $E_{ij} = \{|\phi_i| = |\phi_j| = 1\}$, $i \neq j$. Assuming that E_{ij}^c contains a set of positive measure on which one function, say ϕ_i, satisfies $1 \geq |\phi_i| > 0$, and the other, ϕ_j, satisfies $|\phi_j| < 1$, we have

$$K = \int_0^1 |\phi_i \phi_j|^2 = \int_{E_{ij}} + \int_{E_{ij}^c} < \int_{E_{ij}} |\phi_i|^2 + \int_{E_{ij}^c} |\phi_i|^2 = K,$$

a contradiction. Thus $\phi_i = \phi_j = 0$ a.e. on E_{ij}^c. Set $E = \bigcap E_{ij}$.

(b) There is an n_0 such that $\int |\Psi_n|^2 = K$ if $n \geq n_0$. Then there is a k_0 such that $\int |\phi_i|^2 = \int |\phi_i \phi_j|^2 = K$ for $i, j \geq k_0$, $i \neq j$. Thus there is a set E, $m(E) = K$, such that $|\phi_n| \equiv 1$ on E and $\phi_n = 0$ a.e. on E^c for $n \geq k_0$. Then for *every* i and large j, $\int_E |\phi_i|^2 = \int_0^1 |\phi_i \phi_j|^2 = K$, implying $|\phi_i| = 1$ a.e. on E for every i and establishing (i) and (ii).

Finally, if $i \neq j$ and k is sufficiently large $\int_E \Psi_i \Psi_j = \int_0^1 (\Psi_i \phi_k)(\Psi_j \phi_k) = \int_0^1 \Psi_{i'} \Psi_{j'} = 0$ since $i' \neq j'$, yielding (iii). ∎

If A is a metric automorphism of $[0, 1]$ taking $[0, K]$ into E, we see that $\{\Psi_n \circ A(Kx)\}$ is a W-system on $[0,1]$ and $|\Psi_n \circ A(Kx)| = 1$ a.e. for all n. Thus we can reduce the study of the original system on E to the study of a W-system living on $[0, 1]$.

Now we turn to the main theorem, whose proof will be accomplished through a series of lemmas.

If t is an integer in $[0, 2^k - 1]$, then there is a unique representation

$$t = a_{k-1}2^0 + \cdots + a_0 2^{k-1}$$

with $a_\nu = 0$ or 1. Let $E_k^t = \{x \mid \phi_\nu(x) = e^{i\pi a_\nu},\ \nu = 0, \ldots, k-1\}$. Then $[0, 1] - \bigcup_{t=0}^{2^k-1} E_k^t$ is a zero set and the sets E_k^t are pairwise disjoint and measurable. We have, further,

LEMMA 2. *$m(E_k^t) = 1/2^k$ for all k and $t = 0, 1, \ldots, 2^k - 1$.*

Proof. For $k = 1$ we have

$$0 = \int_0^1 \phi_0 = \int_{E_1^0} \phi_0 + \int_{E_1^1} \phi_0 = m(E_1^0) - m(E_1^1)$$

implying $m(E_1^0) = m(E_1^1) = 1/2$. Suppose for fixed k, $m(E_k^t) = 1/2^k$ for each t. If we knew

$$(***) \qquad \int_{E_k^t} \phi_k = 0 \qquad \text{for} \qquad t = 0, \ldots, 2^k - 1,$$

then since $E_k^t \cap \{\phi_k = 1\} = E_{k+1}^{2t}$ and $E_k^t \cap \{\phi_k = -1\} = E_{k+1}^{2t+1}$, we would have

$$m(E_{k+1}^{2t}) = m(E_{k+1}^{2t+1}) = 1/2m(E_k^t) = 1/2^{k+1}.$$

In order to demonstrate (***), we consider Ψ_ν for $2^k > \nu = b_0 2^0 + \cdots + b_{k-1}2^{k-1}$. Then

$$\Psi_\nu = \phi_0^{b_0} \cdot \phi_1^{b_1} \cdot \ldots \cdot \phi_{k-1}^{b_{k-1}} = \prod_{s=0}^{k-1} e^{i\pi a_s b_s} \qquad \text{on} \qquad E_k^t,$$

where t is as above. Now $w_\nu = r_0^{b_0} \cdot \ldots \cdot r_{k-1}^{b_{k-1}}$ and for each $x_t \in (t/2^k, (t+1)/2^k)$, $w_\nu(x_t) = \prod_{s=0}^{k-1} e^{i\pi a_s b_s}$. Since ϕ_k is orthogonal to Ψ_ν for $\nu < 2^k$, we have

$$\sum_{t=0}^{2^k-1} \int_{E_k^t} \Psi_\nu \phi_k = 0$$

or $\sum_{t=0}^{2^k-1} w_\nu(x_t) \int_{E_k^t} \phi_k = 0$, $\nu = 0, \ldots, 2^k - 1$. This is a system of 2^k homogeneous equations in the unknowns $\int_{E_k^t} \phi_k$.

Consider the coefficient matrix $(w_\nu(x_t))$ and suppose that, for some i,

$$w_i(x_t) = \sum_{\substack{\nu=0 \\ \nu \neq i}}^{2^k-1} \alpha_\nu w_\nu(x_t), \quad t = 0, 1, \ldots, 2^k - 1.$$

Since w_ν is constant on $(t/2^k, (t+1)/2^k)$ for $\nu \leq 2^k - 1$, we have, except perhaps for finitely many x,

$$w_i(x) = \sum_{\substack{\nu=0 \\ \nu \neq i}}^{2^k-1} \alpha_\nu w_\nu(x).$$

From the orthogonality of $\{w_n\}$ we have at once that $\alpha_\nu=0$ for all ν. Hence $(w_\nu(x_t))$ is nonsingular and (***) must hold. ■

We now define a function on [0, 1] by means of the dyadic representation of its values, $y(x)=.\alpha_0\alpha_1\cdots$ with $\alpha_\nu=(1/i\pi)\log\phi_\nu(x)$, $\nu=0,1,2,\ldots$. The set of x for which $\alpha_\nu\neq 0$ or 1 for some ν is clearly a zero set. We will show that the set $Z=\{x\,|\,\text{for some } \nu_0,\ \alpha_\nu\equiv 0 \text{ or } \alpha_\nu\equiv 1 \text{ for } \nu>\nu_0\}$ is also a zero set.

If $y(x)=.a_0a_1\cdots a_{k-1}00\cdots=y_0$ with $a_{k-1}=1$, then either

$$\text{(i)}\qquad \alpha_0=a_0,\ldots,\alpha_{k-1}=a_{k-1},\quad \alpha_\nu=0 \text{ for } \nu\geq k$$

or

$$\text{(ii)}\qquad \alpha_0=a_0,\ldots,\alpha_{k-2}=a_{k-2},\quad \alpha_{k-1}=0,\quad \alpha_\nu=1 \text{ for } \nu\geq k.$$

If (i) holds, then $x\in E_n^{2^ny_0}$ for $n\geq k$. If (ii) holds, then $x\in E_n^t$ for $n\geq k$ and $t=\alpha_0 2^{n-1}+\alpha_1 2^{n-2}+\cdots+\alpha_{n-1}2^0$. The set of dyadic representations of forms (i) and (ii) is countable, call it $\{r_i\}$. For each i and each large n there are two sets $E_n^{t_1}$ and $E_n^{t_2}$ of measure $1/2^n$ such that $y(x)=r_i$ implies $x\in\mathcal{E}_n^i=E_n^{t_1}\cup E_n^{t_2}$. For any $\epsilon>0$ we may choose n_i such that $m(\bigcup\mathcal{E}_{n_i}^i)\leq 2\sum 1/2^{n_i}<\epsilon$, implying $m(Z)=0$.

Except for a set of measure zero, we have then for $n=2^{k_1}+2^{k_2}+\cdots 2^{k_s}$, $0\leq k_1<k_2<\cdots<k_s$,

$$\begin{aligned} w_n\circ y(x) &= w_n(.\alpha_0\alpha_1\cdots)=r_{k_1}(.\alpha_0\alpha_1\cdots)\cdot\ldots\cdot r_{k_s}(.\alpha_0\alpha_1\cdots) \\ &= \phi_{k_1}(x)\cdot\ldots\cdot\phi_{k_s}(x)=\Psi_n(x). \end{aligned}$$

We have further

LEMMA 3. *For every measurable* $E\subset[0,1]$, $y^{-1}(E)$ *is measurable and* $m(y^{-1}(E))=m(E)$.

Proof. Let $y_0=.a_0a_1\cdots a_{k-1}00\cdots$. Then

$$[y_0,1]=\bigcup_{i=0}^{2^k(1-y_0)}[y_0+i/2^k,\ y_0+(i+1)/2^k].$$

Except for a set of x of measure zero, $y(x)\in[y_0+i/2^k, y_0+(i+1)/2k]$ if and only if $x\in E^{2^ky_0+i}$. Thus we have, modulo a zero set, $\{x\,|\,y(x)\geq y_0\}=\bigcup_{t=2^ky_0}^{2^k-1}E_k^t$ and y is a measurable function.

If an integer t is less than 2^i, it is clear from the above that

$$m(y^{-1}[t/2^i,\ (t+1)/2^i])=1/2^i.$$

For a measurable set $E\subset[0,1]$ and any $\epsilon>0$, there is an open set $G=\bigcup I_k$, where the I_k are nonoverlapping intervals of the form $[t/2^i,\ (t+1)/2^i]$, such that $G\supset E$ and $m(G)<m(E)+\epsilon$. Then $m(y^{-1}(E))\leq m(y^{-1}(G))\leq\sum m(y^{-1}(I_k))=m(G)<m(E)+\epsilon$, implying $m(E)=m(y^{-1}(E))$. ■

A sequence $\{f_n\}$ of bounded measurable functions is said to be *maximal* if there is a set Z of measure zero such that if $x_1,x_2\notin Z$ and $f_n(x_1)=f_n(x_2)$ for every n, then $x_1=x_2$. Rényi [4] showed maximality to be a sufficient condition that the system

$\{f_1^{m_1} \cdot f_2^{m_2} \cdot \ldots \cdot f_n^{m_n}\}$, $m_i = 0, 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, be closed in L^2. That maximality is also necessary has been shown by Gundy [2] and Waterman [6].

Clearly $\{\phi_n\}$ is maximal if and only if y is almost everywhere 1–1. We have further

LEMMA 4. *If* $\{\phi_n\}$ *is maximal, there is a metric automorphism* η *on* [0, 1] *such that* $\eta(x) = y(x)$ *a.e.*

Proof. For each n, let $\tilde{\phi}_n = \phi_n$ a.e. with $\{\tilde{\phi}_n = 1\}$, $\{\tilde{\phi}_n = -1\}$ *Borel* sets whose union is [0, 1]. Clearly $\{\tilde{\phi}_n\}$ is maximal and so there is a Borel zero set $\mathfrak{N}_1$ such that $x_1, x_2 \in \mathfrak{N}_1^c$ and $\tilde{\phi}_n(x_1) = \tilde{\phi}_n(x_2)$ for every n imply $x_1 = x_2$. Let $\eta^*(x) = .\beta_0\beta_1\cdots$, $\beta_\nu = (1/i\pi)\log\tilde{\phi}_\nu(x)$. Then there is a Borel zero set $\mathfrak{N}_2$ such that $x \in \mathfrak{N}_2^c$ implies that $\beta_\nu = 0$ or 1 for each ν and $\beta_\nu \not\equiv 0$ or 1 from some ν_0 onward.

Let $\mathfrak{N} = \mathfrak{N}_1 \cup \mathfrak{N}_2$, $B = \mathfrak{N}^c$, and $B^* = \eta^*(B)$. Then η^* is 1–1 on B. If $\tilde{E}_k^t$ are defined relative to $\{\tilde{\phi}_n\}$ as E_k^t were for $\{\phi_n\}$, then for y_0 as above we see that

$$\{x \mid \eta^* \geq y_0\} \cap B = \bigcup_{t=2^k y_0}^{2^k-1} \tilde{E}_k^t - \mathfrak{N},$$

where the $\tilde{E}_k^t$ are Borel sets. Thus η^* is a Borel measurable function on B and B^* is a Borel set. It follows from the Kuratowski theorem (see [3]) that $\eta^{*^{-1}}$ is Borel measurable on B^*. Clearly $\eta^*(x) = y(x)$ a.e. Thus for measurable E,

$$m(y^{-1}(E) \Delta \eta^{*^{-1}}(E)) = 0$$

implying that $\eta^{*^{-1}}$ is measure preserving. Thus η^* is also measure preserving.

Let $\eta^{*^1} = \eta^*$ and $\eta^{*^n} = \eta^* \circ \eta^{*^{n-1}}$ for $n > 1$. Similarly, let η^{*^0} denote the identity and let $\eta^{*^n} = n^{*^{-1}} \circ \eta^{*^{n+1}}$ for $n < -1$.

Then $B_0 = \bigcap_{-\infty}^{\infty} \eta^{*^n}(B)$ is a Borel set of measure one and η^* is a 1–1 mapping of B_0 onto B_0. We define $\eta(x) = \eta^*(x)$, $x \in B_0$, $\eta(x) = x$, $x \in B_0^c$, obtaining the metric automorphism we sought.

REFERENCES

1. G. Alexits, *Convergence problems of orthogonal series,* Pergamon, New York, 1961.
2. R. F. Gundy, *Complete systems in* L_2 *and a theorem of Rényi.* Michigan Math. J. 12 (1965), 161–167.
3. K. R. Parthasarathy, *Probability measures on metric spaces,* pp. 15–22, Academic Press, New York, 1967.
4. A. Rényi, *On a conjecture of H. Steinhaus.* Ann. Polon. Math. 25 (1952), 279–287.
5. D. Waterman, *W-systems are the Walsh functions.* Bull. Amer. Math. Soc. 75 (1969), 139–142.
6. ———, *On a problem of Steinhaus.* Studia Sci. Math. Hungar. 5 (1970), 97–98.

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