FINITE-DIFFERENCE INEQUALITIES AND AN EXTENSION OF LYAPUNOV'S METHOD

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1. INTRODUCTION

Many authors have investigated the boundedness and stability properties of differential equations by considering one-sided estimates of solutions. It is natural to expect that an estimate of the lower bound for the rate at which the solutions approach the origin or the invariant set would yield interesting refinements of stability notions. V. I. Zubov's notion of a uniform attractor (see [6] and [7]) is a refinement of such a nature.

The purpose of this paper is to prove some finite-difference inequalities that have a wide range of applications in the study of finite-difference equations. Incorporating an idea used by Zubov in stability theorems [6], [7], we introduce the concepts of mutual stability and boundedness; our two-sided estimates ensure that the motion remains in tube-like domains.

2. DEFINITIONS AND BASIC THEOREMS

Let G be an open set in an n-dimensional vector space R^n with norm $\|x\|$, let I denote the set of nonnegative integers, and let the function $h(k, x): I \times G \to R^n$ be continuous in x, for each k. Consider the finite-difference equation

$$\Delta x(k) = h(k, x(k)), \tag{2.0}$$

where x and h can be scalars or vectors, $h(k, 0) = 0$, and $\Delta x(k) = x(k+1) - x(k)$.

A function $x(k) = x(k; k_0, x_0)$ is called a solution of the difference equation (2.0) if it satisfies the three conditions:

(a) $x(k; k_0, x_0)$ is defined for $k_0 \le k \le k_0 + \beta$, for some positive integer β or for all $k \ge k_0$,

(b) $x(k_0; k_0, x_0) = x_0$ (we call this the *initial vector*),

(c) $\Delta x(k; k_0, x_0) = h(k, x(k; k_0, x_0))$ for $k_0 \le k \le k_0 + \beta - 1$ or for all $k \ge k_0$.

Hereafter, we assume that a solution to (2.0) exists and is uniquely defined for all $k \ge k_0$ by the initial vector x_0, and that this solution is continuous with respect to the initial vector x_0. More specifically, we assume that if $\{x_n\}$ is a sequence of vectors with $x_n \to x_0$ as $n \to \infty$, then the solutions through x_n converge to the solution through x_0:

$$x(k; k_0, x_n) \to x(k; k_0, x_0) \quad \text{as } n \to \infty .$$

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On each finite interval J, this convergence is then uniform with respect to k, since at most finitely many k belong to J.

The following fundamental theorem on finite-difference inequalities is useful for our subsequent discussion.

THEOREM 1. *Let the scalar function* $h(k, x): I \times G \to R^1$ *be continuous and monotone increasing in* x. *Let* $x(k)$ *and* $y(k)$ *be real-valued continuous functions defined on* I *such that* $(k, x(k)), (k, y(k)) \in I \times G$, *and* $x(0) \leq y(0)$. *Assume further that*

$$\Delta x(k) \leq h(k, x(k)) \quad \text{and} \quad \Delta y(k) \geq h(k, y(k)) \tag{2.1}$$

for all $k \in I$. *Then*

$$x(k) \leq y(k) \quad (k \in I). \tag{2.2}$$

Proof. If the assertion (2.2) is false, let β denote the least nonnegative integer k for which $x(k) > y(k)$. Then $x(k) \leq y(k)$ for $0 \leq k < \beta$, and this implies that

$$x(\beta - 1) \leq y(\beta - 1) \tag{2.3}$$

and

$$x(\beta) > y(\beta). \tag{2.4}$$

Using (2.3) and (2.4), we obtain the inequality

$$\Delta x(\beta - 1) > \Delta y(\beta - 1). \tag{2.5}$$

The relations (2.1) and (2.5) yield the further inequality

$$h(\beta - 1, x(\beta - 1)) > h(\beta - 1, y(\beta - 1)).$$

This is a contradiction, in view of (2.3) and the monotoneity of $h(k, x)$, and the result follows.

The following theorem is an extension to finite-difference equations of Theorem 1 in [5], which in turn is a generalization of a lemma due to R. Bellman [2].

THEOREM 2. *Suppose that the scalar functions* $W_1(k, r)$ *and* $W_2(k, r)$ *are defined, continuous, and nonnegative on* $I \times G$ $(G \subset R^1)$ *and monotone increasing in* r, *and that*

$$W_2(k, m(k)) \leq \Delta m(k) \leq W_1(k, m(k)). \tag{2.6}$$

For $k \geq 0$, *let* $u(k) = u(k; 0, u_0)$ *and* $v(k) = v(k; 0, v_0)$ *be the solutions of the systems*

$$\Delta u(k) = W_1(k, u(k)), \quad u(0) = u_0, \tag{2.7}$$

$$\Delta v(k) = W_2(k, v(k)), \quad v(0) = v_0, \tag{2.8}$$

and suppose $v_0 \leq m(0) \leq u_0$. *Then*

$$v(k) \leq m(k) \leq u(k) \tag{2.9}$$

for all $k \geq 0$.

Proof. Applying Theorem 1 to the second part of (2.6) and to (2.7), we obtain the right half of (2.9). A similar argument yields the left half.

The following comparison theorem is useful in the study of solutions of the finite-difference equations

$$\Delta x(k) = f(k, x(k)), \qquad x(0) = x_0, \tag{2.10}$$

$$\Delta y(k) = g(k, y(k)), \qquad y(0) = y_0, \tag{2.11}$$

where x, y, f, and g can be scalars or vectors and f(k, x), g(k, y) are continuous functions defined on $I \times G$.

THEOREM 3. *Let the functions* $W_1(k, r)$ *and* $W_2(k, r)$ *be defined as in Theorem* 2. *Suppose further that the functions* f(k, x) *and* g(k, y) *of* (2.10) *and* (2.11) *satisfy the condition*

$$W_2(k, \|x - y\|) \leq \|f(k, x) - g(k, y)\| \leq W_1(k, \|x - y\|) \tag{2.12}$$

for all $k \geq 0$. *Let* u(k) *and* v(k) *be solutions of* (2.7) *and* (2.8). *Let* x(k) *and* y(k) *be solutions of the equations* (2.10) *and* (2.11), *and assume that* $v_0 \leq \|x_0 - y_0\| \leq u_0$. *Then*

$$v(k) \leq \|x(k) - y(k)\| \leq u(k) \qquad \text{for } k \geq 0. \tag{2.13}$$

Proof. Let $m(k) = \|x(k) - y(k)\|$; then $m(k+1) = \|x(k+1) - y(k+1)\|$. By (2.12),

$$\Delta m(k) \leq \|f(k, x(k)) - g(k, y(k))\| \leq W_1(k, \|x(k) - y(k)\|),$$

that is,

$$\Delta m(k) \leq W_1(k, m(k)). \tag{2.14}$$

Applying Theorem 1 to (2.7) and (2.14), we obtain the right half of the inequality (2.13). The proof of the left half of the inequality is similar.

3. MUTUALLY EQUISTABLE, ATTRACTING, AND MUTUALLY EQUIBOUNDED SOLUTIONS

The following definitions serve to unify our results on mutual stability and boundedness.

Let $x(k) = x(k; k_0, x_0)$ and $y(k) = y(k; k_0, y_0)$ denote solutions of (2.10) and (2.11).

(a_1) Two solutions of the equations (2.10) and (2.11) are *mutually equistable* if for each $\varepsilon_1 > 0$ and each $k_0 \in I$ it is possible to find positive functions $d_1 = d_1(k_0, \varepsilon_1)$, $d_2 = d_2(k_0, \varepsilon_1)$, and $\varepsilon_2 = \varepsilon_2(k_0, \varepsilon_1)$, continuous in ε_1 for each k, such that $\varepsilon_2 < d_2 \leq d_1 < \varepsilon_1$, and such that $\varepsilon_2 < \|x(k) - y(k)\| < \varepsilon_1$ for $k \geq k_0$, whenever $d_2 \leq \|x_0 - y_0\| \leq d_1$.

(a_2) Two solutions of the equations (2.10) and (2.11) are *mutually attracting* if the following two conditions hold.

(i) In (a_1), it is possible to find $d_1 = d_1(k_0, \varepsilon_1) > 0$ such that $\|x(k) - y(k)\| < \varepsilon_1$ for $k \geq k_0$, whenever $\|x_0 - y_0\| \leq d_1$.

(ii) If $\varepsilon_1 > 0$, $\alpha_1 > 0$, $0 < \alpha_2 < \alpha_1$, and $k_0 \in I$, it is possible to find positive functions $\varepsilon_2 = \varepsilon_2(k_0, \varepsilon_1, \alpha_1, \alpha_2)$, $T_1 = T_1(k_0, \varepsilon_1, \alpha_1, \alpha_2)$, and

$$T_2 = T_2(k_0, \varepsilon_1, \alpha_1, \alpha_2)$$

such that $T_1 \leq T_2$, $\varepsilon_2 < \varepsilon_1$, $\varepsilon_2 < \alpha_2$, and such that $\varepsilon_2 < \|x(k) - y(k)\| < \varepsilon_1$ whenever $k \in I$, $k \in [k_0 + T_1, k_0 + T_2]$, and $\alpha_2 \leq \|x_0 - y_0\| \leq \alpha_1$.

(a_3) Two solutions of the equations (2.10) and (2.11) are *mutually equibounded* if for all $\alpha_1 > 0$, $0 < \alpha_2 \leq \alpha_1$, $k_0 \in I$, it is possible to find positive functions $\beta_1 = \beta_1(k_0, \alpha_1, \alpha_2)$ and $\beta_2 = \beta_2(k_0, \alpha_1, \alpha_2)$, continuous in α_1 for each k_0, such that $\beta_2 < \beta_1$, $\beta_2 < \alpha_2$, and $\beta_2 < \|x(k) - y(k)\| < \beta_1$, whenever $k \geq k_0$ and $\alpha_2 \leq \|x_0 - y_0\| \leq \alpha_1$.

Corresponding to the definition (a_1), we formulate the definition (a_1*) with respect to the system (2.7) and (2.8).

Let u(k) and v(k) be solutions of the equations (2.7) and (2.8).

(a_1*) For each $\eta_1 > 0$ and each $k_0 \in I$, there exist positive functions $\delta_1 = \delta_1(k_0, \eta_1)$, $\delta_2 = \delta_2(k_0, \eta_1)$, and $\eta_2 = \eta_2(k_0, \eta_1)$, continuous in η_1 for each k_0, such that $\eta_2 < \delta_2 \leq \delta_1 < \eta_1$ and such that $\eta_2 < v(k) \leq u(k) < \eta_1$ whenever $k \geq k_0$ and $\delta_2 \leq v_0 \leq u_0 \leq \delta_1$.

Similarly, we can formulate definitions (a_2*) and (a_3*).

THEOREM 4. *Suppose the hypotheses of Theorem* 3 *are satisfied. If the difference equations* (2.7) *and* (2.8) *satisfy condition* (a_1*) *or* (a_2*) *or* (a_3*), *then the solutions of* (2.10) *and* (2.11) *are mutually equistable or mutually attracting or mutually equibounded, respectively.*

This theorem follows directly from Theorem 3 and our definitions.

4. AN EXTENSION OF LYAPUNOV'S DIRECT METHOD

Let the scalar function V(k, x, y) be defined, nonnegative, and continuous on $I \times G \times G$, where $G \subset R^n$, and suppose that for each $k \in I$, $V(k, x, y) = 0$ if and only if $x = y$. Let $\Delta V(k, x, y)$ denote the expression

$$\Delta V(k, x, y) = V(k + 1, x + f(k, x), y + g(k, y)) - V(k, x, y).$$

LEMMA. *Suppose the function* V(k, x, y) *satisfies the condition*

$$W_2(k, V(k, x, y)) \leq \Delta V(k, x, y) \leq W_1(k, V(k, x, y)),$$

where the scalar functions W_1 *and* W_2 *satisfy the hypotheses in Theorem* 2. *If* x(k) *and* y(k) *are solutions of* (2.10) *and* (2.11) *such that* $v_0 \leq V(k_0, x_0, y_0) \leq u_0$, *then*

$$v(k) \leq V(k, x(k), y(k)) \leq u(k) \qquad (k \geq k_0),$$

where u(k) *and* v(k) *are solutions of* (2.7) *and* (2.8).

The proof of this lemma is analogous to the proof of Theorem 3. We omit the details.

For our further discussion, suppose that the function V(k, x, y) satisfies the two conditions

$$(4.1)\quad \begin{cases} b(\|x - y\|) \leq V(k, x, y) \leq a(\|x - y\|)\,, \\ \text{where } a(r) \text{ and } b(r) \text{ are continuous, strictly} \\ \text{increasing functions for } r \geq 0 \text{ with } a(0) = b(0) = 0\,, \end{cases}$$

$$(4.2)\quad b(r) \to \infty \quad \text{as } r \to \infty\,.$$

Then we have the following theorems on mutual stability and boundedness of the solutions of (2.10) and (2.11).

THEOREM 5. *Suppose the hypothesis of the lemma holds. Assume that the function* $V(k, x, y)$ *satisfies condition* (4.1). *Then* $(a_1{*})$ *implies that the solutions of* (2.10) *and* (2.11) *are mutually equistable.*

Proof. Suppose $\varepsilon_1 > 0$, $k_0 \in I$, and $\eta_1 = b(\varepsilon_1) > 0$. Since $(a_1{*})$ holds, there exist positive functions $\delta_1 = \delta_1(k_0, \eta_1)$, $\delta_2 = \delta_2(k_0, \eta_1)$, and $\eta_2 = \eta_2(k_0, \eta_1)$ such that $\eta_2 < \delta_2 \leq \delta_1 < \eta_1$, and

$$(4.3)\quad \eta_2 < v(k) \leq u(k) < \eta_1\,,$$

whenever $k \geq k_0$, and

$$(4.4)\quad \delta_2 \leq v_0 \leq u_0 \leq \delta_1\,.$$

By the continuity of a at $r = 0$, we can choose an $\varepsilon_2 > 0$ such that $a(\varepsilon_2) \leq \eta_2$ and $\varepsilon_2 < \varepsilon_1$. Let $x(k)$ and $y(k)$ be solutions of (2.10) and (2.11) such that

$$(4.5)\quad v_0 \leq V(k_0, x_0, y_0) \leq u_0\,.$$

Then it follows from the lemma that

$$(4.6)\quad v(k) \leq V(k, x(k), y(k)) \leq u(k) \quad (k \geq k_0)\,,$$

where $u(k)$ and $v(k)$ are solutions of (2.7) and (2.8). Further, (4.1), (4.5), and (4.4) show that there exist two positive functions $d_1(k_0, \varepsilon_1)$ and $d_2(k_0, \varepsilon_1)$ such that

$$(4.7)\quad d_2 \leq \|x_0 - y_0\| \leq d_1$$

whenever

$$(4.8)\quad \delta_2 \leq V(k_0, x_0, y_0) \leq \delta_1\,,$$

and vice versa. Thus, whenever $d_2 \leq \|x_0 - y_0\| \leq d_1$, it follows from the lemma that (4.6) is true. Now we claim that $\varepsilon_2 < \|x(k) - y(k)\| < \varepsilon_1$ for $k \geq k_0$, provided (4.7) holds.

Suppose, on the contrary, that there exist solutions $x(k)$ and $y(k)$ of (2.10) and (2.11), satisfying (4.8), such that $\|x(k_1) - y(k_1)\| \geq \varepsilon_1$ or $\|x(k_1) - y(k_1)\| \leq \varepsilon_2$ for some $k_1 > k_0$. Using the inequalities (4.1), (4.6), and (4.3), we arrive at the contradiction

$$b(\varepsilon_1) \leq V(k_1, x(k_1), y(k_1)) \leq u(k_1) < b(\varepsilon_1)\,.$$

On the other hand, if $\|x(k_1) - y(k_1)\| \leq \varepsilon_2$, we obtain the contradiction

$$a(\varepsilon_2) \geq V(k_1, x(k_1), y(k_1)) \geq v(k_1) > \eta_2 \geq a(\varepsilon_2)$$

from the inequalities (4.1), (4.6), and (4.3). This proves that (a_1) follows from (a_1*).

THEOREM 6. *Suppose the hypothesis of the lemma holds, and assume that the function* V(k, x, y) *satisfies condition* (4.1). *Then* (a_2*) *implies that the solutions of* (2.10) *and* (2.11) *are mutually attracting.*

Proof. Suppose $\varepsilon_1 > 0$, $k_0 \in I$, and $\alpha_1 > 0$. Choose an α_2 so that $0 < \alpha_2 \leq \alpha_1$. Let x_0 and y_0 be such that $\alpha_2 \leq \|x_0 - y_0\| \leq \alpha_1$. Then, because V(k, x, y) = 0 and by (4.1), it is possible to find positive numbers $\hat{\alpha}_1 = \hat{\alpha}_1(\alpha_1)$ and $\hat{\alpha}_2 = \hat{\alpha}_2(\alpha_2)$ such that

$$\hat{\alpha}_2 \leq V(k_0, x_0, y_0) \leq \hat{\alpha}_1 . \tag{4.9}$$

Let (a_2*) hold, so that (i*) and (ii*) of (a_2*) are simultaneously true. Then, if $\eta_1 = b(\varepsilon_1) > 0$, $k_0 \in I$, and $0 < \hat{\alpha}_2 \leq \hat{\alpha}_1$, there exist positive numbers $T_1 = T_1(k_0, \hat{\alpha}_1, \hat{\alpha}_2, \eta_1)$, $T_2 = T_2(k_0, \hat{\alpha}_1, \hat{\alpha}_2, \eta_1)$, and $\eta_2 = \eta_2(\eta_1)$ such that $\eta_2 < \eta_1$, $\eta_2 < \hat{\alpha}_2$, $T_1 \leq T_2$, and such that

$$\eta_2 < v(k) \leq u(k) < \eta_1, \qquad k \in [k_0 + T_1, k_0 + T_2] \tag{4.10}$$

whenever

$$\hat{\alpha}_2 \leq v_0 \leq u_0 \leq \hat{\alpha}_1 . \tag{4.11}$$

Also, let x(k) and y(k) be solutions of (2.10) and (2.11) such that

$$v_0 \leq V(k_0, x_0, y_0) \leq u_0 .$$

Then, as in Theorem 5, condition (4.6) is satisfied whenever (4.5) holds. Now, whenever $\alpha_2 \leq \|x_0 - y_0\| \leq \alpha_1$, it follows from (4.9) that (4.11) is valid.

Choose an ε_2 such that $a(\varepsilon_2) \leq \eta_2$, $\varepsilon_2 < \alpha_2$, and $\varepsilon_2 < \varepsilon_1$. Proceeding as in the proof of Theorem 5, we can prove that (ii) is true with this ε_2 and T_1, T_2 as in the earlier part of this proof.

The same argument of the proof of Theorem 5 shows also that (i*) implies (i). The two conditions (i) and (ii) together show that (a_2) is satisfied. Hence the proof is complete.

THEOREM 7. *Assume that the function* V(k, x, y) *satisfies the hypothesis of the lemma, as well as conditions* (4.1) *and* (4.2). *Then* (a_3*) *implies that the solutions of* (2.10) *and* (2.11) *are mutually equibounded.*

The proof is similar to the proof of Theorems 5 and 6, and we leave the details to the reader.

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