PSEUDO-LINEAR SPHERES

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INTRODUCTION

This work has been motivated by an analysis made on the pseudo-free actions constructed by Montgomery and Yang [10]. More generally, we study pseudo-linear S^1 spheres; that is, circle actions on cohomology spheres such that the fixed point set associated with each subgroup of S^1 is again a cohomology sphere.

Section 1 provides our main tool which is a refinement of a theorem of Atiyah and Segal. This result is interesting in its own right. It essentially says that $K^1(X \times_G EG) = 0$ for a compact connected Lie group G acting on a space X which has $K^1(X) = 0$. This generalizes the well known fact that $K^1(BG) = 0$. In sections 2 and 3 we define an equivariant Euler characteristic geometrically on the pseudo-linear spheres and show that this class determines, and is determined by the algebraic and homotopical structure of these spaces. Theorem 2.5 is the central result that allows one to work with this invariant. We show in particular that a nontrivial S^1 map between pseudo-linear spheres implies divisibility of the Euler characteristics and this in turn has immediate geometrical consequences. In section 4 we show that twice the tangent bundle of a smooth pseudo-linear sphere is equivariantly stably trivial. Also, as a consequence of the algebraic machinery developed, we prove in section 5 a conjecture of Ted Petrie on homotopy complex projective spaces in a restricted situation [11].

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1. COMPLETION

Let G be a compact Lie group and X a locally compact G space. $K_G^*(X)$ denotes complex equivariant K-theory with compact support and $R(G) = K_G^0(\text{point})$ is the complex representation ring of G [15].

Atiyah and Segal [3,(5.1)] proved that if $K_G^*(X)$ is finitely generated over R(G) and $K^*(X) = 0$ then $\hat{K}_G^*(X) = 0$, where $\wedge$ denotes completion with respect to the ideal I_G of all elements in R(G) of virtual dimension zero:

$$\hat{K}_G^*(X) = \lim_{\substack{\leftarrow \\ n}} \frac{K_G^*(X)}{(I_G)^n K_G^*(X)}$$

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A stronger statement holds if we assume that G is connected.

THEOREM 1.1. *Let* G *be a compact connected Lie group and* X *a locally compact* G *space such that* $K_G^*(X)$ *is finitely generated over* R(G). *If* K(X) = 0 *then* $\hat{K}_G(X) = 0$.

Before proving this theorem we mention two special cases having noteworthy properties and a counterexample if connectedness is omitted.

(1.2) If $G = S^1$ then under the assumption of Theorem 1.1 the following stronger statement holds: $\hat{K}_G(X) = 0$ if and only if the forgetful homomorphism $K_G(X) \to K(X)$ is trivial.

(1.3) Let G be a compact connected Lie group and X a compact space with $K^1(X) = 0$. Given an action of G on X with a finite number of isotropy subgroups it follows from Theorem 1.1 that $\hat{K}_G^1(X) = \hat{K}_G(X \times \mathbb{R}) = 0$. Let $\mathscr{K}^*$ denote inverse limit K-theory. From [3] we have

$$\mathscr{K}^1(X \times_G EG) = \hat{K}_G^1(X) = 0$$

This is a generalization of the well-known fact that $\mathscr{K}^1(BG) = 0$.

(1.4) If G is not connected then the theorem need not hold. For consider the nontrivial action of $\mathbb{Z}_2$ on the real numbers $\mathbb{R}$. Then $K(\mathbb{R}) = 0$, $K_{\mathbb{Z}_2}^*(\mathbb{R})$ is finite over $R(\mathbb{Z}_2)$ but $\hat{K}_{\mathbb{Z}_2}(\mathbb{R}) \neq 0$.

We recall some facts on localization and completion which will be needed in the proof of Theorem 1.1.

(1.5) NAKAYAMA'S LEMMA [2,(2.5)]. *Let* R *be a ring,* I *an ideal of* R *and* M *a finitely generated* R *module. If* IM = M *then* $(1 + I)^{-1}M = 0$.

If G is a compact Lie group let I_G be the kernel of the augmentation homomorphism $R(G) \to \mathbb{Z}$; that is, the ideal of all elements of virtual dimension zero. If M is an R(G) module let $\hat{M}$ denote completion with respect to I_G and $S^{-1}M$ localization with respect to the multiplicatively closed set $S = 1 + I_G$. Since R(G) is Noetherian we have

LEMMA 1.6 [2,(10.17)]. *Let* M *be a finitely generated* R(G) *module. Then* $\hat{M} = 0$ *if and only if* $S^{-1}M = 0$.

Proof of Theorem 1.1. We shall first prove the theorem for a torus by induction on its dimension. For this we construct an exact sequence relating different dimensional torii. The general case will follow from a Weyl group argument.

Let T denote a torus of dimension $n \geq 0$ and V the complex one-dimensional $S^1 \times T$ module on which T acts trivially and S^1 acts as complex multiplication. V will be identified with the complex numbers and the set S(V) of unit vectors with S^1. V represents the element $t \in R(S^1 \times T) = R(T)[t, t^{-1}]$ and its Euler class is $1 - t \in R(S^1 \times T)$.

We assume the theorem proved for T and wish to prove it for $S^1 \times T$. Let X be a locally compact $S^1 \times T$ space such that $K_{S^1 \times T}^*(X)$ is finite over $R(S^1 \times T)$ and K(X) = 0. Denote by X′ the $S^1 \times T$ action on X induced by $S^1 \times T \xrightarrow{\pi_2} T \xrightarrow{i} S^1 \times T$.

There is then an $S^1 \times T$ homeomorphism $S(V) \times X' \to S(V) \times X$ given by

$$(z,x) \to (z, z^{-1}x), \quad z \in S(V) = S^1, \quad x \in X' = X.$$

Consider the exact sequence associated with the complex $S^1 \times T$ vector bundle $V \times X \to X$

$$\begin{array}{ccc} K^*_{S^1\times T}(V \times X) & \xrightarrow{\ i^*\ } & K^*_{S^1\times T}(X) \\ & \nwarrow \delta \quad \swarrow \pi^* & \\ & K^*_{S^1\times T}(S(V) \times X) & \end{array} \tag{1.7}$$

By the Thom isomorphism theorem $K^*_{S^1\times T}(V \times X) = K^*_{S^1\times T}(X)$ and i^* becomes multiplication by the Euler class $1 - t$. Also

$$K^*_{S^1\times T}(S(V) \times X) = K^*_{S^1\times T}(S(V) \times X') = K^*_T(X)$$

[15]. The action of T on X in the right hand term of the equation is given by the inclusion $T \subset S^1 \times T$. π^* then becomes the forgetful homomorphism, thus 1.7 translates into

$$\begin{array}{ccc} K^*_{S^1\times T}(X) & \xrightarrow{\ 1-t\ } & K^*_{S^1\times T}(X) \\ & \nwarrow \delta_1 \quad \swarrow \varepsilon & \\ & K^*_T(X) & \end{array} \tag{1.8}$$

We have assumed that $K^*_{S^1\times T}(X)$ is finite over $R(S^1 \times T)$. Now from (1.8) it follows that $K^*_T(X)$ is also finite over $R(T)$ so we may apply our induction hypothesis. We thus have that $\hat{K}^*_T(X) = 0$.

Recall that localization is an exact functor and note that $1 + (1 - t) \subset 1 + I_{S^1\times T}$ in $R(S^1 \times T)$, and $S^{-1}K^*_T(X) = (1 + I_{S^1\times T})^{-1}K^*_T(X)$. Here we view $K^*_T(X)$ as an $R(S^1 \times T)$ module via the inclusion $T \subset S^1 \times T$. If we apply S^{-1} to (1.8) we have

$$\begin{array}{ll} (1 - t)S^{-1}K_{S^1\times T}(X) = S^{-1}K_{S^1\times T}(X) & \text{if and only if} \\ S^{-1}\varepsilon : S^{-1}K_{S^1\times T}(X) \to S^{-1}K_T(X) & \text{is trivial.} \end{array} \tag{1.9}$$

By induction and Lemma 1.6, $S^{-1}K_T(X) = 0$ so $S^{-1}\varepsilon$ is trivial. Now applying (1.5) and (1.6) to (1.9) gives that $\hat{K}_{S^1\times T}(X) = 0$. This proves the theorem for the torus. We prove now the general case. Let $T \subset G$ be a maximal torus and assume that $K^*_G(X)$ is finite over $R(G)$. From [15] one sees that the inclusion $i: T \to G$ induces an injection

$$K^*_G(X) \xrightarrow{\ i^*\ } K^*_T(X) \tag{1.10}$$

making $K_T^*(X)$ a finitely generated module over $K_G^*(X)$. Therefore $K_T^*(X)$ is finite over R(G), hence over R(T). Let $\alpha \in S_T$ and $\alpha^* = \prod_{\sigma \in W} \sigma\alpha \in S_G$ where W denotes the Weyl group. Now, if $\alpha i^* x = 0$ for some $x \in K_G(X)$ then $\alpha^* x = 0$. Hence

$$S^{-1} K_G(X) = 0.$$

2. PSEUDO-LINEAR SPHERES

In this section we study pseudo-linear spheres, the main object being to relate geometrical and algebraic information. We shall be working in a finite G-CW category [6] so that obstruction theory can be done. It is known that all compact smooth actions are G-homotopy equivalent to finite G-CW-complexes.

To begin with, we need the following fact due to Atiyah.

LEMMA 2.1. *Let* G *be a compact abelian Lie group,* H *a closed subgroup and* X *a finite* G-CW-*complex. Then the projection* $G \xrightarrow{\pi} G/H$ *induces an isomorphism of* R(G) *modules*

$$\pi^*: K_{G/H}(X^H) \otimes_{R(G/H)} R(G) \to K_G^*(X^H)$$

Sketch of proof. The idea is to compare two cohomology theories. First one has to show that R(G) is a flat R(G/H) module so that $K_{G/H}^*(\ ) \otimes_{R(G/H)} R(G)$ is a cohomology theory. To prove this one induces on the orders of G and H and uses the induction properties of flatness [2, pp. 29–35]. Next one has to show that π^* is an isomorphism for spaces of the form G/K where $H \leq K \leq G$. This is immediate from [1, p.79]. This isomorphism extends to all finite G-CW-complexes.

Definition. A finite G-CW-complex Σ will be called a pseudo-linear G sphere if for every closed subgroup $H \leq G$, the fixed point set Σ^H is a cohomology sphere.

Examples. (i) If M is an orthogonal representation of G then the sphere S(M) is called a linear sphere; clearly $S(M)^H = S(M^H)$.

(ii) Free and semi-free (free outside the fixed point set) circle actions on spheres.

(iii) The most interesting examples are the pseudo-free spheres constructed by Montgomery and Yang [9], [10]. An action of S^1 on a homotopy sphere is pseudo-free if the orbit space has only a finite number of singularities and for each nontrivial subgroup of S^1 the fixed point set is empty or has dimension 1.

(iv) Nondifferentiable examples are produced by taking joins of the above. If X, Y are G spaces then the join $X * Y$ is $X \times I \times Y \cup X \cup Y$ with identification $(x, 0, y) \sim x$, $(x, 1, y) \sim y$. In particular we define $SX = S^0 * X$ to be the (unreduced) suspension of X.

(v) The spheres $Z(\omega)$ constructed in [7, p. 126] are not T^2 pseudo-linear spheres. This can be seen by proving that the conclusion of proposition 2.3 below is not satisfied.

Let T denote a torus and $\mathscr{P}$ a prime ideal of R(T). Then there is a unique subgroup H of T such that $\mathscr{P}$ is the inverse image via the restriction $R(T) \to R(H)$ of some prime ideal of R(H) and H is minimal with respect to this property. If M is an R(T) module we let M_H denote M localized at a fixed but arbitrary prime ideal of R(T) associated to the subgroup H. Thus by $M_H = N_H$ we mean $M_{\mathscr{P}} = N_{\mathscr{P}}$ for each $\mathscr{P}$ associated with H.

According to [15] we have that the inclusion $X^H \subset X$ induces an isomorphism

$$K_T^*(X)_H = K_T^*(X^H)_H \tag{2.2}$$

for every locally compact T space X and every closed subgroup H of T.

PROPOSITION 2.3. *Let Σ be a pseudo-linear* G *sphere where* G *is a compact connected Lie group. If Σ is even-dimensional then* $K_G^1(\Sigma) = 0$. *If Σ is odd-dimensional then projection onto a point induces an epimorphism* $R(G) \to K_G(\Sigma)$.

Proof. Assume first that Σ is even dimensional and note that by (1.10) it suffices to prove the proposition for $G = T$ a torus. Also, it is enough to show it for effective actions, for if H is the principal isotropy group of Σ then by (2.1)

$$K_T^*(\Sigma) = K_{T/H}^*(\Sigma) \otimes_{R(T/H)} R(T)$$

where T/H is again a torus. Moreover we may induce on the number of isotropy subgroups. Hence by (2.2)

$$K_T^1(\Sigma)_H = K_T^1(\Sigma^H)_H = 0$$

for H nontrivial. Since $K^1(\Sigma) = 0$, we have by Theorem 1.1 that $S^{-1}K_T^1(\Sigma) = 0$. If $\mathscr{P}$ is a prime ideal of R(T) associated to the trivial subgroup e, then

$$(1 + I_T) \cap \mathscr{P} = \emptyset.$$

Hence $K_T^1(\Sigma)_e = 0$. But then $K_T^1(\Sigma)_{\mathscr{P}} = 0$ for every prime ideal $\mathscr{P}$ of R(T), and $K_T^1(\Sigma) = 0$.

If Σ is an odd-dimensional G sphere then $K_G^1(S\Sigma) = 0$ by the above argument. The proposition now follows from the cofibration $\Sigma \to p \to S\Sigma$.

Let Σ be a pseudo-linear S^1 sphere and H a closed subgroup of S^1. Since the Euler characteristic $\chi(\Sigma) = \chi(\Sigma^{S^1})$ and since $(\Sigma^H)^{S^1} = \Sigma^{S^1}$ we have that

$$\dim \Sigma^H - \dim \Sigma^{S^1}$$

is even. Here we let $\dim \emptyset = -1$. Dimension is cohomology dimension. Note that $\dim \Sigma^H - \dim \Sigma^{S^1}$ is always nonnegative [4, p.375]. Denote by $\phi_n \in \mathbb{Z}[t]$ the n-th cyclotomic polynomial in t. Since $R(S^1) = \mathbb{Z}[t, t^{-1}]$ we may define an invariant of Σ in terms of the cyclotomic polynomials and the dimensions of Σ at the various fixed point sets.

Definition. Let $2\nu_H = \dim \Sigma^H - \dim \Sigma^{S^1}$ and define the invariant $\Phi(\Sigma) \in R(S^1)$ by

$$\Phi(\Sigma) = \prod_{H<S^1} (\phi_{|H|})^{\nu_H}$$

where $|H|$ denotes the order of H.

Note that Φ is multiplicative with respect to joins:

$$\Phi(\Sigma * \Gamma) = \Phi(\Sigma)\Phi(\Gamma) \tag{2.4}$$

In particular it is stable: $\Phi(S\Sigma) = \Phi(\Sigma)$.

Example. If Σ is a smooth pseudo-linear S^1 sphere with nonempty fixed point set, let N denote the normal bundle of the fixed point set restricted to some fixed point. N then admits a complex structure as an S^1 module and $\Phi(\Sigma) = \lambda_{-1}N$ up to a unit of $R(S^1)$. Here $\lambda_{-1}N = \Sigma(-1)^i \Lambda^i N$ is the Euler class of N.

If Σ^{S^1} is not empty then an orientation on Σ^{S^1} gives the following identification [15]

$$\tilde{K}^n_{S^1}(\Sigma^{S^1}) = \tilde{K}^n(\Sigma^{S^1}) \otimes R(S^1) = \mathbb{Z} \otimes R(S^1) = R(S^1)$$

where $n \equiv \dim \Sigma \equiv \dim \Sigma^{S^1} \bmod 2$. As the main application of Theorem 1.1 we have the following algebraic interpretation of Φ:

THEOREM 2.5. *Let Σ be a pseudo-linear S^1 sphere of dimension n. Then:*

(i) *If $\Sigma^{S^1} = \emptyset$, then $K^1_{S^1}(\Sigma) = 0$ and the projection $\Sigma \to$ point induces an exact sequence*

$$0 \to (\Phi(\Sigma)) \to R(S^1) \to K^0_{S^1}(\Sigma) \to 0.$$

(ii) *If $\Sigma^{S^1} \neq \emptyset$, then $\tilde{K}^{n+1}_{S^1}(\Sigma) = 0$ and the inclusion $\Sigma^{S^1} \xrightarrow{i} \Sigma$ induces on injection*

$$\tilde{K}^n_{S^1}(\Sigma) \xrightarrow{i^*} \tilde{K}^n_{S^1}(\Sigma^{S^1}) = R(S^1)$$

onto the ideal $(\Phi(\Sigma)) \subset R(S^1)$.

Proof. Assume first that Σ is even-dimensional (hence $\Sigma^{S^1} \neq \emptyset$). By Lemma 2.3 $K^1_{S^1}(\Sigma) = 0$ thus the exact sequence (1.8) is reduced to

$$0 \to K_{S^1}(\Sigma) \xrightarrow{1-t} K_{S^1}(\Sigma) \xrightarrow{\varepsilon} K(\Sigma) \to 0 \tag{2.6}$$

where $\tilde{K}(\Sigma) \cong \mathbb{Z}$. Let $g \in \tilde{K}_{S^1}(\Sigma)$ denote an element with $\varepsilon(g)$ a generator of $\tilde{K}(\mathbb{Z})$. Multiplication by g induces a homomorphism $g: R(S^1) \to \tilde{K}_{S^1}(\Sigma)$. Putting this together with the above exact sequence gives a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & \ker g & \xrightarrow{1-t} & \ker g & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R(S^1) & \xrightarrow{1-t} & R(S^1) & \longrightarrow & Z \to 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \tilde{K}_{S^1}(\Sigma) & \xrightarrow{1-t} & \tilde{K}_{S^1}(\Sigma) & \xrightarrow{\varepsilon} & \tilde{K}(\Sigma) \to 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \operatorname{coker} g & \xrightarrow{1-t} & \operatorname{coker} g & \longrightarrow & 0 &
\end{array}$$

By Nacayama's lemma $S^{-1}\ker g = S^{-1}\operatorname{coker} g = 0$, and so $S^{-1} g$ is an isomorphism. This implies that

$$g_e: R(S^1)_e \to \tilde{K}_{S^1}(\Sigma)_e \tag{2.7}$$

is an isomcrphism. In particular $\tilde{K}_{S^1}(\Sigma)_e$ is torsion free. We can use induction on the dimension of the sphere and (2.1) to show that $\tilde{K}_{S^1}(\Sigma)_H$ is torsion free for all $e < H \leq S^1$. But this means that $\tilde{K}_{S^1}(\Sigma)$ is torsion free.

Next we show that, in fact, $\tilde{K}_{S^1}(\Sigma)$ is free of rank 1 over $R(S^1)$. Since $\tilde{K}_{S^1}(\Sigma)$ is torsion free and $\tilde{K}_{S^1}(\Sigma^{S^1}) \cong R(S^1)$ is torsion free, it follows from (2.2) with $H = S^1$, that the inclusion $\Sigma^{S^1} \subset \Sigma$ induces an injection of $R(S^1)$ modules

$$\tilde{K}_{S^1}(\Sigma) \subset R(S^1)$$

so that $\tilde{K}_{S^1}(\Sigma)$ can be thought of as an ideal of $R(S^1)$. It follows from (2.7) that $\tilde{K}_{S^1}(\Sigma)_e$ is a principal ideal of $R(S^1)_e$. Also, by induction we have that $\tilde{K}_{S^1}(\Sigma)_H$ is principal for $e < H \leq S^1$, but then it is free of rank 1.

To finish the proof of the theorem in the even dimensional case it remains to identity $\tilde{K}_{S^1}(\Sigma)$ as a submodule of $\tilde{K}_{S^1}(\Sigma^{S^1})$. Let $K^*_{S^1}(X; \mathbb{Q}) = K^*_{S^1}(X) \otimes \mathbb{Q}$. If $K^*_{S^1}(X; \mathbb{Q})$ is finite over $R(S^1) \otimes \mathbb{Q}$ there is a natural ring isomorphism

$$\hat{K}^*_{S^1}(X; \mathbb{Q}) \cong \varprojlim_n K^*(X \times_{S^1} S^{2n-1}; ;\mathbb{Q})$$

[3] through which $1 - t \in K_{S^1}(p, \mathbb{Q})$ gets identified with the virtual Hopf bundle in $K^*(\mathbb{C}P^\infty; \mathbb{Q})$. Completion is taken with respect to the kernel of the forgetful homomorphism $R(S^1) \otimes \mathbb{Q} \to \mathbb{Q}$. If we define $\hat{H}^*_{S^1}(X; \mathbb{Q}) = \varprojlim H^*(X \times_{S^1} S^{2n-1}; \mathbb{Q})$ we have a Chern character isomorphism $ch: \hat{K}^*_{S^1}(X; \mathbb{Q}) \to \hat{H}^*_{S^1}(X; \mathbb{Q})$ so that $1 - t$ goes to the power series $1 - e^c \in \hat{H}^*_{S^1}(p; \mathbb{Q})$ in the first Chern class c of the Hopf

bundle over $\mathbb{C}P^{\infty}$. It is easy to see that $\tilde{H}^*_{S^1}(\Sigma)$ is free of rank 1 over $H^*_{S^1}(p)$, with generator $h \in \tilde{H}^{\dim\Sigma}_{S^1}$. Similarly let $h_0 \in \tilde{H}^{\dim \Sigma^{S(1)}}$ denote a generator. The inclusion $i: \Sigma^{S^1} \to \Sigma$ induces a commutative diagram

$$\begin{array}{ccc} \hat{K}^*_{S^1}(\Sigma;\mathbb{Q}) & \xrightarrow{\ \text{ch}\ } & \hat{H}^*_{S^1}(\Sigma;\mathbb{Q}) \\ \Big\downarrow {\scriptstyle i^*} & & \Big\downarrow {\scriptstyle i^*} \\ \hat{K}^*_{S^1}(\Sigma^{S^1};\mathbb{Q}) & \xrightarrow{\ \text{ch}\ } & \hat{H}^*_{S^1}(\Sigma^{S^1};\mathbb{Q}). \end{array}$$

If g, g_0 are the generators of $\tilde{K}^*_{S^1}(\Sigma)$ and $\tilde{\Sigma}^*_{S^1}(\Sigma^{S^1})$ respectively over $R(S^1)$ then $\text{ch}g = u_1 h$, $\text{ch}g_0 = u_2 h_0$ and $i^*h = u_3 c^{\nu} h_0$ where $2\nu = \dim\Sigma - \dim\Sigma^{S^1}$ and u_1, u_2, u_3 are units in $\hat{H}^*_{S^1}(p;\mathbb{Q})$. Since $\text{ch}(1-t) = 1 - e^c$ and $(1-e^c)/c$ is a unit we have

$$\begin{aligned} \text{ch}\, i^* g &= i^*\, \text{ch}g = i^*(u_1 h) = u_1 u_3 c^{\nu} h_0 \\ &= (u_1 u_2^{-1} u_3 c^{\nu}/(1-e^c)^{\nu})\, \text{ch}((1-t)^{\nu} g_0), \end{aligned}$$

so $i^* g = u(1-t)^{\nu} g_0$ where u is a unit in $K^*_{S^1}(p;\mathbb{Q})$. It is easy to see that u is actually in $R(S^1)$ and is not divisible by $1 - t$.

We assume that S^1 acts effectively on Σ and use induction on the dimension of the sphere. Suppose $(i^* g)_{\mathscr{P}} = (\Phi(\Sigma) g_0)_{\mathscr{P}}$ up to a unit of $R(S^1)_{\mathscr{P}}$ for every principal prime ideal $\mathscr{P} \subset R(S^1)$ not associated with the trivial group. But the only principal prime ideal associated with the trivial group is $(1-t)$. Since $R(S^1)$ is a U.F.D. we have by the above argument and the definition of Φ that

$$i^* g = \Phi(\Sigma) g_0$$

up to a unit of $R(S^1)$. This proves the theorem in the even-dimensional case.

For the odd-dimensional case assume first that the fixed point set Σ^{S^1} is not empty and consider the cofibration $\Sigma \to p \to S\Sigma$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{K}^1_{S^1}(\Sigma) & \xrightarrow{\ \cong\ } & \tilde{K}^0_{S^1}(S\Sigma) \\ \Big\downarrow & & \Big\downarrow {\scriptstyle i^*} \\ \tilde{K}^1_{S^1}(\Sigma^{S^1}) & \xrightarrow{\ \cong\ } & \tilde{K}^0_{S^1}(S\Sigma^{S^1}). \end{array}$$

If g, g_0 are generators of $\tilde{K}^1_{S^1}(\Sigma)$, $\tilde{K}^1_{S^1}(\Sigma^{S^1})$ respectively then $i^* g = \Phi(S\Sigma) g_0$ up to a unit of $R(S^1)$ but by (2.4) $\Phi(\Sigma) = \Phi(S\Sigma)$ so we are done.

If $\Sigma^{S^1} = \emptyset$ then the cofibration gives a commutative diagram

$$\begin{array}{ccccccc} 0 \to K^1_{S^1}(\Sigma) \to \tilde{K}^0_{S^1}(S\Sigma) & \to & R(S^1) & \xrightarrow{\pi^*} & K^0_{S^1}(\Sigma) \to 0 \\ \downarrow i^* & & \downarrow 1 & & \\ 0 \to \tilde{K}^0_{S^1}(S\Sigma^{S^1}) & \to & R(S^1) & \longrightarrow & 0 \end{array}$$

Since i^* is injective, $\Sigma^1_{S^1}(\Sigma) = 0$ and $\ker \pi^* = (\Phi(\Sigma))$.

As a consequence of theorem 2.5 we can define an equivariant orientation. This will turn out to be stable so we may assume $\Sigma^{S^1} \neq \emptyset$. A generator $g \in K^n_{S^1}(\Sigma)$ is an S^1 orientation for Σ if $i^*g = \Phi(\Sigma)$ in $K^n_{S^1}(\Sigma^{S^1})$. g depends only on an orientation on Σ^{S^1}. Since the forgetful homomorphism $\tilde{K}^n_{S^1}(\Sigma) \to \tilde{K}^n(\Sigma)$ is surjective by (2.6), $\varepsilon(g)$ determines an orientation on Σ.

COROLLARY 2.8. *Let Σ be a pseudo-linear S^1 sphere. Then:*

(i) *An orientation on Σ^{S^1} determines an S^1 orientation on Σ.*

(ii) *An S^1 orientation on Σ determines an orientation on Σ.*

3. MAPS OF PSEUDO-LINEAR SPHERES

Maps of pseudo-linear spheres determine strong algebraic and homotopical relations between the spheres. We define a degree of a map and study its consequences.

Let $f:\Sigma \to \Gamma$ be an S^1 map between pseudo-linear spheres, then an S^1 degree, $\deg_{S^1} f \in R(S^1)$, is uniquely defined by $f^*h = (\deg_{S^1} f) \cdot g$ where g,h are the respective S^1 orientations.

LEMMA 3.1. *Let $f:\Sigma \to \Gamma$ be an S^1 map between pseudo-linear spheres. Then:*

(i) $\deg_{S^1} f = (\deg f^{S^1}) \cdot \dfrac{\Phi(\Gamma)}{\Phi(\Sigma)}$.

(ii) $\deg f = (\deg f^{S^1}) \cdot \lim\limits_{t \to 1} \dfrac{\Phi(\Gamma)}{\Phi(\Sigma)}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{K}^n_{S^1}(\Gamma^{S^1}) & \xrightarrow{\deg f^{S^1}} & \tilde{K}^n_{S^1}(\Sigma^{S^1}) \\ \uparrow i^* & & \uparrow i^* \\ \tilde{K}^n_{S^1}(\Gamma) & \xrightarrow{f^*} & \tilde{K}^n_{S^1}(\Sigma) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ \tilde{K}^n(\Gamma) & \xrightarrow{\deg f} & \tilde{K}^n(\Sigma). \end{array}$$

We have that $(\deg f^{S^1})\Phi(\Gamma) = (\deg f^{S^1}) i^*h = i^*f^*h = (\deg_{S^1} f)\Phi(\Sigma)$. $\deg f^{S^1}$ is an integer so $\Phi(\Sigma)\,|\,\Phi(\Gamma)$ if $\deg f^{S^1} \neq 0$. From the lower part of the diagram we get $\deg f \cdot \varepsilon f^*(h) = \varepsilon(\deg_{S^1} f)$.

COROLLARY 3.2. *Let* $f:\Sigma \to \Gamma$ *be an* S^1 *map between pseudo-linear spheres and assume that* $\deg f^{S^1} \neq 0$. *Then:*

(i) $\dim \Sigma^H \leq \dim \Gamma^H$ *for all* $H \leq S^1$.

(ii) *If* $\dim \Sigma = \dim \Gamma$, *then* $\deg f$ *is a nonzero multiple of* $\deg f^{S^1}$.

Proof. This is immediate from Lemma 3.1 and the definition of Φ.

COROLLARY 3.3. *The degree of an* S^1 *map* $f:\Sigma \to \Gamma$ *between fixed point free pseudo-linear spheres depends only on the spaces:*

$$\deg f = \lim_{t \to 1} \frac{\Phi(\Gamma)}{\Phi(\Sigma)}$$

In particular, if $\dim \Sigma = \dim \Gamma$ *then* $\deg f$ *is nonzero.*

Next we construct S^1 maps between pseudo-linear spheres. But we need that the fixed point sets at the various subgroups of S^1 be homotopy spheres.

THEOREM 3.4. *Let* Σ, Γ *be pseudo-linear* S^1 *spheres with* $\dim \Sigma^{S^1} = \dim \Gamma^{S^1}$. *Assume that* Γ^H *is a homotopy sphere for all* $H \leq S^1$. *Then:*

(i) *If* $\Phi(\Sigma)\,|\,\Phi(\Gamma)$, *then any* S^1 *map* $f:\Sigma^{S^1} \to \Gamma$ *can be equivariantly extended to* Σ.

(ii) *Let* $f,g:\Sigma \to \Gamma$ *be* S^1 *maps with* $\deg f^{S^1} \neq 0$. *Then* f^{S^1} *is homotopic to* g^{S^1} *if and only if* f *is* S^1 *homotopic to* g.

Proof. The obstructions lie in

$$H^*(\Sigma^H/S^1, \Sigma_H/S^1; \pi_{*-1}(\Gamma^H))$$
$$H^*(\Sigma^H/S^1, \Sigma_H/S^1; \pi_*(\Gamma^H))$$

[12] for (i) and (ii) respectively. Here $\Sigma_H \subset \Sigma^H$ is the subset of Σ^H of all points whose isotropy group contains H properly. Since $\Phi(\Sigma)\,|\,\Phi(\Gamma)$ and $\dim \Sigma^{S^1} = \dim \Gamma^{S^1}$ we have that $\dim \Sigma^H \leq \dim \Gamma^H$ for all $H \leq S_1$ hence

$$\dim(\Sigma H/S^1 - \Sigma_H/S^1) < \dim \Gamma^H$$

and all groups are zero.

COROLLARY 3.5. *Let* Σ, Γ *be pseudo-linear* S^1 *spheres, and* Σ^H, Γ^H *homotopy spheres for all* $H \leq S^1$. *If* $f:\Sigma \to \Gamma$ *is an* S^1 *map then* f *is an* S^1 *homotopy equivalence if and only if* $\deg_{S^1} f = \pm 1$.

Let $p \in \Sigma$ be a fixed point. If the action on Σ is smooth then

$$\dim(\tau_p \Sigma)^H = \dim \Sigma^H \qquad \text{for all } H \leq S^1,$$

so that $\Phi(\Sigma) = \Phi(S(\tau_p \Sigma))$, where τ denotes the tangent bundle and S the sphere bundle. This gives

COROLLARY 3.6. *If Σ is a smooth pseudo-linear S^1 sphere with $\Sigma^{S^1} \neq \emptyset$ then there is an S^1 module V and an S^1 map $f: \Sigma \to S(V)$ with $\deg_{S^1} f = 1$. In particular if Σ^H is a homotopy sphere for all $H \leq S^1$ then f is an S^1 homotopy equivalence.*

Remark. If $\Sigma^{S^1} = \Phi$ then exotic examples do exist [10].

4. THE TANGENT BUNDLE

We have shown that algebraically and homotopically pseudo-linear S^1 spheres behave essentially as linear spheres. The question arises whether they do so from the differentiable view point. Specifically, is the tangent bundle of a smooth pseudo-linear sphere stably trivial? (see [13] for cyclic p-groups). In what follows we show that twice the tangent bundle is stably trivial.

PROPOSITION 4.1. *If Σ is a smooth pseudo-linear G sphere, where G is a compact connected Lie group then $\tau\Sigma \otimes \mathbb{C}$ is stably trivial as a G complex vector bundle.*

Proof. Note that it is enough to show the proposition for $G = T$ a torus. If Σ is odd-dimensional then by proposition 2.3 we know that every bundle is stably trivial so we may assume Σ to be even-dimensional. Also one can generalize the first part of the proof of theorem 2.5 to show that $K_T(\Sigma)$ is a torsion free R(T) module. But this implies that

$$K_T(\Sigma) \xrightarrow{i^*} K_T(\Sigma^T)$$

is injective. So it suffices to show that $\tau\Sigma \otimes \mathbb{C}|_{\Sigma^T} = \nu\Sigma^T \otimes \mathbb{C} \oplus \tau\Sigma^T \otimes \mathbb{C}$ is stably trivial where $\nu\Sigma^T$ denotes the normal bundle of Σ^T in Σ. The right hand summand is always stably trivial and the left hand summand is stably trivial by [5].

5. COMPLEX PROJECTIVE SPACES

Ted Petrie [11] conjectured that if a homotopy complex projective space admits a circle action with isolated fixed point set then it has the same Pontrjagin classes as the standard projective space. Here is a situation where the conjecture holds.

Let $f: X \to \mathbb{C}P^n$ be a homotopy equivalence where X admits a smooth circle action with a nonempty set of isolated fixed points. Denote by $\eta \to X$ the pullback of the Hopf bundle over $\mathbb{C}P^n$. The S^1 action lifts to an action over η. Also, there is a natural action of $T^1 = S^1$ on η given by complex multiplication. Since both actions commute we get a $T^2 = S^1 \times T^1$ action on η [11].

COROLLARY 5.1. *In the above situation if the total space of the sphere bundle $S(\eta)$ is a pseudo-linear T^2 sphere then f preserves the Pontrjagin classes.*

Proof. Since $S(\eta)$ is odd-dimensional it follows from Proposition 2.3 that the canonical homomorphism $R(T^2) \to K_{T^2}(S(\eta))$ is surjective. We interpret this as follows. Since T^1 acts freely on $S(\eta)$, $K_{T^2}(S(\eta)) = K_{S^1}(X)$. Also, if M denotes the standard nontrivial irreducible T^1 module then $M \times_{S^1} S(\eta) = \eta$. This implies that the canonical homomorphism $R(S^1) \to K_{S^1}(X)$ extends to an epimorphism

$$R(S^1)[{}^{-}\eta, \eta^{-1}] \to K_{S^1}(X).$$

The result now follows from [11].

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