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ON GROUP INDUCED ORDERINGS, MONOTONE FUNCTIONS, AND CONVOLUTION THEOREMS[1]

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Orderings defined by compact groups of linear transformations acting on vector spaces are studied. In some cases, these orderings induce orderings on convex cones similar to those defined by reflection groups. In these cases the monotone functions can be conveniently characterized. Convolution theorems for monotone functions are discussed.

1. Introduction. Majorization, as defined by Hardy, Littlewood and Pólya (1934), has been an extremely important notion in the theory and applications of many types of inequalities. The recent work of Marshall and Olkin (1979) contains an extensive discussion of majorization and its application to many branches of mathematics including probability and statistics. Although not essential for understanding this paper, the reader may find it useful to glance through Part I of Marshall and Olkin (1979).

To motivate the situation to be considered here, first recall the permutation group definition of majorization (see Rado (1952)). Let $\mathcal{P}_n$ be the group of $n \times n$ permutation matrices acting on $\mathcal{R}^n$. For $x,y \in \mathcal{R}^n$, x is majorized by y (written as $x \leqslant y$) means that x is in the convex hull of the set $\{gy|g\in\mathcal{P}_n\}$ (the $\mathcal{P}_n$-orbit of y). A careful study of the pre-order $\leqslant$ (using the terminology in Marshall and Olkin (1979), p. 13) has resulted in a useful and important characterization of the real valued functions f which are decreasing or increasing in the pre-order of majorization (see Schur (1923), Ostrowski (1952)). A recent result of Marshall and Olkin (1974), which has had applications in probability and statistics, shows that the convolution of two decreasing (in the pre-order of majorization) functions is again a decreasing function.

In this paper, we begin a systematic study of pre-orderings defined on vector spaces which arise in much the same way that majorization arises. Let G be any closed group of $n \times n$ orthogonal matrices. Using G, rather than $\mathcal{P}_n$, define a pre-order on $\mathcal{R}^n$ as follows: $x \leqslant y$ iff x is in the convex hull of $\{gy|g\in G\}$. The examples in the next section show that there are a number of groups G which give useful and interesting orderings. Based on the known majorization results, it seems rather natural to ask for conditions on G for which (i) it is possible to characterize the class of decreasing real valued functions on $\mathcal{R}^n$. (ii) the convolution result of Marshall and Olkin (1974) continues to hold.

This paper is mainly concerned with (i), but (ii) is discussed rather incompletely. Here is a brief outline of the paper. In Section 2, group induced orderings are defined on inner product spaces. The geometry which prevails in the permutation group case is described and is shown to hold in a number of interesting cases. It is this geometry which is used in Section 3 to give a characterization of the decreasing functions. The results of Marshall, Walkup and Wets (1967) on cone orderings are used extensively in Section 3. In Section

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4, the convolution type results are discussed with special attention being given to a necessary condition for the Marshall and Olkin (1974) result to hold. It is shown that this necessary condition does not hold for any finite rotation group acting on $\mathcal{R}^2$.

2. Group Induced Orderings. To set notation, let $(V,(\cdot,\cdot))$ be a finite dimensional real inner product space and let G be a closed subgroup of the group of orthogonal transformations $\mathcal{O}(V)$, of $(V,(\cdot,\cdot))$. The compact group G defines a pre-order on V as follows:

(2.1) For $x,y \in V$, write $x \leq y$ to mean x is in the convex hull of $\{gy|g\in G\}$.

Thus, $x \leq y$ means that x is in the convex hull of the G-orbit of y. To say $\leq$ is a pre-order means that for all $x,y,z\in V$, (i) $x \leq y$ and $y \leq z$ implies $x \leq z$ and (ii) $x \leq x$.

These two conditions are easily checked. The dependence of $\leq$ on G will usually be supressed as G will remain fixed through much of our discussion.

Definition 1. *A function $f:V \to \mathcal{R}^1$ is decreasing (increasing) if $x \leq y$ implies $f(x) \geq f(y)$ ($f(x) \leq f(y)$). A set $B \subseteq V$ is called* monotone *if the indicator of B, say I_B, is decreasing.*

Our first task is to give an analytic rather than geometric description of $\leq$. To this end, recall the following.

PROPOSITION 1. *Let A be a non-empty subset of V and let C be the closed convex set generated by A. Then, $x\in C$ iff. for all $u\in V$,*

$$(u,x) \leq \sup_{z\in A}(u,z). \tag{2.2}$$

Proof. Without loss of generality, $C \neq V$ since otherwise the right hand side of (2.2) is $+\infty$ and the assertion is trivial. If $x = \Sigma\, \alpha_i z_i$ with $z_i\in A$, $0\leq\alpha_i\leq 1$, and $\Sigma\alpha_i = 1$, then $(u,x) = \Sigma\alpha_i(u,z_i) \leq \sup_{z\in A}(u,z)$ so (2.2) holds for convex combinations of elements of A. But, every point in C is the limit of such convex combinations so continuity implies that (2.2) holds for all $x\in C$. Conversely, assume (2.2) holds and write C as the intersection of all the closed half spaces which contains it—say $C = \bigcap_\alpha H_\alpha$ where $H_\alpha = \{y|(h_\alpha, y) \leq k_\alpha\}$ with $\|h_\alpha\| = 1$ and $k_\alpha\in\mathcal{R}^1$. Since $A \subseteq C \subseteq H_\alpha$ for all α, we have $(h_\alpha, z) \leq k_\alpha$ for all $z\in A$. If x satisfies (2.2), then $(h_\alpha, x) \leq \sup_{z\in A}(h_\alpha, z) \leq k_\alpha$ so $x\in H_\alpha$ for all α. Hence $x\in \bigcap_\alpha H_\alpha = C$. □

Given $y\in V$, let $C(y)$ denote the convex hull of $\{gy|g\in G\}$. The compactness of G implies $C(y)$ is compact. Since $x \leq y$ means $x \in C(y)$, Proposition 1 with $A = \{gy|g\in G\}$ shows that $x \leq y$ iff for all $u\in V$

$$(u,x) \leq \sup_{g\in G}(u,gy). \tag{2.3}$$

For $u,y\in V$, consider

$$m(u,y) \equiv \sup_{g\in G}(u,gy). \tag{2.4}$$

defined on $V \times V$. The following properties of m are easily verified.

$$\begin{aligned} &\text{(i)}\ \ m(c_1u,c_2y) = c_1c_2m(u,y) \text{ for } c_1,c_2 \geq 0\\ &\text{(ii)}\ \ m(g_1u,g_2y) = m(u,y) \qquad \text{for } g_1,g_2\in G\\ &\text{(iii)}\ \ m(u,y) = m(y,u)\\ &\text{(iv)}\ \ m(u,\cdot) \text{ is convex on } V \end{aligned} \tag{2.5}$$

That the pre-order $\leq$ is completely determined by m is the content of

PROPOSITION 2. *For $x,y \in V$, $x \leq y$ iff.*

$$m(u,x) \leq m(u,y) \quad \textit{for all } u\in V. \tag{2.6}$$

Proof. If $x \leqslant y$, then (2.3) shows that for all $u \in V$,

$$(u,x) \leqslant m(u,y). \tag{2.7}$$

Since $m(gu,y) = m(u,y)$, (2.7) implies that for $g \in G$,

$$(gu,x) \leqslant m(u,y), \tag{2.8}$$

so $m(u,x) = \sup_{g \in G}(gu,x) \leqslant m(u,y)$. If (2.6) holds, the inequality $(u,x) \leqslant m(u,x)$ together with (2.3) shows that $x \leqslant y$. □

In a number of important examples, Proposition 2 can be used to provide a useful analytic characterization of $\leqslant$. First, (2.5) (ii) shows that m is determined by its values on the quotient space V/G. In other words, m is a function of a maximal invariant under the action of G on V. Let τ be such a maximal invariant (see Lehmann (1959), Ch. 6). *Assume* that $\tau(x) \in \{gx | g \in G\}$, and let $\mathcal{T} \subseteq V$ be the range of τ. Thus, $\tau(x) = \tau(gx)$ for all $x \in V$, $g \in G$ and $\tau(x_1) = \tau(x_2)$ implies that $x_1 = gx_2$ for some $g \in G$. From (2.5) (ii), we see that

$$m(u,y) = m(\tau(u), \tau(y)) \tag{2.9}$$

for all $u, y \in V$. This implies

PROPOSITION 3. *For $x, y \in V$, $x \leqslant y$ iff $m(\tau(u), \tau(x)) \leqslant m(\tau(u), \tau(y))$ for all $u \in V$.*

Proof: This is immediate from Proposition 2 and (2.9). □

For all of the interesting examples that I know, there is a natural choice for τ which results in $\mathcal{T}$ being a convex cone (such $\mathcal{T}$'s are often called fundamental regions—see Benson and Grove (1971), p. 27). The following assumption is to hold for the remainder of this section:

(A.1) The maximal invariant τ has a range $\mathcal{T} \subseteq V$ which is a convex cone, and $\tau(x) \in \{gx | g \in G\}$.

The key to analyzing a number of important examples is being able to calculate the restriction of m to $\mathcal{T}$. Many of these examples are special cases of the following result.

PROPOSITION 4. *For $\beta, \gamma \in \mathcal{T}$, suppose that $m(\beta, \gamma) = (\beta, \gamma)$—that is, assume m restricted to $\mathcal{T} \times \mathcal{T}$ is just the inner product on $V \times V$. Then $x \leqslant y$ iff*

$$(\beta, \tau(x)) \leqslant (\beta, \tau(y)) \quad \text{for all } \beta \in \mathcal{T} \tag{2.10}$$

Proof. This is an immediate consequence of Proposition 3 and the assumption that m restricted to $\mathcal{T} \times \mathcal{T}$ is just the inner product. □

Recall that a subset T of $\mathcal{T}$ *spans* $\mathcal{T}$ *positively* if every element of $\mathcal{T}$ can be written as a positive linear combination of a finite number of elements of T. Further, T is called a *frame* if T spans $\mathcal{T}$ positively, but no proper subset of T does. The following result is clear.

COROLLARY 1. *Under the assumption of Proposition 4, if T spans $\mathcal{T}$ positively, then $x \leqslant y$ iff.*

$$(t, \tau(x)) \leqslant (t, \tau(y)) \quad \textit{for all } t \in T. \tag{2.11}$$

Before discussing a characterization of the decreasing functions, we first introduce the examples alluded to above. At this point it is appropriate to mention the recent work of Jensen (1984) whose examples coincide with some here. Jensen considers orderings (sometimes pre-orders, lattice orders, etc.) on a set (corresponding to our $\mathcal{T}$) and then lifts the ordering to the whole space via an invariance requirement. Aside from applications, Jensen's main concern is the effect of the lifting but he does not attempt to identify general situations where the lifted ordering is equivalent to the type of group induced ordering discussed above. However, the overlap of Jensen's and our examples show that closely related ideas generated the two works. The important special case treated by Proposition 4 and Corollary 1 is not discussed in Jensen. Under a rather weak assumption, this case leads to a complete description of the G-decreasing functions (see Section 3).

The first three examples here are also discussed briefly in Jensen (1984).

Example 2.1. Take $V = \mathcal{R}^n$ with the usual inner product and let $\mathcal{D}_n$ be the group of coordinate sign changes. Elements of $\mathcal{D}_n$ can be represented as $n \times n$ diagonal matrices whose diagonal elements are ± 1. Let $\mathcal{F} = \{x | x_i \geq 0, i = 1, \ldots, n\}$ so a frame for $\mathcal{F}$ is $T = \{\epsilon_1, \ldots, \epsilon_n\}$ where ϵ_i is the i-th unit vector in $\mathcal{R}^n$. A convenient choice for τ is $(\tau(x))_i = |x_i|$, $i = 1, \ldots n$; $\tau(x)$ is the vector of absolute values of the coordinates of $x \in \mathcal{R}^n$. For $\beta, \gamma \in \mathcal{F}$, $m(\beta, \gamma) = \sup_{g \in G} \beta' g \gamma = \beta' \gamma$. The definition of m gives the first equality while the second follows from the non-negativity of the coordinates of β and γ. Thus Corollary 1 is applicable and yields $x \leqslant y$ iff $\epsilon_i' \tau(x) \leqslant \epsilon_i' \tau(y)$, $i = 1, \ldots, n$ which is equivalent to $|x_i| \leqslant |y_i|$, $i = 1, \ldots, n$.

Example 2.2. Again take $V = \mathcal{R}^n$ with the usual inner product and take G to be the group $\mathcal{P}_n$ of permutations acting on $\mathcal{R}^n$. Let

$$\mathcal{F} = \{x | x_1 \geqslant \ldots \geqslant x_n, \mathbf{x} \in \mathcal{R}^n\}$$

and let e_i be the vector whose first i coordinates are 1 and the rest of the coordinates are $0, i = 1, \ldots, n$. It is not hard to show that $T = \{e_1, \ldots, e_n, -e_n\}$ is a frame for $\mathcal{F}$. A classical rearrangement result due to Hardy, Littlewood and Pólya (1952, p. 261) shows that $m(\beta, \gamma) = \beta' \gamma$ for $\beta, \gamma \in \mathcal{F}$. Let $\tau(x)$ be the vector of the ordered values of x so $\tau(x) \in \mathcal{F}$. These ordered values are denoted by $x_{(i)}$, $i = 1, \ldots, n$ so $x_{(1)} \geqslant x_{(2)} \geqslant \ldots \geqslant x_{(n)}$. A direct application of Corollary 1 shows that $x \leqslant y$ iff $e_i' \tau(x) \leqslant e_i' \tau(y)$, $i = 1, \ldots, n-1$, and $e_n' \tau(x) = e_n' \tau(y)$. Thus, $x \leqslant y$ iff.

$$(*) \qquad \Sigma_1^k x_{(i)} \leqslant \Sigma_1^k y_{(i)}, k = 1, \ldots, n-1 \text{ and } \Sigma_1^n x_i = \Sigma_1^n y_i.$$

Of course, this is the traditional ordering of majorization discussed at length in Marshall and Olkin (1979). For this example, that (*) is equivalent to saying x is in the convex hull of $\{gy | g \in \mathcal{P}_n\}$ was observed by Rado (1952).

Example 2.3. We use the notation established in Examples 2.1 and 2.2. Take $V = \mathcal{R}^n$ and take G to be the group generated by $\mathcal{D}_n \bigcup \mathcal{P}_n$. Take $\mathcal{F}$ to be

$$\mathcal{F} = \{\mathbf{x} | x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0\}$$

and note that $T = \{e_1, \ldots, e_n\}$ is a frame for $\mathcal{F}$. For $x \in \mathcal{R}^n$, let $|x|_{(i)}$ denote the i-th largest value of $\{|x_j|, j = 1, \ldots, n\}$, and let $\tau(x) \in \mathcal{F}$ be the vector with i-th coordinate $|x|_{(i)}$. Then τ is a maximal invariant for this example. Combining the results of Examples 2.1 and 2.2 shows that $m(\beta, \gamma) = \beta' \gamma$ for $\beta, \gamma \in \mathcal{F}$. Corollary 1 shows that $x \leqslant y$ iff $e'_i \tau(x) \leq e'_i \tau(y)$ for $i = 1, \ldots, n$ which is equivalent to $\Sigma_1^k |x|_{(i)} \leqslant \Sigma_1^k |y|_{(i)}$, $k = 1, \ldots, n$. This is usually called the sub-majorization ordering although terminology is not consistent in this case (see Marshall and Olkin (1979)).

Before discussing the next three examples, some notation is required. Given a real symmetric $p \times p$ matrix $\mathbf{x}$ let $\mu_1(\mathbf{x}) \geqslant \ldots \geqslant \mu_p(\mathbf{x})$ denote the p ordered eigenvalues of $\mathbf{x}$. Given an $n \times p$ real matrix $\mathbf{x}$, let $\lambda_1(\mathbf{x}) \geqslant \ldots \geqslant \lambda_p(\mathbf{x}) \geqslant 0$ denote the singular values of $\mathbf{x}$ (if $n < p$, then necessarily the last $p - n$ singular values are zero). Thus, $\lambda_i(\mathbf{x}) = (\mu_i(\mathbf{x}'\mathbf{x}))^{1/2}$ where $\mathbf{x}'$ is the transpose of $\mathbf{x}$. A useful result due to von Neumann (1937) and Fan (1951) is

THEOREM 1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be real* $n \times k$ *matrices. Then*

$$\sup_{\Gamma,\Delta} \operatorname{tr} \Gamma \mathbf{A} \Delta \mathbf{B}' = \Sigma_1^k \lambda_i(\mathbf{A}) \lambda_i(\mathbf{B})$$

where the sup is over all $\Gamma \in \mathcal{O}_n$ *and* $\Delta \in \mathcal{O}_k$.

A discussion of and variations on this Theorem can be found in Marshall and Olkin (1979, p. 514).

Examples 2.4. In this example, $V = \mathcal{S}_p$—the vector space of all $p \times p$ real symmetric matrices and the inner product is

$$(\mathbf{x}_1, \mathbf{x}_2) = tr\, \mathbf{x}_1 \mathbf{x}_2$$

where *tr* denotes trace. The group G is the group of $p \times p$ orthogonal matrices, $\mathcal{O}_p$, and the group action is $\mathbf{x} \to \Gamma\ \mathbf{x}\ \Gamma'$ for $\Gamma \in \mathcal{O}_p$. Clearly G is a subgroup of $\mathcal{O}(V)$ for this example. Let

$$\mathcal{T} = \{\mathbf{x} | x_{11} \geq \ldots \geq x_{pp}, x_{ij} = 0 \quad \text{for } i \neq j\}$$

where x_{ij} is the i,j element of $\mathbf{x}$. A convenient choice for τ is to let $\tau(\mathbf{x})$ be the diagonal matrix in $\mathcal{S}_p$ with diagonal elements $(t(\mathbf{x}))_{ii} = \mu_i(\mathbf{x})$, $i = 1, \ldots, p$. A frame for $\mathcal{T}$ can be constructed as follows. Let $\mathbf{t}_i \in \mathcal{S}_p$ have the first i diagonal elements equal to one and all the remaining elements of $\mathbf{t}_i$ equal to zero, $i = 1, \ldots, p$. Also, let $\mathbf{t}_{p+1} = -\mathbf{t}_p$. That T $= \{\mathbf{t}_1, \ldots, \mathbf{t}_p, \mathbf{t}_{p+1}\}$ is a frame for $\mathcal{T}$ follows easily (see Example 2.2). We now claim that $m(\mathbf{u},\mathbf{y}) = (\mathbf{u},\mathbf{y})$ for $\mathbf{u},\mathbf{y} \in \mathcal{T}$. To see this, first choose δ large enough so that $\mathbf{u} + \delta\mathbf{I}$ and $\mathbf{y} + \delta\mathbf{I}$ have positive diagonal elements. Then

$$(*) \quad m(\mathbf{u},\mathbf{y}) = \sup_\Gamma tr\, \mathbf{u}\Gamma\mathbf{y}\Gamma' = \sup_\Gamma \{tr(\mathbf{u}+\delta\mathbf{I})\Gamma(\mathbf{y}+\delta\mathbf{I})\Gamma'\} - \delta tr(\mathbf{y}) - \delta tr(\mathbf{x}) + \delta^2\mathbf{p}$$
$$= m(\mathbf{u}+\delta\mathbf{I}, \mathbf{y}+\delta\mathbf{I}) - \delta tr(\mathbf{y}) - \delta tr(\mathbf{u}) + \delta^2\mathbf{p}.$$

Since $\mathbf{u}+\delta\mathbf{I}$ and $\mathbf{y}+\delta\mathbf{I}$ have positive diagonals and are in $\mathcal{T}$,

$$\lambda_i(\mathbf{u}+\delta\mathbf{I}) = \mu_i(\mathbf{u}+\delta\mathbf{I}), i = 1, \ldots, p$$

and the same holds for $\mathbf{y}$ in place of $\mathbf{u}$. With $n = k = p$, $\mathbf{A} = \mathbf{u}+\delta\mathbf{I}$ and $\mathbf{B} = \mathbf{y}+\delta\mathbf{I}$. Theorem 1 implies that

$$m(\mathbf{u}+\delta\mathbf{I}, \mathbf{y}+\delta\mathbf{I}) \leq \Sigma_1^p \mu_i(\mathbf{u}+\delta\mathbf{I}) \mu_i(\mathbf{y}+\delta\mathbf{I}).$$

Since $\mu_i(\mathbf{u}+\delta\mathbf{I}) = u_{ii} + \delta$ and $\mu_i(\mathbf{y}+\delta\mathbf{I}) = y_{ii} + \delta$, there is obviously equality in the above inequality (just take $\Gamma = I$ in the definition of m). Substituting this into (*) and a bit of algebra show that $m(\mathbf{u},\mathbf{y}) = tr(\mathbf{uy}) = (\mathbf{u},\mathbf{y})$. Hence Corollary 1 is applicable and yields that $\mathbf{x} \leq \mathbf{y}$ iff

$$\Sigma_1^k \mu_i(\mathbf{x}) \leq \Sigma_1^k \mu_i(\mathbf{y}), k = 1, \ldots, p-1 \text{ and } \Sigma_1^p \mu_i(\mathbf{x}) = \Sigma_1^p \mu_i(\mathbf{y}).$$

In other words, $\mathbf{x} \leq \mathbf{y}$ iff the vector of eigenvalues of $\mathbf{x}$ is majorized by the vector of eigenvalues of $\mathbf{y}$. This result was established by Karlin and Rinott (1981) using a different argument.

Example 2.5. For this example, V is the vector space $\mathcal{L}_{p,n}$ of all $n \times p$ real matrices with inner product $(\mathbf{x}_1, \mathbf{x}_2) = tr(\mathbf{x}_1\mathbf{x}_2')$. For notational simplicity, it is assumed $n \geq p$; the contrary case is handled by a similar argument. The group G is $\mathcal{O}_n \times \mathcal{O}_p$ which acts on $\mathcal{L}_{p,n}$ by $\mathbf{x} \to \Gamma\mathbf{x}\Delta'$ for $\Gamma \in \mathcal{O}_n$ and $\Delta \in \mathcal{O}_p$. The convex cone $\mathcal{T}$ is $\mathcal{T} = \{\mathbf{x} | x_{11} \geq \ldots \geq x_{pp} \geq 0, x_{ij} = 0$ for all $i \neq j)$ where x_{ij} is the i,j element of $\mathbf{x}$. The maximal invariant τ is defined to be: $\mathbf{u} = \tau(\mathbf{y})$ is the element of $\mathcal{T}$ with $u_{ii} = \lambda_i(y)$, $i = 1, \ldots, p$. That τ is a maximal invariant is a consequence of the singular value decomposition theorem (Eckart and Young, 1939). To evaluate m on $\mathcal{T}$, consider $\mathbf{u},\mathbf{y} \in \mathcal{T}$. Then

$$m(\mathbf{u},\mathbf{y}) = \sup_{\Gamma,\Delta} tr(\mathbf{u}\Delta\mathbf{y}'\Gamma') = \Sigma_1^p \lambda_1(\mathbf{u}) \lambda_i(\mathbf{y})$$

by Theorem 1. Since $\mathbf{u},\mathbf{y} \in \mathcal{T}$, it follows that $\lambda_i(\mathbf{u}) = u_{ii}$ and $\lambda_i(\mathbf{y}) = y_{ii}$. Thus, for

$\mathbf{u},\mathbf{y} \in \mathcal{T}$, $m(\mathbf{u},\mathbf{y}) = \Sigma_1^p \mu_{ii} y_{ii} = (\mathbf{u},\mathbf{y})$. To apply Corollary 1, we first need a frame of $\mathcal{T}$. Let $\mathbf{t}_i \in \mathcal{T}$ have its first i diagonal elements equal to one and all other elements equal to zero. Then it is easy to see that $\mathrm{T} = \{\mathbf{t}_1, \ldots, \mathbf{t}_p\}$ is a frame for $\mathcal{T}$. A direct application of Corollary 1 shows that $\mathbf{x} \leq \mathbf{z}$ iff.

$$\Sigma_1^k \lambda_i(\mathbf{x}) \leq \Sigma_1^k \lambda_i(\mathbf{z}), k = 1, \ldots, p.$$

In other words, $\mathbf{x} \leq \mathbf{z}$ iff the vector of singular values of $\mathbf{x}$ is submajorized by the vector of singular values of z (see Example 2.3 for a discussion of submajorization).

Example 2.6. As in Example 2.5, take V to be $\mathcal{L}_{p,n}$ with the inner product $(\mathbf{x}_1, \mathbf{x}_2) = tr(\mathbf{x}_1\mathbf{x}_2')$ and again assume for convenience that $n \geq p$. Consider the group $G = \mathcal{O}_n$ which acts on $\mathcal{L}_{p,n}$ by $\mathbf{x} \to \Gamma\mathbf{x}$. Let $\mathcal{S}_p^+$ denote the convex cone of positive semi-definite $p \times p$ real matrices and for $\mathbf{s} \in S_p^+$ let $\mathbf{s}^{1/2}$ denote the unique element in S_p^+ which satisfies $\mathbf{s}^{1/2}\mathbf{s}^{1/2} = \mathbf{s}$. For this example, let $\mathcal{T}$ be $\mathcal{T} = \{\mathbf{x}|\mathbf{x} = \binom{\mathbf{s}}{0}, \mathbf{s} \in \mathcal{S}_p^+\}$ and set $\tau(\mathbf{x}) = \binom{(\mathbf{x}'\mathbf{x})^{1/2}}{0} \in \mathcal{T}$. That $\tau(\mathbf{x})$ is a maximal invariant follows from Vinograde (1950). To characterize the group induced ordering, we first calculate m using Theorem 1. For $\mathbf{y} \in \mathcal{L}_{p,n}$,

$$m(\mathbf{u},\mathbf{y}) = \sup_\Gamma tr(\mathbf{u}\mathbf{y}'\Gamma') = \sup_\Gamma tr(\Gamma\mathbf{u}\mathbf{y}')$$

where the sup is over $\mathcal{O}_n$. Now, apply Theorem 1 with $n = k$, $\mathbf{A} = \mathbf{u}\mathbf{y}'$ and $\mathbf{B} = \mathbf{I}_n$ to see that $m(\mathbf{u},\mathbf{y}) = \Sigma_1^n \lambda_i(\mathbf{u}\mathbf{y}') = \Sigma_1^p \lambda_i(\mathbf{u}\mathbf{y}')$. The second equality holds since $\lambda_i(\mathbf{u}\mathbf{y}') = 0$ for $i > p$. In this example, the assumption that m restricted to $\mathcal{T}$ is the inner product, does not hold. However a description of the order can be given in terms of the Loewner ordering on $\mathcal{S}_p$. For $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}_p$, write $\mathbf{s}_1 \leq_L \mathbf{s}_2$ if $\mathbf{s}_2 - \mathbf{s}_1 \in \mathcal{S}_p^+$ (see Loewner (1934)).

LEMMA 1. $\mathbf{x} \leq \mathbf{y}$ *iff* $\mathbf{x}'\mathbf{x} \leq_L \mathbf{y}'\mathbf{y}$.

Proof. Assume $\mathbf{x} \leq \mathbf{y}$ so $m(\mathbf{u},\mathbf{x}) \leq m(\mathbf{u},\mathbf{y})$ for all $\mathbf{u} \in \mathcal{L}_{p,n}$.
Pick $\mathbf{u} = \alpha\beta'$ where $\alpha \in \mathcal{R}^n$, $\alpha'\alpha = 1$ and $\beta \in \mathcal{R}^p$. Then

$$m(\mathbf{u},\mathbf{y}) = \Sigma_1^p \lambda_i(\alpha\beta'\mathbf{y}') = \Sigma_1^p \mu_i^{1/2}(\alpha\beta'\mathbf{y}'\mathbf{y}'\beta\alpha') = (\beta'\mathbf{y}'\mathbf{y}'\beta)^{1/2}$$

since $\alpha\beta'\mathbf{y}'$ has rank one and $\alpha'\alpha = 1$. A similar expression holds for $\mathbf{x}$ so $(\beta'\mathbf{x}'\mathbf{x}\beta)^{1/2} \leq (\beta'\mathbf{y}'\mathbf{y}'\beta)^{1/2}$ for all $\beta \in \mathcal{R}^p$ which implies that $\mathbf{x}'\mathbf{x} \leq_L \mathbf{y}'\mathbf{y}$. Conversely, assume $\mathbf{x}'\mathbf{x} \leq_L \mathbf{y}'\mathbf{y}$ so for all $u \in \mathcal{L}_{p,n}$, $\mathbf{u}\mathbf{x}'\mathbf{x}\mathbf{u}' \leq_L \mathbf{u}\mathbf{y}'\mathbf{y}\mathbf{u}'$. This implies that (see Bellman (1960), p. 115) $\mu_i(\mathbf{u}\mathbf{x}'\mathbf{x}\mathbf{u}') \leq \mu_i(\mathbf{u}\mathbf{y}'\mathbf{y}\mathbf{u}')$, $i = 1, \ldots, p$ so $\lambda_i(\mathbf{u}\mathbf{x}') \leq \lambda_i(\mathbf{u}\mathbf{y}')$ for $i = 1, \ldots, p$. Hence $m(\mathbf{u},\mathbf{x}) \leq m(\mathbf{u},\mathbf{y})$ for all $\mathbf{u} \in \mathcal{L}_{p,n}$ so $\mathbf{x} \leq \mathbf{y}$ by Proposition 2. □

The result has a number of interesting consequences.

PROPOSITION 5. *The closed convex hull of* $\mathcal{O}_n$ *in* $\mathcal{L}_{n,n}$ *is* $\{\psi|\psi \in \mathcal{L}_{n,n}, \psi'\psi \leq_L I_n\}$.

Proof. In Lemma 1, take $n = p$ and $\mathbf{y} = \mathbf{I}_n$. Then $\mathbf{x} \leq \mathbf{I}_n$ means that $\mathbf{x}$ is in the convex hull of $\{\Gamma|\Gamma \in \mathcal{O}_n\}$ and by Lemma 1, this is equivalent to $\mathbf{x}'\mathbf{x} \leq_L I_n$. □

PROPOSITION 6. *For* $\mathbf{x},\mathbf{y} \in \mathcal{L}_{p,n}$, $\mathbf{x}'\mathbf{x} \leq_L \mathbf{y}'\mathbf{y}$ *iff.* $\mathbf{x} = \psi\mathbf{y}$ *where* $\psi \in \mathcal{L}_{n,n}$ *satisfies* $\psi'\psi \leq_L I_n$.

Proof. From Lemma 1, $\mathbf{x}'\mathbf{x} \leq_L \mathbf{y}'\mathbf{y}$ *iff* $\mathbf{x} \leq \mathbf{y}$. Also, $\mathbf{x} \leq \mathbf{y}$ *iff.* $\mathbf{x}$ is in the convex hull of $\{\Gamma\mathbf{y}|\Gamma \in \mathcal{O}_n\}$. By Proposition 5, this convex hull is just $\{\psi\mathbf{y}|\psi'\psi \leq_L I_n\}$. □

This completes our discussion of Example 2.6.

The reader should compare Examples 2.5 and 2.6 with the treatment in Jensen (1984). There is some overlap but the results do complement each other. The result in Proposition 6 is an extension of Vinograde's (1950) result and can be derived rather easily from Vinograde's result. The final example in this section is rather simple but shows that in some cases very little is gained from Proposition 2.

Example 2.7. In this example, let V be $\mathcal{R}^2$ and let G be the group generated by g_0 which is rotation through $\pi/2$ in the counterclockwise direction in $\mathcal{R}^2$. Thus, G has four elements which are $\{g_0^0, g_0, g_0^2, g_0^3\}$ and G is Abelian. Take $\mathcal{J}$ to be

$$\mathcal{J} = \{\mathbf{x} | x_1 > 0, x_2 \geq 0\} \cup \{0\}$$

and let $\tau(\mathbf{x})$ be the unique vector in $\mathcal{J}$ of the form $g_0^i\mathbf{x}$ for $i = 0, 1, 2, 3$. The calculation of m is easy and of very little help. About all one can say here is that $\mathbf{x} \leq \mathbf{y}$ iff $\mathbf{x}$ is in the convex hull of $\{g_0^i y | \ i = 0, 1, 2, 3\}$. Exactly the same remarks are in order when G is the group generated by the rotation through $2\pi/k$ ($k = 3, 4, \ldots$). Namely, m is easy to describe, but of no help in describing the ordering. Note that Proposition 4 (and hence Corollary 1) cannot be used in this example. However, if instead of a finite rotation group, we use a finite dihedral group (see Benson and Grove (1971), p. 7) acting on $\mathcal{R}^2$, then with the obvious choice for $\mathcal{J}$, Proposition 4 and Corollary 1 do apply directly.

3. The Decreasing Functions. In this section, we apply results of Marshall, Walkup and Wets (1967) to describe the decreasing functions of some of the group induced orderings discussed earlier. As in the last section, it is assumed that $(V, (\cdot,\cdot))$ is an inner product space acted on by a closed group $G \subseteq O(V)$, τ is a maximal invariant function whose range is the convex cone $\mathcal{J}$ with $\tau(x) \in \{gx | g \in G\}$. The problem considered here is to give a useful analytic condition on $f: V \to R$ so that $x \leq y$ implies that $f(x) \geq f(y)$.

A solution to the problem just described will be given in the case that, when restricted to $\mathcal{J}$, the group induced ordering $\leq$ is a cone ordering. To be more precise, let D be a subset of an inner product space $(W, (\cdot,\cdot))$ and let $K \subseteq W$ be a fixed convex cone. A *cone ordering* induced on D by K is a relation $<$ defined by $x < y$ iff $y - x \in K$. A function $f: D \to R$ is decreasing on $(D, <)$ if $x < y$ implies $f(x) \geq f(y)$, for $x, y \in D$.

THEOREM 2 (Marshall, Walkup and Wets (1967)). *Suppose D is convex with a non-empty interior and $f: D \to R$ is continuous at the boundary of D. Let T be a frame for K. Then f is decreasing on $(D, <)$ iff.*

(3.1) $\quad f(x + \lambda t) \leq f(x)$ *for all $x \in D$ and all $t \in T$ and $\lambda > 0$ such that $x + \lambda t \in D$.*

Corollary 2 (Marshall, Walkup, Wets (1967). *In addition to the assumptions in Theorem 1, assume that f has a differential $df: D^\circ \to W$ on the interior of D. Then f is decreasing on $(D, <)$, iff*

(3.2) $$(df(x), t) \leq 0$$

for all $t \in T$ and $x \in D^\circ$.

Before applying these results to the problem at hand, a few preliminaries are needed. Given the convex cone $\mathcal{J}$ which is the range of τ, let M be the subspace of V which is generated by $\mathcal{J}$. Thus, $\mathcal{J}$ is a convex cone with a non-empty interior in the inner product space $(M, (\cdot,\cdot))$. Also, let

$$\mathcal{J}^* = \{x | x \in M, (x,y) \geq 0 \quad \text{for all } y \in \mathcal{J}\}$$

so $\mathcal{J}^*$ is the dual cone (in M) of $\mathcal{J}$. Of course, $\mathcal{J}^*$ is also a convex cone. The following result shows that the group induced ordering is in fact a cone ordering on $\mathcal{J}$ in the special case treated in Proposition 4.

PROPOSITION 7. *For $\beta, \gamma \in \mathcal{J}$, suppose that $m(\beta, \gamma) = (\beta, \gamma)$. Then, for $u, v \in \mathcal{J}$, $u \leq v$ iff $v - u \in \mathcal{J}^*$.*

Proof. Since $u, v \in \mathcal{T}$, $\tau(u) = u$ and $\tau(v) = v$. Thus, Proposition 4 implies that $u \leq v$ iff

$$(\beta, u) \leq (\beta, v) \quad \text{for all } \beta \in \mathcal{T} \text{ which holds iff.} \tag{3.3}$$

$$(\beta, v-u) \geq 0 \quad \text{for all } \beta \in \mathcal{T}, \tag{3.4}$$

and this is equivalent to the assertion that $v - u \in \mathcal{T}^*$. □

The above result gives a sufficient condition that $\leq$ be a cone ordering on $\mathcal{T}$. Further the cone ordering is determined by the convex cone $\mathcal{T}^*$. In examples 2.1 through 2.5, m on $\mathcal{T} \times \mathcal{T}$ is the inner product so Proposition 7 applies directly.

PROPOSITION 8. *Assume that the pre-ordering on $\mathcal{T}$ is the cone ordering defined by $\mathcal{T}^*$. Let T^* be a frame for $\mathcal{T}^*$. Then a function $f: V \to R^1$ which is continuous at the boundary of $\mathcal{T}$ is decreasing iff for each $x \in \mathcal{T}$ and $t \in T^*$,*

$$f(x + \lambda t) \leq f(x)$$

for all $\lambda > 0$ such that $x + \lambda t \in \mathcal{T}$.

Proof. Apply Theorem 2 with $D = \mathcal{T}$ and $K = \mathcal{T}^*$. □

COROLLARY 3. *Let the assumptions of Proposition 8 hold. Also assume that f has a differential $df: \mathcal{T}^0 \to M$. Then f is decreasing iff. for all $t \in T^*$,*

$$(df(x), t) \leq 0 \tag{3.4}$$

for $x \in \mathcal{T}^*$.

Proof. This is immediate from Corollary 2. □

The application of Corollary 3 to Examples 2.1, 2.2, and 2.3 is quite easy and the essential details can be found in Marshall, Walkup and Wets (1967). A discussion of Example 2.4 is much the same as that for Example 2.5 which we now give.

Example 2.5 continued. The notation and results given in Example 2.5 are assumed. First, the subspace M generated by $\mathcal{T}$ is the space of all $n \times p$ real matrices $\mathbf{u}$ with $u_{ij} = 0$ for $i \neq j$, so M is p dimensional. The dual cone $\mathcal{T}^* \subseteq M$ is

$$\mathcal{T}^* = \{\mathbf{u} | \mathbf{u} \in M, \Sigma_1^k u_{ii} \geq 0, k = 1, \ldots, p\}.$$

A frame T^* for $\mathcal{T}^*$ consists of $t_1, \ldots, t_p \in M$ where:

> t_i has its i,i diagonal 1, its $(i+1), (i+1)$ diagonal -1 and all other elements of t_i are 0, for $i = 1, \ldots, p-1$;
>
> t_p has its p,p diagonal 1 and all other elements 0.

This follows from Proposition 1 in Marshall, Walkup and Wets (1967). Let $f: \mathcal{L}_{p,n} \to R^1$ be a $\mathcal{O}_n \times \mathcal{O}_p$ invariant function and let $\bar{f}$ denote the restriction of f to $\mathcal{T}$. When $\bar{f}$ has a differential, then f is decreasing iff

$$(t, d\bar{f}(u)) \leq 0, t \in T^* \text{ and } u \in \mathcal{T}$$

which is equivalent to

$$\delta\bar{f}/\delta u_{11} \leq \delta\bar{f}/\delta u_{22} \leq \ldots \leq \delta\bar{f}/\delta u_{pp} \leq 0.$$

In Example 2.6, the group ordering is not an $\mathcal{T}^*$ cone ordering on $\mathcal{T}$, but is an $\mathcal{T}^*$ cone ordering in a different coordinate system. To be more precise, Lemma 1 shows that $x \leq y$ iff. $x'x \leq_L y'y$. The Loewner ordering $\leq_L$ is a cone ordering on $\mathcal{S}_p^+$. Thus, a decreasing function f on $\mathcal{L}_{p,n}$ in Example 2.6 can be characterized by first writing it as $f(x) = \bar{f}(x'x)$ and then using the Marshall, Walkup and Wests ((1967), Example 4) results.

4. Remarks on the Convolution Theorem. Again consider the general situation of an inner product space $(V(\cdot,\cdot))$ acted on by a compact group $G \subseteq \mathcal{O}(V)$. As usual, $\leq$ denotes the pre-order defined by G.

Definition 2. If for every two compact monotone sets A and B, the function

$$\psi(y) = \int_V I_A(x) I_B(y-x)\,dx$$

is decreasing (see Definition 1), then we say the convolution theorem (CT) holds for G.

It is a standard approximation argument to show that CT implies that for suitably smooth, integrable and decreasing f_1, f_2, the convolution

$$f(y) = (f_1 * f_2)(y) = \int_V f_1(x) f_2(y-x)\,dx$$

is again decreasing. Hence the term convolution theorem. This result has many applications in the area of probability inequalities—for example, see Marshall and Olkin (1974), Eaton and Perlman (1977), Marshall and Olkin (1979) and Eaton (1982).

CT was established for $V = R^n$ and $G = \mathcal{P}_n$ by Marshall and Olkin (1974). This result was extended to all reflection groups by Eaton and Perlman (1977). Examples of reflection groups are the groups considered in Examples 2.1, 2.2 and 2.3. When the group G acts transitively on $\{x | x \in V, \|x\| = 1\}$, then $x \leq y$ means that $\|x\| \leq \|y\|$ and all the decreasing functions have the form $x \to \eta(\|x\|)$ where η is decreasing on $[0, \infty)$. CT obviously holds for such cases. In summary, here is a listing of some groups for which CT is known to hold:

(i) All finite and infinite closed reflection groups (see Eaton and Perlman (1977)).

(ii) Any group G which acts transitively on $\{x | x \in V, \|x\| = 1\}$.

(iii) A product $G_1 \times G_2 \times \ldots \times G_k$ acting on the direct sum $V_1 \otimes V_2 \otimes \ldots \otimes V_k$. The action is coordinatewise, $(g_1, g_2, \ldots, g_k)(x_1, x_2, \ldots, x_k) = (g_1x_1, g_2x_2, \ldots, g_kx_k)$, where G_i acting on V_i is of the type (i) or (ii) above.

These are the only groups that I know for which CT holds. The remainder of this section is devoted to a discussion of a necessary condition on G in order that CT hold. Some examples are given where CT does not hold.

Recall that $x \leq y$ means $x \in C(y)$ where $C(y)$ is the convex hull of $\{gy | g \in G\}$. Also, a set B is monotone iff for all $x \in B$, $C(x) \subseteq B$.

Definition 3. Given any set A, let

$$\mathcal{S}(A) = \bigcup_{x \in A} C(x).$$

PROPOSITION 9. *The set $\mathcal{S}(A)$ is the smallest monotone set which contains A.*

Proof. To show $\mathcal{S}(A)$ is monotone, consider $u \in \mathcal{S}(A)$ so $u \in C(x)$ for some $x \in A$. Since $C(x)$ is monotone, $C(u) \subseteq C(x)$ so $C(u) \subseteq \mathcal{S}(A)$ which shows $\mathcal{S}(A)$ is monotone. Now, assume $B \supseteq A$ and B is monotone. If $x \in A$ then $x \in B$ so $C(x) \subseteq B$ as B is monotone. Hence $\bigcup_{x \in A} C(x) \subseteq B$. □

Here are some properties of $\mathcal{S}$ which are easily verified:

(i) $\mathcal{S}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha \mathcal{S}(A_\alpha)$

(4.1) (ii) $\mathcal{S}(A_1 + A_2) \subseteq \mathcal{S}(A_1) + \mathcal{S}(A_2)$

(iii) A compact implies $\mathcal{S}(A)$ compact

In (4.1), the sign + denotes the usual Minkowski sum of two sets.

Next is a necessary condition for CT to hold.

PROPOSITION 10. *Assume that for each $x \neq 0$, $C(x)$ has a non-empty interior (For a discussion of this condition, see Eaton and Perlman (1977)). In order that CT hold, it is necessary that*

(4.2) $A + y \subseteq \mathcal{S}(A + x)$ for all $y \in C(x)$, for all $x \in V$, and for all compact monotone sets A.

Proof. Assume that (4.2) does not hold for some x, $y \in C(x)$ and A. Then, A must have a non-zero element and x must be non-zero. Let $z \in A$ with $z \neq 0$. Since $\phi \neq (C(z))^\circ \subseteq A$, the set A has a non-empty interior. Hence $A + y$ has a non-empty interior and the open set $N = (A + y)^\circ \cap (\mathcal{S}(A + x))^C$ is not empty. With l denoting Lebesgue measure, we have

$$l(A) = l(A + y) > l((A + y) \cap \mathcal{S}(A + x)),$$

since N is open and non-empty. For $u \in V$ let

$$\Psi(u) = \int I_{\mathcal{S}(A+x)}(w)\, I_A(w-u)\, dw.$$

Since $A+x \subseteq \mathcal{S}(A + x)$,

$$\Psi(x) = l((A + x) \cap \mathcal{S}(A + x)) = l(A + x) = l(A).$$

However,

$$\Psi(y) = l((A+y) \cap \mathcal{S}(A+x)) < l(A)$$

so $\Psi(y) < \Psi(x)$ and CT does not hold. □

PROPOSITION 11. *Each of the following conditions is equivalent to (4.2):*

(4.3) $A + C(x) \subseteq \mathcal{S}(A+x)$ for all $x \in V$ and for all compact monotone sets A,

(4.4) $C(z) + C(x) \subseteq \mathcal{S}(C(z) + x)$ for all $x, z \in V$,

(4.5) $\mathcal{S}(C(z) + x)$ is a convex set for all $x, z \in V$.

Proof. Clearly (4.3) implies (4.2). Conversely, if (4.2) holds, then

$$A + C(x) = \bigcup_{y \in C(x)} (A + y) \subseteq \mathcal{S}(A + x)$$

so (4.3) holds. Clearly (4.3) implies (4.4). To show (4.4) implies (4.3) first observe that when A is a monotone set,

$$A + C(x) = \bigcup_{z \in A} (C(z) + C(x)).$$

Since A is monotone, $A = \bigcup_{z \in A} C(x)$ so (4.1)(i) and (4.4) imply that

$$\mathcal{S}(A + x) = \mathcal{S}(\bigcup_{z \in A}(C(z) + C(x)) = \bigcup_{z \in A} \mathcal{S}(C(z) + x) \supseteq \bigcup_{x \in A}(C(z) + C(x)) = \mathbf{A} + C(x).$$

Hence (4.3) holds. To show (4.4) and (4.5) are equivalent, first assume (4.4) holds. Since $C(z) + C(x)$ is monotone, Proposition 9 implies that

(*) $$\mathcal{S}(C(z) + x) \subseteq C(z) + C(x).$$

Thus, when (4.4) holds there is equality in (*). But $C(z) + C(x)$ is convex as both $C(z)$ and $C(x)$ are convex so (4.5) holds. Conversely, assume that (4.5) holds and consider $u \in C(z)$ and $v \in C(x)$. It must be shown that $u + v \in \mathcal{S}(C(z) + x)$. Since $\mathcal{S}(C(z) + x) = \mathcal{S}(C(z) + gx))$ for all $g \in G$, it follows that $u + gx$ is in $\mathcal{S}(C(z) + x)$ for all $g \in G$ as $u \in C(z)$. But, if $\mathcal{S}(C(z) + x)$ is convex, this implies that all convex combinations (over $g \in G$) of $u + gx$ are also in $\mathcal{S}(C(z) + x)$. Since $v \in C(x)$, v can be represented as a convex combination of $\{gx | g \in G\}$ so $u + v$ is a convex combination of $\{u + gx | g \in G\}$. Hence (4.4) holds. □

The following example shows that CT does not hold for any finite rotation group acting on $\mathcal{R}^2$. It will be shown that condition (4.5) does not hold for these cases.

Example 4.1. Fix an integer $k \geqslant 3$ and let $\theta = 2\pi/k$. The case of $k = 2$ is trivial. Let G be the group generated by $g = g_\theta$ which is rotation (in the counter-clockwise direction) through the angle θ. Thus $G = \{I, g, \ldots, g^{k-1}\}$ has k elements. Let $z = \binom{1}{0}$ and let x be

$$x = 5(g_\eta z)$$

where $\eta = \theta/2$, so x has length 5. Then $u = z + x \in C(z) + x$ and has coordinates

$$\binom{u_1}{u_2} = \binom{x_1 + 1}{x_2}.$$

Applying g^{k-1} to the set $C(z) + x$ shows that the vector

$$\bar{u} = \binom{\bar{u}_1}{\bar{u}_2} = \binom{x_1 + 1}{-x_2} = g^{k-1}(x + gz)$$

is in the set $\mathcal{S}(C(z) + x)$. Hence, if $\mathcal{S}(C(z) + x)$ is to be convex, the vector

$$v = \frac{u + \bar{u}}{2} = \binom{x_1 + 1}{0}$$

must be in $\mathcal{S}(C(z) + x)$. However, a carefully drawn picture will convince the reader that v is not in $C(w)$ for any $w \in C(z) + x$. The case of $k = 4$ is a good starting point to see why $\mathcal{S}(C(z) + x)$ is not convex for the particular choices of z and x above (see Figure 1). Thus CT does not hold for any of the finite rotation groups acting on $\mathcal{R}^2$. However, CT does hold for the finite dihedral groups acting on $\mathcal{R}^2$ as these are reflection groups (see Benson and Grove (1971)).

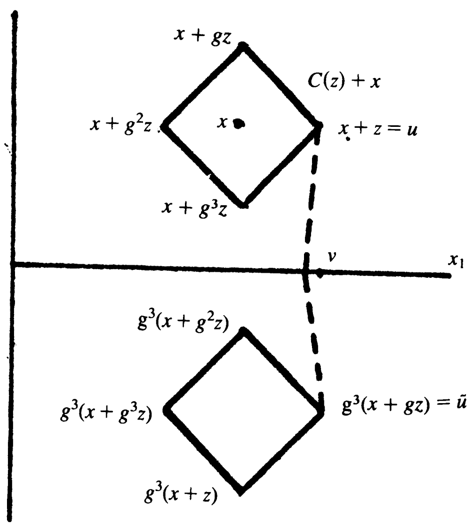

FIGURE 1. Case of $k = 4$. The dashed line gives the right most boundary of $\mathcal{S}(C(z) + x)$.

The final result of this section shows that the necessary condition (4.2) is satisfied for the situation considered in Proposition 4. More precisely, again assume that τ is a maximal invariant with a convex cone $\mathcal{T}$ as its range and $\tau(x) \in \{gx | g \in G\}$.

PROPOSITION 12. As in Proposition 4, assume that for $u, v \in \mathcal{T}$, $m(u,v) = (u,v)$. Then, for $u, v \in \mathcal{T}$

$$C(u) + C(v) = C(u + v) \tag{4.6}$$

and condition (4.2) holds.

Proof. Since $u + v \in C(u) + C(v)$ and since $C(u) + C(v)$ is monotone, it is clear that $C(u + v) \subseteq C(u) + C(v)$. Now, suppose $z \in C(u) + C(v)$ so $z = \gamma + \delta$ with $\gamma \in C(u)$ and $\delta \in C(v)$. Using the relations given by (2.5) and the results of Propositions 2, 3 and 4, we have

$$
\begin{aligned}
(4.7)\quad m(w,z) &= m(w,\gamma + \delta) \leq m(w,\gamma) + m(w,\gamma) \leq m(w,u) + m(w,v) = \\
&m(\tau(w),u) + m(\tau(w),v) = (\tau(w),\, u + v) = m(\tau(w),\, u + v) = m(w,u + v)
\end{aligned}
$$

for any $w \in V$. By Proposition 2, this implies that $z \leq u + v$ so $z \in C(u + v)$. Hence $C(u) + C(v) \subseteq C(u + v)$ so (4.6) holds. To show (4.2) holds, (4.4) will be verified. First observe that $C(z) = C(\tau(z))$ and

$$\mathcal{S}(C(z) + x) = \mathcal{S}(C(\tau(z)) + \tau(x)) \quad \text{for } z,x \in V.$$

Thus, by (4.6), we have

$$
\begin{aligned}
&C(z) + C(x) = C(\tau(z)) + C(\tau(x)) = \\
&C(\tau(z) + \tau(x)) \subseteq \mathcal{S}(C(\tau(z)) + \tau(x)) = \\
&\mathcal{S}(C(z) + x)
\end{aligned}
$$

so (4.4) and hence (4.2) holds. □

The above result shows that (4.2) holds for Examples 2.1–2.5 although CT is only known to hold for Examples 2.1–2.3. It is not known whether (4.2) holds for Example 2.6. Whether or not CT holds for Example 2.5 is an important unresolved problem.

The implications of (4.6) concerning the group G are not known, but are probably important in understanding when CT holds. Both these implications and useful conditions for CT to hold would be welcome contributions.

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