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Families of Affine Planes: The Existence of a Cylinder

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Introduction

Dolgachev and Weisfeiler [9, (3.8.5)] formulated the following.

Conjecture. *Let $f: X \to S$ be a flat affine morphism of smooth schemes with every fiber isomorphic* (*over the residue field*) *to an affine space. Then f is locally trivial in the Zariski topology.*

In the characteristic-0 case, this conjecture is known to be true (under much weaker assumptions) for morphisms of relative dimension 1 ([24; 23, Thm. 2]; see also [30, Thm. 2] and [5; 6; 10]). Another proof based on the Rosenlicht–Chevalley–Grothendieck theory of special algebraic groups [2; 37] was indicated by Danilov; see [9]. The known partial positive results in higher relative dimensions (see e.g. [30; 38] and [4, (3.9)–(3.10)]) deal only with families over a 1-dimensional base with 2-dimensional fibers, under an extra assumption that the generic fiber is the affine plane as well. In this paper we show that the latter assumption holds over any base. To simplify consideration, we restrict it to smooth, quasi-projective varieties defined over $\mathbb{C}$ (actually, Theorem 0.1 remains true over any algebraically closed field of characteristic 0).

We say that a family $f: X \to S$ of quasi-projective varieties *contains a cylinder* if, for some Zariski open subset S_0 of S, there is a commutative diagram

$$\begin{tikzcd} f^{-1}(S_0) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & S_0 \times \mathbb{C}^k \arrow[dl, "\mathrm{pr}_1"] \\ & S_0 & \end{tikzcd}$$

where φ is an isomorphism. (In general, by a *cylinder over U* we mean a Cartesian product $U \times \mathbb{C}^k$ where $k > 0$.)

Our main result is the following theorem.

Theorem 0.1. *A smooth family $f: X \to S$ with general fibers isomorphic to $\mathbb{C}^2$ contains a cylinder $S_0 \times \mathbb{C}^2$.*

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See [14; 29; 31; 39] for statements of this type concerning affine surfaces with logarithmic Kodaira dimension $-\infty$. We do not know if the theorem remains true in higher relative dimensions.

A theorem of Sathaye ([38], which fixed a preliminary incomplete version in [22]; cf. also [4, (3.9)–(3.10)]), together with Theorem 0.1, proves the following.

COROLLARY 0.2. *The Dolgachev–Weisfeiler conjecture is, indeed, true for families of affine planes over smooth curves.*

Recall that, for an affine domain R (over $\mathbb{C}$) and a prime ideal $\mathfrak{p}$ of R, the residue field of R at $\mathfrak{p}$ is $K(\mathfrak{p}) := R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$, where $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$. Note that sometimes the assumption of the Dolgachev–Weisfeiler conjecture is addressed in a more restrictive form, not only for closed points of S but for all its points. Namely, one supposes the existence of isomorphisms

$$A \otimes_R K(\mathfrak{p}) \simeq K(\mathfrak{p})^{[n]} \quad \forall \mathfrak{p} \in \operatorname{Spec} R.$$

The next corollary shows that, at least for $n = 2$, this additional assumption is fulfilled automatically.

COROLLARY 0.3. *Let A be an affine domain over R. If*

$$A \otimes_R K(\mathfrak{p}) \simeq K(\mathfrak{p})^{[2]} \tag{1}$$

for any maximal ideal $\mathfrak{p}$ of R, then this is so for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. Denote $S := \operatorname{Spec} R$ and $X := \operatorname{Spec} A$, and let $f : X \to S$ be the morphism induced by the inclusion $R \hookrightarrow A$. Note first that if $\mathfrak{p} = 0$ is the zero ideal and so $K(\mathfrak{p})$ is the fraction field of R (i.e., for the generic point of S), then condition (1) is nothing but the existence of a cylinder of f. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, denote $S_{\mathfrak{p}} = V(\mathfrak{p}) = \operatorname{Spec} R^{\mathfrak{p}}$ and $X_{\mathfrak{p}} = \operatorname{Spec} A^{\mathfrak{p}} \subset X$, where $R^{\mathfrak{p}} := R/\mathfrak{p}$ and $A^{\mathfrak{p}} := A \otimes_R R^{\mathfrak{p}}$. If $M \supset \mathfrak{p}$ is a maximal ideal of R then $M_{\mathfrak{p}} := MR^{\mathfrak{p}}$ is a maximal ideal of $R^{\mathfrak{p}}$, and any such ideal arises in that way. Our assumption implies that $A^{\mathfrak{p}} \otimes_{R^{\mathfrak{p}}} K(M_{\mathfrak{p}}) \simeq K(M_{\mathfrak{p}})^{[2]}$, that is, the fibers of $X_{\mathfrak{p}} \to S_{\mathfrak{p}}$ over the closed points $M_{\mathfrak{p}} \in S_{\mathfrak{p}}$ are isomorphic to the affine plane. In view of this observation, the existence of a cylinder of $f_{\mathfrak{p}} := f|_{X_{\mathfrak{p}}} : X_{\mathfrak{p}} \to S_{\mathfrak{p}}$ provided by Theorem 0.1 simply means that (1) is fulfilled for the algebra $A^{\mathfrak{p}}$ over $R^{\mathfrak{p}}$ with the zero ideal $\mathfrak{p}' = 0$ of the algebra $R^{\mathfrak{p}}$. In turn, the latter means that (1) is fulfilled for $\mathfrak{p}$. $\square$

On the other hand, Theorem 0.1 provides one of the principal ingredients in the proof of the following statement (see [21, Lemma III]).

THEOREM [21]. *A polynomial p on $\mathbb{C}^3$ with general fibers isomorphic to $\mathbb{C}^2$ is a variable of the polynomial algebra $\mathbb{C}^{[3]}$ (that is, $\mathbb{C}^{[3]} \simeq \mathbb{C}[p]^{[2]}$). In particular, all its fibers are isomorphic to $\mathbb{C}^2$.*

Up to Theorem 0.1, this result was observed in [28, Prop. 4.3] and [38, Cor. on p. 60] for polynomials with only smooth (or at least factorial [30]) fibers; now we see that this condition is superfluous.

Let us give a brief outline of the proof. It uses the following simple observation (Lemma 3.2): Let $\bar{f}: V \to S$ be a $\mathbb{P}^n$-family over a quasi-projective base, and let $D \subseteq V$ be an irreducible smooth divisor that meets every fiber $V_s := \bar{f}^{-1}(s) \simeq \mathbb{P}^n$ $(s \in S)$ transversally along a hyperplane; then the family (V, D) over S is locally trivial in the Zariski topology. Thus, to prove Theorem 0.1 it suffices to complete a given $\mathbb{C}^2$-family $f: X \to S$ obtaining a $\mathbb{P}^2$-family $\bar{f}: V \to S$ with an irreducible divisor $D = V \setminus X$ as before. We start with an arbitrary relative completion $\bar{f}: V \to S$ with an SNC (simple normal crossing) divisor $D = V \setminus X$, and then we contract successively the superfluous irreducible components E of D in order to obtain a minimal relative SNC completion. Owing to a relative version of the Castelnuovo–Enriques–Kodaira contraction theorem (Theorem 3.2), E can be smoothly contracted if, for every $s \in S$, the irreducible components $(C_{s,i})_{i=1,\dots,n}$ of $E_s := E \cap V_s$ are disjoint (-1)-curves in the surface V_s. That is, $(C_{s,i})_{i=1,\dots,n}$ must correspond to a set of (-1)-vertices of the weighted dual graph $\Gamma(D_s)$ (where $D_s := D \cap V_s$) that contains no pair of neighbors.

Combinatorially, Γ does not change when s varies in an appropriate Zariski open subset $U \subseteq S$. For a fixed $s \in U$ there is a natural monodromy representation $\pi_1(U) \to \operatorname{Aut}\Gamma(D_s)$ that acts transitively on the set $(C_{s,i})_{i=1,\dots,n}$, leaving it invariant. We show (Proposition 2.2) that the dual graph Γ of an arbitrary nonminimal SNC completion of $\mathbb{C}^2$ possesses an orbit of $\operatorname{Aut}\Gamma$ that consists of (-1)-vertices and has no pair of neighbors. Thus, we can minimize Γ by then contracting it (via equivariant contractions) to a minimal linear chain known as a Ramanujam–Morrow graph. By Theorem 3.2, this minimization can be realized geometrically.

The advantage of a Ramanujam–Morrow graph Γ is that (with one simple exception) the group $\operatorname{Aut}\Gamma$ is trivial; henceforth, for any irreducible component E of D, the curve $E_s = E \cap V_s$ is irreducible as well. Moreover, Γ can be transformed to a single-vertex graph via a sequence of blow-ups and blow-downs. Since the monodromy action is trivial, these blow-ups and blow-downs can be done simultaneously over U, thus yielding the desired single-component SNC completion as in Lemma 3.2, which completes the proof.

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1. A Contraction Theorem

The main result of this section (Theorem 1.3) is a relative version of the classical Castelnuovo–Enriques–Kodaira contraction theorem. In the analytic setting it follows from the Moishezon–Nakano–Fujiki theorem (see [1; 13; 26; 32; 34] and especially [12, Rem. 3]), whereas in the projective setting it follows from the theorem on contraction of extremal rays as given in [25, Thm. 3-2-1] (cf. also [3; 7; 20; 27]). Actually, the particular version that we need is much simpler, so we provide a proof along the lines of the Castelnuovo–Enriques–Kodaira original approach [26, Apx.; 17, Sec. 4.1, p. 154].

As usual, the structure sheaf of an algebraic variety X is denoted by $\mathcal{O}_X$. If L is a line bundle on X and if Y is a subvariety of X, then $\mathcal{O}_Y(L)$ denotes the sheaf of germs of section of L over Y. We begin with the following lemma (cf. [15, Sec. 7.6; 35, Thm. 4.7]).

LEMMA 1.1. *Let $\rho\colon E \to S$ be a smooth proper morphism of smooth quasi-projective varieties with fibers $E_s := \rho^{-1}(s)$ $(s \in S)$. For a line bundle L on E we denote $\mathcal{L} := \mathcal{O}_E(L)$ and $\mathcal{L}_s := \mathcal{O}_{E_s}(L)$. Suppose that*

(o) *$H^q(E_s, \mathcal{L}_s) = 0$ for all $s \in S$ and all $q \geq 1$.*

Then, for any Zariski open affine subset $S_0 \subset S$ and $E_{S_0} := \rho^{-1}(S_0)$, we have:

(a) *$H^q(E_{S_0}, \mathcal{L}) = 0$ for all $q \geq 1$, and*

(b) *for every point $s \in S_0$, the restriction homomorphism*

$$H^0(E_{S_0}, \mathcal{L}) \to H^0(E_s, \mathcal{L}_s)$$

is surjective; furthermore,

(c) *the sheaf $\rho_*\mathcal{L}$ is locally free and generated by a vector bundle (say, $\xi(L)$) over S with fibers $\xi(L)_s = H^0(E_s, \mathcal{L}_s)$, $s \in S$.*

Proof. (a) Note that by [19, Prop. III.9.2.c], $\mathcal{L}$ is a flat $\mathcal{O}_S$-module. In virtue of the assumption (o), for every $s \in S$ and every $q \geq 1$ the natural homomorphism

$$R^q\rho_*\mathcal{L} \otimes_{\mathcal{O}_S} k(s) \to H^q(E_s, \mathcal{L}_s) = 0$$

is an isomorphism, and the coherent sheaf $R^q\rho_*\mathcal{L}$ is locally free [15, Thm. 4; 19, Thm. III.12.11.a, Ex. II.5.8.c; 8, Prop. II.3.7]; here $k(s) \simeq \mathbb{C}$ denotes the residue field of a closed point $s \in S$. Thus we have

$$R^q\rho_*\mathcal{L} = 0 \quad \forall q \geq 1.$$

The Leray spectral sequence gives now isomorphisms

$$H^q(E, \mathcal{L}) \simeq H^q(S, \rho_*\mathcal{L}) \quad \forall q \geq 1 \tag{2}$$

[35, (5.16)]. For a Zariski open affine subset $S_0 \subset S$, by Serre's vanishing theorem [19, Thm. III.3.7] we have

$$H^q(S_0, \rho_*\mathcal{L}) = 0 \quad \forall q \geq 1,$$

which together with (2) implies (a).

(b) Since $R^1\rho_*\mathcal{L} = 0$, it follows that for every point $s \in S_0$ the homomorphism

$$\rho_*\mathcal{L} \otimes_{\mathcal{O}_{S_0}} k(s) \to H^0(E_s, \mathcal{L}_s) \tag{3}$$

is surjective, whence it is an isomorphism [19, Thm. III.12.11]. On the other hand, since S_0 is affine we have an isomorphism

$$(\rho_*\mathcal{L})|_{S_0} \simeq H^0(E_{S_0}, \mathcal{L})^{\sim}, \tag{4}$$

where $M^{\sim}$ denotes the $\mathcal{O}_{S_0}$-module generated by an $H^0(S_0, \mathcal{O}_{S_0})$-module M [19, Prop. III.8.5]. Now (3) and (4) yield (b).

(c) By (o) we have $h(s) := \dim H^0(E_s, \mathcal{L}_s) = \chi(E_s, \mathcal{L}_s)$, where the Euler characteristic is locally constant on S [15, Thm. 5; 8, Prop. II.3.8]. Now the isomorphism in (2) and [19, Ex. II.5.8.c] imply that $\rho_*\mathcal{L}$ is a locally free sheaf with $\rho_*\mathcal{L} \simeq \mathcal{O}_S(\xi(L))$, where $\xi(L)_s = H^0(E_s, \mathcal{L}_s)$, $s \in S$. The proof is completed. □

COROLLARY 1.2. *The statements of Lemma 1.1 remain true if one replaces the assumption* (o) *by any one of the following two.*

(o′) *For every $s \in S$, the line bundle $L|_{E_s} - K_{E_s}$ on the fiber E_s is ample, where K_{E_s} is the canonical bundle on E_s.*

(o″) *For every $s \in S$ and for each irreducible component C of the fiber E_s, $C \simeq \mathbb{P}^n$ $(n \geq 1)$ and $L|_C \simeq \mathcal{O}(l)$ with $l \geq 0$.*

Indeed, by the Kodaira–Nakano vanishing theorem, any one of the conditions (o′) and (o″) implies (o) [17, Sec. 1.2, p. 154].

THEOREM 1.3. *Let $\pi : V \to S$ be a smooth proper morphism of smooth quasi-projective varieties, and let $E \subset V$ be an irreducible smooth divisor* (*proper over S*) *that meets every fiber $V_s = \pi^{-1}(s)$ $(s \in S)$ transversally. Let $E_s := V_s \cap E = \bigcup_{i=1}^m C_{s,i}$ be the decomposition into irreducible components* (*ordered arbitrarily for every $s \in S$*).

(a) *Assume that, for every $s \in S$ and each $i = 1, \dots, m$, $C_{s,i} \simeq \mathbb{P}^n$ $(n \geq 1)$ with the conormal bundle $J_{C_{s,i}}/J^2_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, where $J_{C_{s,i}}$ is the ideal sheaf of $C_{s,i}$ in V_s.* (In other words, by Kodaira's contraction theorem [26] we assume that each irreducible component $C_{s,i}$ can be contracted in the fiber V_s into a smooth point.) *Then there is a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & S & \end{array}$$

where W is a smooth quasi-projective variety, π' is a smooth morphism, and φ is an S-contraction of the divisor E on V onto a smooth subvariety $A \subset W$ étale over S.

(b) *Let E' be another smooth divisor on V* (*proper over S*) *that meets every fiber V_s $(s \in S)$ and the divisor E transversally, and let $E'_s := E' \cap V_s = \bigcup_{j=1}^{m'} C'_{s,j}$ be the decomposition into irreducible components. Assume that, for every $s \in S$ and for every pair i, j such that $C_{s,i} \cap C'_{s,j} \neq \emptyset$, this intersection becomes a hyperplane under an isomorphism $C_{s,i} \simeq \mathbb{P}^n$ as in* (a)*. Then $\varphi(E')$ is a divisor on W* (*proper over S*) *whose singularities are at worst transversal intersections* (*along A*) *of several smooth branches.*

Proof. (a) We may assume in the sequel that the base S is connected. Fix a very ample line bundle H on V. For an arbitrary point $s_0 \in S$, letting $L = mH$ with m sufficiently big, we may assume that the line bundle $L|_{V_{s_0}} - K_{V_{s_0}}$ on the fiber V_{s_0} over s_0 is ample; hence, by the Kodaira vanishing theorem (see [17, Sec. 1.2]), we have $H^q(V_{s_0}, \mathcal{L}_{s_0}) = 0$ for all $q \geq 1$ (here $\mathcal{L}_s := \mathcal{O}_{V_s}(L)$). Since $\mathcal{L}$ is a flat $\mathcal{O}_S$-module, by the semi-continuity theorem [19, Thm. III.12.8] it follows that

$$H^q(V_s, \mathcal{L}_s) = 0 \quad \forall q \geq 1 \tag{5}$$

for every point s in a neighborhood S_0 of the point s_0. Thus, for m_0 large enough and $L = L_0 := m_0 H$, (5) holds for every point $s \in S$.

Since the divisor E is irreducible, the monodromy of the smooth family $\pi|_E : E \to S$ acts transitively on the set of irreducible components $\{C_{s,i}\}_{i=1}^m$ of the fiber E_s $(s \in S)$. Hence all these components (regarded as cycles of V) are algebraically (and then also numerically) equivalent. Thus $k := \deg(L_0|_{C_{s,i}})$ does not depend on s, i. Consider the (Cartier) divisors $L_j := L_0 + jE = m_0 H + jE$ on V $(j \in \mathbb{Z})$. Under our assumptions ($C_{s,i} \simeq \mathbb{P}^n$ and $[C_{s,i}]|_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$), for every $s \in S$ and each $i = 1, \dots, m$ we have $L_j|_{C_{s,i}} \simeq \mathcal{O}_{\mathbb{P}^n}(l)$ with $l := k - j$, so

$$H^q(E_s, (\mathcal{L}_j)_s) = 0 \quad \forall q \geq 1,\ \forall j = 0, \dots, k. \tag{6}$$

Now the same argument as in the proof of the Castelnuovo–Enriques–Kodaira theorem [26, Apx.; 17, p. 477] shows that:

(i) for every $j = 0, \dots, k$ and for every $s \in S$, the restriction map

$$H^0(V_s, (\mathcal{L}_j)_s) \to H^0(E_s, (\mathcal{L}_j)_s)$$

is surjective;

(ii) the linear system $|L_k|_{V_s}|$ of divisors on V_s is base point free; and

(iii) the associated morphism $\varphi_s : V_s \to \mathbb{P}^{h-1} = \mathbb{P}(H^0(V_s, (\mathcal{L}_k)_s)^*)$ (with $h := h^0(V_s, (\mathcal{L}_k)_s)$) yields a contraction of the irreducible components $C_{s,i}$ $(i = 1, \dots, m)$ of the divisor $E_s \subset V_s$ into m distinct smooth points.

For the convenience of readers we sketch this argument. For each $j = 1, \dots, k$, consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_{V_s}(L_{j-1}) \to \mathcal{O}_{V_s}(L_j) \to \mathcal{O}_{E_s}(L_j) \to 0$$

and the corresponding long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(V_s, (\mathcal{L}_{j-1})_s) &\to H^0(V_s, (\mathcal{L}_j)_s) \\ &\to H^0(E_s, (\mathcal{L}_j)_s) \to H^1(V_s, (\mathcal{L}_{j-1})_s) \to \cdots . \end{aligned} \tag{7}$$

It follows from (6) and (7) that, for every $q \geq 1$, the natural homomorphisms

$$H^q(V_s, (\mathcal{L}_0)_s) \to H^q(V_s, (\mathcal{L}_1)_s) \to \cdots \to H^q(V_s, (\mathcal{L}_k)_s)$$

are surjective, and so by (5) all these groups vanish; in particular,

$$H^q(V_s, (\mathcal{L}_{k-1})_s) = H^q(V_s, (\mathcal{L}_k)_s) = 0 \quad \forall q \geq 1,\ \forall s \in S. \tag{8}$$

Now (7) implies (i).

Since the line bundle $L_k|_{V_s \setminus E_s} \simeq L_0|_{V_s \setminus E_s}$ is very ample, the restriction $\varphi_s|_{V_s \setminus E_s}$ gives an embedding. Furthermore, the restriction $L_k|_{E_s}$ is a trivial bundle and so (ii) and (iii) easily follow.

By Lemma 1.1(c), (8) implies that the dimension $h := h^0(V_s, (\mathcal{L}_k)_s)$ is constant on S and that $\xi(L_k) = \left(\bigcup_{s \in S} H^0(V_s, (\mathcal{L}_k)_s) \to S\right)$ is a rank-h vector bundle on S. Thus,

$$\varphi : V \to \mathbb{P}(\xi(L_k)^*), \quad \varphi|_{V_s} = \varphi_s,$$

is clearly a proper morphism onto a closed subvariety $W := \varphi(V)$ of (the total space of) the projective bundle $\mathbb{P}(\xi(L_k)^*)$; actually it consists of contracting the divisor $E \subset V$ onto a smooth subvariety $A \subset W$ étale (and m-sheeted) over S under the projection $\pi' := \mathrm{pr}|_W$, where $\mathrm{pr}\colon \mathbb{P}(\xi(L_k)^*) \to S$ is the standard projection. By [19, Prop. II.7.10(b)], $\mathbb{P}(\xi(L_k)^*)$ is a quasi-projective variety, whence so is W.

The morphism $\varphi\colon V \setminus E \to W \setminus A \hookrightarrow \mathbb{P}^{h-1}$ is an embedding (indeed, so is the morphism given by the line bundle $L_k|_{V\setminus E} \simeq L_0|_{V\setminus E}$). Hence $W \setminus A$ is a smooth variety. To show that the variety W itself is smooth, we proceed locally using local trivializations of the vector bundle $\xi(L_k)$. Fix a point $s \in S$ together with an affine neighborhood S_0 of s in S and an index $i_0 \in \{1, \dots, m\}$. Since by (i) the restriction map $H^0(V_s, (\mathcal{L}_{k-1})_s) \to H^0(E_s, (\mathcal{L}_{k-1})_s)$ is surjective, we can find $n+1$ sections $\xi_{s,0}, \dots, \xi_{s,n} \in H^0(V_s, (\mathcal{L}_{k-1})_s)$ that are linearly independent when restricted to sections of $L_{k-1}|_{C_{s,i_0}} \simeq \mathcal{O}_{\mathbf{P}^n}(1)$. Fix also a section η_0 of the line bundle $[E]$ over V that vanishes on E, another one $\eta_{s,1} \in H^0(V_s, (\mathcal{L}_k)_s)$ that does not vanish on C_{s,i_0}, and a basis $\sigma_{s,j}$ $(j = 1, \dots, h)$ of $H^0(V_s, (\mathcal{L}_k)_s)$. Shrinking the neighborhood S_0 (if necessary) we may assume that $\sigma_{s,j} = \sigma_j|_{V_s}$, where $\sigma_j \in H^0(V_{S_0}, \mathcal{L}_k)$ $(j = 1, \dots, h)$ (see Lemma 1.1(b)) and, for any point $s' \in S_0$, the restrictions $\sigma_j|_{V_{s'}} \in H^0(V_{s'}, (\mathcal{L}_k)_{s'})$ $(j = 1, \dots, h)$ still form a basis. Decomposing the sections $\eta_{s,1}, (\eta_0|_{V_s}) \cdot \xi_{s,l} \in H^0(V_s, (\mathcal{L}_k)_s)$ $(l = 0, \dots, n)$ by the basis $\sigma_{s,j}$ $(j = 1, \dots, h)$, we may extend them to sections, say, $\eta_1, \xi_0, \dots, \xi_n \in H^0(V_{S_0}, \mathcal{L}_k)$ decomposed by the system $\sigma_j \in H^0(V_{S_0}, \mathcal{L}_k)$ $(j = 1, \dots, h)$ with the same coefficients. Then the ratios

$$z_0' := \frac{\xi_0}{\eta_1}, \; \dots, \; z_n' := \frac{\xi_n}{\eta_1} \tag{9}$$

can be pushed down to regular functions (say) $z_0, \dots, z_n$ in a neighborhood of the point $c_{s,i_0} := \varphi(C_{s,i_0}) \in A$ in W that give a local coordinate system on the fiber W_s with center at the point c_{s,i_0} (cf. [26, Apx.; 17, p. 477]). Clearly, they still give a local coordinate system on the fiber $W_{s'}$ around the point $c_{s',i_0} := \varphi(C_{s',i_0}) \in A$ close enough to c_{s,i_0}. Thus if $(x_1, \dots, x_r)$ (with $r := \dim_{\mathbb{C}} S$) is a local coordinate system at the point $s \in S$, then $(x_1, \dots, x_r, z_0, \dots, z_n)$ define a local coordinate system on W with center at the point c_{s,i_0}; the projection π' in these local coordinates is given as

$$(x_1, \dots, x_r, z_0, \dots, z_n) \mapsto (x_1, \dots, x_r).$$

This proves (a).

(b) Let M be an algebraic vector bundle on an algebraic variety Z, and let L, L' be two transversal vector subbundles of M, so that $M/(L \cap L') \simeq L/(L \cap L') \oplus L'/(L \cap L')$. Then we have $M/L \simeq L'/(L \cap L')$. Letting $M := TV|_{E\cap E'}$, $L := TE|_{E\cap E'}$, and $L' = TE'|_{E\cap E'}$, and using our previous observation, we obtain an isomorphism of the normal bundles

$$N_{(E\cap E')/E'} \simeq N_{E/V}. \tag{10}$$

By assumption, for every $s \in S$ and for every pair (i, j) with $C_{s,i} \cap C'_{s,j} \neq \emptyset$, we have $(C_{s,i}, C_{s,i} \cap C'_{s,j}) \simeq (\mathbb{P}^n, \mathbb{P}^{n-1})$ and $N_{C_{s,i}/V_s} \simeq \mathcal{O}_{\mathbb{P}^n}(-1)$, so it follows from

(10) that $N_{(C_{s,i}\cap C'_{s,j})/E'_s} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. This allows us to apply the arguments of (a) (which are valid also for $n = 0$), replacing the pair (V, E) by the pair $(E', E\cap E')$. Hence the restriction $\varphi|_{E'}$ is an S_0-contraction of the divisor $E\cap E'$ on E'. The image $\varphi(E')$ is a divisor in W proper over S_0 that has only smooth branches (in fact, for $n \geq 2$ the image itself is smooth because then, assuming that E' is smooth, for any $i \in \{1, \dots, m\}$ we have $C_{s,i} \cap C'_{s,j} \neq \emptyset$ for at most one value of j). Since for every $s \in S$ the fiber $E'_s := E' \cap V_s$ is a smooth divisor in V_s and $\varphi_s = \varphi|_{V_s} : V_s \to W_s$ is the blowing up with the (finite) smooth center $A_s := A \cap W_s$, it follows that the intersection of local branches of the (reduced) divisor $\varphi(E'_s) \subset W_s$ at any point of A_s is transversal. Therefore, the branches of the divisor $\varphi(E') \subset W$ that contain the center A also transversally meet each other as well as every fiber W_s $(s \in S_0)$. Now the proof is completed. □

2. Combinatorial Constructions

2.1. TERMINOLOGY AND NOTATION. Let $\pi : V \to S$ be a family of quasi-projective varieties. *Shrinking the base* means passing to a new family $\pi|_{\pi^{-1}(U)}$: $\pi^{-1}(U) \to U$, where U is a Zariski open subset of S; usually we keep the same notation before and after shrinking the base.

By a *smooth family of quasi-projective varieties* we mean a smooth surjective morphism $f : X \to S$ of smooth quasi-projective varieties; hereafter the base S is presumed to be irreducible. Note that any quasi-projective family with a smooth total space can be made smooth by shrinking the base.

We say that a family $\bar{f} : V \to S$ is a *relative completion* of $f : X \to S$ if $\bar{f}$ is a proper morphism, $X \subset V$ is a Zariski open dense subset, and $f = \bar{f}|_X$. It is *of simple normal crossing* (or simply SNC) *type* if $D := V \setminus X$ is a simple normal crossing divisor on V. If the family $\bar{f} : V \to S$ is smooth and if each fiber $V_s := \bar{f}^{-1}(s)$, $s \in S$, meets the divisor D transversally along an SNC divisor $D_s := D \cap V_s \subset V_s$, then we say that (V, D) is a *relative SNC completion* of X. Clearly, any smooth relative completion with an SNC divisor D can be reduced to a relative SNC completion by shrinking the base.

Let $f : X \to S$ be a smooth family with all fibers isomorphic to $\mathbb{C}^2$, and let $\bar{f} : V \to S$ be its relative SNC completion. Then, for every point $s \in S$, the “boundary divisor” D_s is a *rational tree* (on the smooth rational projective surface V_s). The latter means that each irreducible component $C_{s,i}$ of D_s is a smooth rational curve, and the *weighted dual graph* (say, Γ_s) of D_s is a tree (see e.g. [40, Sec. 2]).

Let $v \in \Gamma_s$ be an *at most linear* (-1)-*vertex* of Γ_s (i.e., the valence of v is at most 2 and the weight of v is -1). The Castelnuovo contraction of the corresponding irreducible (-1)-component of D_s leads again to an SNC completion (V'_s, D'_s) of $X_s \simeq \mathbb{C}^2$. The dual graph Γ'_s of D'_s is obtained from Γ_s by the *blowing down* v. The inverse operation on graphs is called a *blowing up*. This blowing up (blowing down) is called *inner* if v is a linear vertex of Γ_s and *outer* if v is terminal. The graph Γ_s is called *minimal* if no contraction is possible; in this case, it is linear [36]. All minimal linear graphs corresponding to minimal SNC completions of $\mathbb{C}^2$ are described in [33] and [36]; we call them the *Ramanujam–Morrow graphs*.

Assume that $\bar{f}\colon (V, D) \to S$ as described previously is a proper and smooth SNC family, and fix a base point $s_0 \in S$. There exists a smooth horizontal connection on V that is tangent along the boundary SNC divisor D (indeed, it can be patched from local smooth connections tangent along D using a smooth partition of unity on V). This provides us with a *geometric monodromy representation*

$$\mu\colon \pi_1(S, s_0) \to \mathrm{Diff}(V_{s_0}, D_{s_0}).$$

We denote by the same letter μ the induced *combinatorial monodromy representation* $\pi_1(S, s_0) \to \mathrm{Aut}\,\Gamma_{s_0}$.

For a vertex $v \in \Gamma_{s_0}$, let $O(v)$ be its μ-orbit. Clearly, two vertices $v, v' \in \Gamma_{s_0}$ belong to the same orbit if and only if, for a certain irreducible component $E = E(v)$ of the boundary divisor $D \subset V$, the corresponding irreducible components $C(v)$ and $C(v')$ of the curve $D_{s_0} \subset V_{s_0}$ are contained in $E_{s_0} := E \cap V_{s_0}$ (note that this fact is stable under shrinking the base). The next important lemma follows easily from Theorem 1.3.

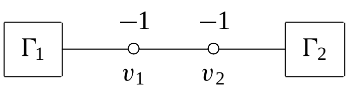
LEMMA 2.1. *In the notation just described, let v be the most linear (-1)-vertex of the graph Γ_{s_0}. Assume that the orbit $O(v) \subset \Gamma_{s_0}$ of v does not contain a pair of neighbors in Γ_{s_0}. Then (possibly after shrinking the base) there is a relative blowing down of the irreducible component $E(v)$ of the divisor D that gives again a relative SNC completion of the family $f\colon X \to S$.*

2.2. EQUIVARIANT CONTRACTIONS. From now on we consider a smooth SNC completion of $\mathbb{C}^2$ by an SNC divisor (say, D_0) with a weighted dual graph Γ. Denote by $O(v)$ the orbit of a vertex v of Γ under the action on Γ of the full automorphism group $\mathrm{Aut}\,\Gamma$. If v is an at most linear (-1)-vertex such that its orbit $O(v)$ does not contain a pair of neighbors in Γ, then all the vertices in $O(v)$ can be simultaneously contracted; we call this an *equivariant contraction* (or an *equivariant blowing down*). The main result of this section is the following proposition.

PROPOSITION 2.2. *A graph Γ as described previously can be contracted to a Ramanujam–Morrow graph by means of equivariant contractions.*

The proof is accomplished via Lemmas 2.3 and 2.4.

LEMMA 2.3. *Let v_1 and v_2 be at most linear (-1)-vertices of Γ that are neighbors and belong to the same orbit (i.e., $v_2 \in O(v_1)$). Draw Γ as follows:*



Then the following statements hold.

(a) *$\alpha(\Gamma_i) = \Gamma_j$ for any $\alpha \in \mathrm{Aut}\,\Gamma$ with $\alpha(v_i) = v_j$, $i, j \in \{1, 2\}$. Moreover, $O(v_1) = \{v_1, v_2\}$, and there is only one pair $\{v_1, v_2\}$ satisfying the assumptions of the lemma.*

(b) *If $w \in \Gamma_1$ is an at most linear (-1)-vertex of Γ, then the orbit $O(w)$ does not contain a pair of neighbors in Γ.*

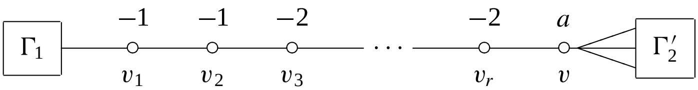
Proof. (a) Denote by $\mathrm{Br}_l(v_i)$ (resp., $\mathrm{Br}_r(v_i)$) the left-hand (resp., right-hand) branch of Γ at v_i; thus $\mathrm{Br}_l(v_1) = \Gamma_1$ and $\mathrm{Br}_r(v_2) = \Gamma_2$. Since $\operatorname{card}\mathrm{Br}_l(v_2) = \operatorname{card}\mathrm{Br}_l(v_1) + 1$, for any $\alpha \in \operatorname{Aut}\Gamma$ with $\alpha(v_1) = v_2$ we have $\alpha(\mathrm{Br}_l(v_1)) = \mathrm{Br}_r(v_2)$, whence $\alpha(\Gamma_1) = \Gamma_2$, $\alpha(v_2) = v_1$, and $\alpha(\Gamma_2) = \Gamma_1$.

In particular, $\operatorname{card}\Gamma_1 = \operatorname{card}\Gamma_2$. Therefore, if $\beta \in \operatorname{Aut}\Gamma$ is such that $\beta(v_1) = v_1$ then $\beta(\Gamma_1) = \Gamma_1$, whence $\beta(v_2) = v_2$ and $\beta(\Gamma_2) = \Gamma_2$. This proves the first statement of (a).

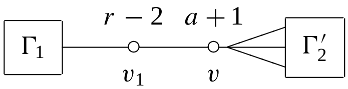
Let (v_1', v_2') be another pair of at most linear (-1)-neighbors of Γ that belong to the same orbit. As we have seen, both the edges $[v_1, v_2]$ and $[v_1', v_2']$ of Γ divide Γ into two parts of equal cardinality, which is only possible if $[v_1, v_2] = [v_1', v_2']$. Thus the pair (v_1, v_2) is unique. It follows that $O(v_1) = \{v_1, v_2\}$, which proves (a). It also follows that $O(w) \cap O(v_1) = \emptyset$, which proves (b). $\square$

LEMMA 2.4. *Suppose that the graph Γ is not minimal. Then Γ has an at most linear (-1)-vertex w such that the orbit $O(w)$ does not contain a pair of neighbors.*

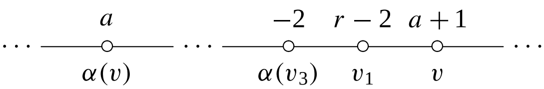
Proof. Let (v_1, v_2) be a pair of at most linear (-1)-neighbors of Γ that belong to the same orbit. Clearly, $\Gamma \neq \{v_1, v_2\}$ and so Γ contains a fragment



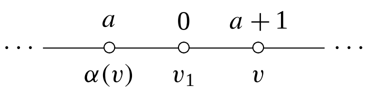
where either $a \neq -2$ or v is a branch vertex of Γ. Contracting the chain $(v_2, v_3, \dots, v_r)$, we obtain the graph



where the vertex v cannot be further contracted. Assume that Γ does not contain at most linear (-1)-vertices other than v_1 and v_2. Then, after this contraction, the resulting graph is minimal (i.e., a Ramanujam–Morrow graph). We show that this is impossible. Indeed, otherwise Γ would be a linear graph admitting an automorphism $\alpha \in \operatorname{Aut}\Gamma$ that interchanges v_1 and v_2 (resp., Γ_1 and Γ_2). After the contraction as before, the image of Γ would contain one of the following fragments:

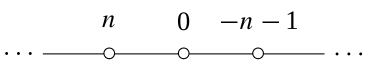


$(r \geq 3)$,



$(r = 2)$.

But a Ramanujam–Morrow graph can have only one positively weighted vertex, and a neighbor of this vertex has zero weight. Hence $a = -1$ in the first fragment just displayed, which contradicts the minimality assumption. Furthermore, the only fragments of a Ramanujam–Morrow graph of length 3 including a zero vertex in the middle are of the form



where $n > 0$ [11, Sec. 3.5; 33; 36]. Thus, neither fragment is possible. By virtue of Lemma 2.3, this concludes the proof. □

Proposition 2.2 and Lemma 2.1 yield the following.

COROLLARY 2.5. *As before, let $f: X \to S$ be a smooth family of quasi-projective varieties with all fibers isomorphic to $\mathbb{C}^2$, and let $\bar{f}: V \to S$ be its relative SNC completion. Then the boundary divisor $D = V \setminus X$ (possibly after shrinking the base) can be contracted providing a new relative SNC completion $\bar{f}^{\min}: V^{\min} \to S$, where for each $s \in S$ the dual graph $\Gamma_s^{\min}$ of the boundary divisor $D_s^{\min} := V_s^{\min} \setminus X_s$ is a Ramanujam–Morrow graph.*

We shall need the following lemma from [11].

LEMMA 2.6 [11, Lemma 3.7]. *Let Γ be a Ramanujam–Morrow graph. Then Γ can be transformed, by a sequence of inner blowing ups and blowing downs, into one of the following graphs:*

$$\overset{1}{\circ} \qquad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{n}{\circ} \quad (n \neq -1) \qquad \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{k-1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\circ} \quad (k \geq 1). \tag{$*$}$$

3. Proof of Theorem 0.1

The next proposition is the key point in the proof of Theorem 0.1.

PROPOSITION 3.1. *Let the assumptions of Corollary 2.5 be fulfilled. Then the family $f: X \to S$ (possibly after shrinking the base) admits a relative SNC completion $\tilde{f}: \tilde{V} \to S$ such that, for every $s \in S$, the boundary divisor $\tilde{D}_s := \tilde{V}_s \setminus X_s$ is irreducible (and so is isomorphic to $\mathbb{P}^1$).*

Proof. Let a relative SNC completion $(V^{\min}, D^{\min})$ be as in Corollary 2.5. Note that, for any Ramanujam–Morrow graph Γ except the following,

$$\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}$$

we have $\operatorname{Aut}\Gamma = \{\mathrm{id}\}$. Let us deal with this exceptional case first. The edge of this graph (invariant under automorphisms) corresponds to a section (say, Σ) of $D^{\min}$ over S (here Σ is just the set of double points of the divisor $D^{\min}$). We can blow up $V^{\min}$ along Σ and then (possibly after shrinking the base) blow down (according to Lemma 2.1) the proper transform(s) of (the irreducible components of) D to arrive at a new relative SNC completion with only irreducible boundary divisors in fibers, as required.

In all other cases, the absence of nontrivial automorphisms implies that:

1. each irreducible component E_i of the boundary divisor $D^{\min}$ meets every fiber $V_s^{\min}$ along an irreducible curve $C_{s,i}$; and
2. the intersection (if non-empty) of two such components $\Sigma_{ij} := E_i \cap E_j$ $(i \neq j)$ is a (smooth) section.

These two properties are stable under blowing up with center at a section that is the intersection of two components of the boundary divisor, as well as under blowing down of a component of the boundary divisor that corresponds to an at most linear (-1)-vertex (it is defined correctly over S by virtue of Lemma 2.1).

The preceding observation and Lemma 2.6 imply that a relative SNC completion $(V^{\min}, D^{\min})$ can be transformed (possibly after shrinking the base) into another one with the dual graph Γ_{s_0} as in $(*)$. If we finally arrive at a relative SNC completion with the dual graph $\Gamma = \Gamma_{s_0}$ as in the third case of $(*)$ and then blow down the (-1)-curve in every fiber, we obtain a relative SNC completion with the dual graph as in the second case of $(*)$. In particular, we may assume that the singular locus Σ of the boundary divisor D is a section.

If $n = 0$ then we deal with the exceptional case, which is already settled.

If $n > 0$ then we can proceed as before, performing first an inner relative blowing up with center at Σ and then an outer relative blowing down. After a sequence of n such "elementary transformations" we obtain a relative SNC completion with $n = 0$ and so can finish the proof as before.

Finally, consider the case where $n \le -2$. In this case we have $\operatorname{Aut}\Gamma = \{\mathrm{id}\}$, so the combinatorial monodromy of the family $\bar f \colon D \to S$ is trivial. Hence the divisor D consists of two smooth irreducible components (say, C_0 and C_1) with $C_{s,0}^2 = 0$ and $C_{s,1}^2 = n \le -2$ for every $s \in S$.

Suppose that there exists a section Σ' of $\bar f|_{C_0} \colon C_0 \to S$ disjoint with $\Sigma := C_0 \cap C_1$. The kinds of elementary transformations appropriate in our case are blowing up with center at Σ' and then blowing down the proper transform of C_0 (by Theorem 1.3, this is possible after shrinking the base). Performing n such elementary transformations (which requires, at each step, the existence of a section as before), we arrive again at a relative SNC completion of the second type with $n = 0$, and so we are done. Thus it only remains to prove the following statement.

CLAIM. *After shrinking the base appropriately, one can find a section Σ' of $\bar f|_{C_0} \colon C_0 \to S$ disjoint with $\Sigma := C_0 \cap C_1$.*

Proof. Letting in Lemma 1.1 $E = C_0$ and $L = [C_1]|_E$ (so that $L|_{E_s} = L|_{C_{s,0}} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ for every $s \in S$) and shrinking the base S to make it affine, by Corollary 1.2(b) we conclude that, for every point $s \in S$, the restriction map

$$H^0(E, \mathcal{L}) \to H^0(E_s, \mathcal{L}_s) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

is surjective. Thus for any points $s_0 \in S$ and $z_0 \in C_{s_0,0} \setminus C_{s_0,1}$ there exists a section $\sigma \in H^0(E, \mathcal{L})$ with $\sigma^*(0) \cdot C_{s_0,0} = z_0$. The divisor $\Sigma' := \sigma^*(0)$ on $E = C_0$ (linearly equivalent to Σ) passes through the point z_0 and meets every fiber $E_s = C_{s,0}$ $(s \in S)$ transversally at one point. Clearly, $Z := \bar f(\Sigma \cap \Sigma') \not\ni s_0$ is a Zariski closed proper subset of the base S. The restrictions of the sections Σ and Σ' onto the Zariski open subset $S_0 := S \setminus Z$ of S are disjoint, as required. This proves the claim. $\square$

Now the proof of Proposition 3.1 is completed. $\square$

Given Corollary 2.5 and Proposition 3.1, the proof of Theorem 0.1 is reduced to the following simple lemma. It is well known that any smooth family with fibers isomorphic to a projective space is locally trivial in the étale topology (and so is a smooth Severi–Brauer variety) [18, Thm. I.8.2]. It is locally trivial in the Zariski topology if and only if this family (or, equivalently, its dual) admits local sections; Lemma 3.2 provides a proof along the lines in [18, II, Sec. 0].

LEMMA 3.2. *Let $\bar{f}: V \to S$ be a proper smooth family over a quasi-projective base with all fibers isomorphic to $\mathbb{P}^n$, and let D be an irreducible smooth divisor on V that meets every fiber V_s $(s \in S)$ transversally, with $D_s \simeq \mathbb{P}^{n-1}$ and $N_{D_s/V_s} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. Then the family (V, D) is locally trivial in the Zariski topology.*

Proof. Fix an arbitrary point $s_0 \in S$. Shrinking the base S to an affine neighborhood of the point s_0 and letting (in Lemma 1.1) $E = V$ and $L = [D]$, by Corollary 1.2 we will have that the restriction map

$$H^0(E, \mathcal{L}) \to H^0(E_{s_0}, \mathcal{L}_{s_0}) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$$

is surjective. Fix sections $\sigma_0, \dots, \sigma_n \in H^0(E, \mathcal{L})$ such that their restrictions to the fiber E_{s_0} are linearly independent and $\sigma_0^*(0) = D$. Shrinking the base further, we may suppose that, for every fiber E_s $(s \in S)$, the restrictions $\sigma_0|_{E_s}, \dots, \sigma_n|_{E_s}$ are linearly independent as well. Then the morphism

$$\varphi: V \to S \times \mathbb{P}^n, \quad z \longmapsto \big(\bar{f}(z), (\sigma_0(z) : \dots : \sigma_n(z))\big),$$

yields a desired trivialization over S. □

Added in proof: After this paper appeared as an MPI-preprint, we were kindly informed by P. Russell that he had found a different proof of our main result.

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