CHAPTER 1

Introduction

In studying the behavior of power functions of various multivariate tests it is essential to develop a distribution theory for noncentral distributions. It is somewhat surprising that this is not straightforward. In fact even intuitively obvious results concerning the power functions of multivariate tests often require extensively elaborate arguments because of this difficulty. For a recent example of this see Olkin and Perlman (1980).

Let us take the noncentral χ^2 distribution and its multivariate analog, the noncentral Wishart distribution, as an example. The density of the noncentral χ^2 distribution is usually written as an infinite series. This series arises from the expansion of the exponential part of the normal density into a power series and its term by term integration with respect to irrelevant variables. In the multivariate case this integration becomes nontrivial involving an integration with respect to the Haar invariant measure on the orthogonal group. Zonal polynomials form an essential tool for studying and expressing this integration. Let us briefly review historical developments of the subject.

The first systematic studies of the noncentral Wishart distribution appeared in Anderson and Girshick (1944) and Anderson (1946). James (1955a, 1955b) introduced the integration with respect to the Haar measure on the orthogonal group explicitly and made further progress. Herz (1955) developed a theory of hypergeometric functions in matrix arguments and expressed the density of the noncentral Wishart distribution using such a function. Then James (1960, 1961a) introduced "zonal polynomials" which have a special invariance property with respect to the Haar measure and expressed the density of the noncentral Wishart distribution as an infinite series involving zonal polynomials. This infinite series provides an explicit infinite series expression for the hypergeometric function introduced by Herz. James also credits Hua (1959) for introducing zonal polynomials in another context. Zonal polynomials provided a unifying tool for the study of noncentral distributions in multivariate analysis and gave rise to an extensive literature on multivariate noncentral distributions particularly following his review article in 1964.

Examples include the distribution of the latent roots of the covariance matrix (James (1960)), the noncentral Wishart distribution (James (1961a,b)), the noncentral multivariate F distribution (James (1964)), the distribution of the roots of multivariate beta matrix (Constantine (1963)), the distribution of the canonical correlation coefficients (Constantine (1963)), the noncentral distribution of Lawley-Hotelling's trace statistic (Constantine (1966), Khatri (1967)), the noncentral distribution of Pillai's trace statistic (Khatri and Pillai (1968)), the distribution of the largest and smallest root of a Wishart or a multivariate beta distribution (Sugiyama (1966, 1967a,b), Khatri (1967, 1972), Khatri and Pillai (1968), Hayakawa (1967, 1969), Pillai and Sugiyama (1969), Krishnaiah and Chang (1971)), the distribution of quadratic functions (Khatri (1966), Hayakawa (1966, 1969), Shah (1970), Crowther and DeWaal (1973)), the distribution of multiple correlation matrix (Srivastava, 1968), the multivariate Dirichlet-type distributions (DeWaal (1970, 1972)), the distribution of some statistics for testing the equality of two covariance matrices and other hypotheses (Pillai and Nagarsenker (1972), Nagarsenker (1978)), distribution of ratio of roots (Pillai, Al-Ani, and Jouris (1969)). Pillai (1975) gives a unified treatment of several statistics.

Zonal polynomials have been found to be useful also for expressing moments of multivariate test statistics under alternative hypotheses as, for example, the moments of the generalized variance (Herz (1955)), likelihood ratio tests (Constantine (1963), Pillai, Al-Ani, and Jouris (1969)), Lawley-Hotelling's trace statistic (Constantine (1966), Khatri (1967)), the sphericity test (Pillai and

Nagarsenker (1971)), correlation matrix (DeWaal (1973)), traces of multivariate symmetric normal matrix (Hayakawa and Kikuchi (1979)). These distributional results are reviewed in Pillai (1976, 1977).

In a decision theoretic context zonal polynomials were used to evaluate risk functions in certain invariant estimation problems (Shorrock and Zidek (1976), Zidek (1978)).

In spite of the use of zonal polynomials in unifying noncentral multivariate distribution theory their theory has been considered difficult to understand. One reason for this is that James' definition of zonal polynomials (1961a) requires a rather extensive background in group representation theory. James' construction of zonal polynomials was based on fairly detailed results of classical group representation theory found in Littlewood (1950) and Weyl (1946) in particular. Compared to the extensive literature on the application of zonal polynomials, treatments of the theoretical basis of zonal polynomials have been rather scarce. Farrell (1976) and Kates (1980) in addition to James' work (1961a,b,1968) should be mentioned as advanced theoretical treatments of zonal polynomials. Farrell's construction of zonal polynomials (1976) is based on the theory of H^*-algebra found in Loomis (1953). Zonal polynomials are a particular case of spherical functions in the sense of Helgason (1962). Kates (1980) gives a thorough modern treatment of zonal polynomials in the framework of Helgason (1962) and other more recent literature on spherical functions and group representations. Unfortunately these treatments have been difficult to understand for many statisticians, including the present author. One basic reason may be that the advanced algebra required for these treatments do not form part of the usual training of mathematical statisticians.

The purpose of this monograph is to present a self-contained readable development of zonal polynomials in the framework of standard multivariate analysis. It is intended for people working with multivariate analysis. No knowledge of extensive mathematics is needed but we assume that the reader is familiar with usual multivariate analysis (e.g. Anderson (1958)) and linear algebra.

In addition to the present work there have appeared recently several elementary treatments of zonal polynomials. Let us briefly discuss these approaches and their relation to the present work. Our starting point is Saw (1977) who derived many properties of zonal polynomials using basic properties of the multivariate normal and Wishart distributions. Unfortunately at several points Saw (1977) refers to Constantine (1963) who in turn makes use of group representation theory. Therefore Saw (1977) is not entirely self-contained. Actually as Saw (1977) suggested, it turns out that only the elementary methods from Constantine (1963) are needed to complete Saw's argument. Furthermore it seems more advantageous to define zonal polynomials differently than Saw and to rearrange his logical steps; his definition of zonal polynomials appears to lack a conceptual motivation. (See Remark 3.4.1 on this point.) In our approach zonal polynomials will be defined as eigenfunctions of an expectation or integral operator. By considering the finite dimensional vector space of homogeneous symmetric polynomials of a given degree we work with vectors and matrices and define zonal polynomials simply as characteristic vectors of a certain matrix. This is done in Section 3.1.

The idea of "eigenfunctions of expected value operators" is investigated in a more general framework in recent works by Kushner and Meisner (1980) and Kushner, Lebow, and Meisner (1981). In the second paper they give a definition of zonal polynomials. They follow James' original idea but mostly use techniques of linear algebra and their approach is very helpful in understanding James' original definition. They consider the space of homogeneous polynomials of (elements of) a symmetric matrix variable $\boldsymbol{A}$ whereas we consider the space of symmetric homogeneous polynomials of the characteristic roots of $\boldsymbol{A}$. In the former approach an extra step is needed to define zonal polynomials by requiring an orthogonal invariance. Our approach in Section 3.1 seems to be more direct.

Another welcome addition to our limited literature of elementary treatments of zonal polynomials is provided by the recent book by Muirhead (1982, Chapter 7), in which zonal polynomials are defined as eigenfunctions of a differential operator, or as solutions to partial differential equations (see James (1968, 1973)). Although Muirhead proceeds informally and many points are illustrated rather than proved, the approach can be made precise. We discuss this in Section 4.5.4. From our prejudiced viewpoint our approach seems more natural, primarily because the differential equation is hard to motivate.

What are the disadvantages of an elementary approach? One disadvantage is that often more detailed computation is needed for deriving various results than in a more abstract approach. This is in a sense a necessary tradeoff in adopting an elementary approach. However, often results in multivariate analysis require fairly heavy computation and the computation in the sequel does not seem too heavy. Another disadvantage is that the best possible results may not be obtainable. In our approach there is one important coefficient (see formula 3.4.12 below) obtained by Jamas (1961a) which we could not obtain. Furthermore our approach does not seem to apply to the recent generalization of zonal polynomials by Davis (Davis (1979, 1980, 1981), Chikuse (1981)). Except for these limitations all major properties of zonal polynomials will be proved, along with many new results. In particular the basic properties of zonal polynomials derived in Chapter 3 are sufficient for their usual applications. A good measure to judge our claim is the remarkable paper by James (1964). It is expository and contains many statements which seem to have never been proved in publication. In the sequel we will often refer to formula numbers in this paper. In summary, despite some limitations our approach seems to be very useful in removing theoretical difficulties associated with zonal polynomials.

Another difficulty of zonal polynomials is their numerical aspect. Explicit expressions for zonal polynomials are not yet known. There are several algorithms for computing their values, but they are not very fast. In Chapter 4 we study numerical aspects of zonal polynomials, especially their coefficients. Apparently some more progress is needed before zonal polynomials can be successfully applied in numerical computations.

In this monograph we will not discuss various applications of zonal polynomials in the study of multivariate noncentral distributions. Once the basic properties of zonal polynomials are established, it does not seem very difficult to express noncentral distributions in terms of zonal polynomials. Furthermore the applications are numerous as indicated above. Zonal polynomials often appear in an infinite series form which can be conveniently classified as hypergeometric function in matrix arguments (Constantine (1963), Herz (1955)). We will not discuss this topic either. Interested readers are referred to Muirhead (1982), where the noncentral distribution theory and the hypergeometric functions are amply treated. See also excellent review articles of Subrahmaniam (1976) and Pillai (1976,1977).

On the other hand we have included a development of complex zonal polynomials (zonal polynomials associated with the complex normal and the complex Wishart distributions). One reason for this is that our approach for the real case almost immediately carries over to the complex case. Another reason is that the existing theory of Schur functions provides explicit expressions for complex zonal polynomials. This becomes apparent if one compares Farrell (1980) and Macdonald (1979). See Chapter 5.

Finally let us briefly describe subsequent chapters. Chapter 2 gives preliminary material on partitions and homogeneous symmetric polynomials. Definitions and notations should be checked since they vary from book to book. In Chapter 3 we define zonal polynomials and derive their major properties. If the reader is not much interested in computational aspects of zonal polynomials the material covered in Chapter 3 should suffice for usual applications. Chapter 4 generalizes and refines the results in Chapter 3. It deals largely with computation, coefficients, etc., of zonal polynomials. The development becomes inevitably more tedious. In Chapter 5 we apply our approach to complex zonal polynomials. We show that complex zonal polynomials are the same as the Schur functions.