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ON SOME DISTRIBUTIONAL PROPERTIES OF QUADRATIC FORMS IN NORMAL VARIABLES AND ON SOME ASSOCIATED MATRIX PARTIAL ORDERINGS

BY J. K. BAKSALARY, J. HAUKE AND G. P. H. STYAN

Tadeusz Kotarbiński Pedagogical University, Zielona Góra, Agricultural University, Poznań, and McGill University, Montréal

We establish two new versions of Cochran's Theorem concerning the distribution of quadratic forms in normal variables. Instead of the usual rank additivity condition we consider two partial orderings among symmetric matrices.

1. Results. Our main purpose in this paper is to establish two new versions of Cochran's Theorem concerning the distribution of quadratic forms in normal variables. Instead of the usual rank additivity condition we consider two matrix partial orderings.

In our first theorem we use the rank subtractivity, or minus, partial ordering of two matrices $\boldsymbol{L}$ and $\boldsymbol{M}$, possibly rectangular, introduced by Hartwig (1980) and defined by

$$\boldsymbol{L} \leq_{rs} \boldsymbol{M} \iff \operatorname{rank}(\boldsymbol{M}-\boldsymbol{L}) = \operatorname{rank}(\boldsymbol{M}) - \operatorname{rank}(\boldsymbol{L}), \tag{1}$$

cf. also Hartwig and Styan (1986).

The equivalence of the rank subtractivity partial ordering with rank additivity for any matrices $\boldsymbol{B}_1, \cdots, \boldsymbol{B}_k$, possibly rectangular, was established by Hartwig (1981), and is

$$\operatorname{rank}\Big(\sum_{i=1}^{k} \boldsymbol{B}_i\Big) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{B}_i) \iff \boldsymbol{B}_i \leq_{rs} \boldsymbol{B} = \boldsymbol{B}_1 + \cdots + \boldsymbol{B}_k \quad \text{for all} \quad i = 1, \cdots, k. \tag{2}$$

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THEOREM 1. *Let $\boldsymbol{A} = \sum_{i=1}^{k} \boldsymbol{A}_i$, where the $\boldsymbol{A}_i$ are $n \times n$ nonrandom symmetric matrices, $i = 1, \cdots, k$, not necessarily nonnegative definite. Let the random $n \times 1$ vector $\boldsymbol{X}$ follow a multivariate normal distribution with mean vector $\mathcal{E}(\boldsymbol{X}) = \boldsymbol{\mu}$, not necessarily $\mathbf{0}$, and covariance matrix $\boldsymbol{V} \geq \mathbf{0}$, not necessarily positive definite, and let $\boldsymbol{W}$ denote the $n \times (n+1)$ partitioned matrix $(\boldsymbol{V} : \boldsymbol{\mu})$. Consider the quadratic forms $Q = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ and $Q_i = \boldsymbol{X}'\boldsymbol{A}_i\boldsymbol{X}, i = 1, \cdots, k$, and the conditions:*

(a) *Q_i is distributed as a chi-squared variable for all $i = 1, \cdots, k$,*
(b) *$Q_1, \cdots, Q_k$ are mutually independent,*
(c) *Q is distributed as a chi-squared variable,*
(d) *$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \leq_{rs} \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}$ for all $i = 1, \cdots, k$.*

Then

$$\text{(a) and (b)} \iff \text{(c) and (d)} \tag{3}$$

and

$$\text{(a) and (c)} \Longrightarrow \text{(b) and (d).} \tag{4}$$

Furthermore, whenever

$$\begin{aligned} \operatorname{rank}(\boldsymbol{W}) = \operatorname{rank}(\boldsymbol{V}), \quad &\text{or} \quad \boldsymbol{A}_i = 0 \quad \text{for all} \quad i = 1, \cdots, k, \\ &\text{or} \quad \boldsymbol{A}_i\boldsymbol{\mu} = 0 \quad \text{for all} \quad i = 1, \cdots, k, \end{aligned} \tag{5}$$

then the condition (d) *simplifies to*

(d$_1$) $$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V} \leq_{rs} \boldsymbol{V}\boldsymbol{A}\boldsymbol{V} \quad \text{for all} \quad i = 1, \cdots, k,$$

and then

$$\text{(b) and (c)} \Longrightarrow \text{(a) and (d}_1\text{).} \tag{6}$$

Whenever

$$\operatorname{rank}(\boldsymbol{A}_i : \boldsymbol{W}) = \operatorname{rank}(\boldsymbol{W}) \qquad \text{for all} \quad i = 1, \cdots, k, \tag{7}$$

then the condition (d) *simplifies to*

(d$_2$) $$\boldsymbol{A}_i \leq_{rs} \boldsymbol{A} \qquad \text{for all} \quad i = 1, \cdots, k.$$

Being valid for arbitrary $\boldsymbol{V} \geq \mathbf{0}$, i.e., nonnegative definite and not necessarily positive definite, and $\boldsymbol{\mu}$ not necessarily $\mathbf{0}$, Theorem 1 is comparable to Theorem 9.3.3 in Rao and Mitra (1971) and generalizes all those versions of Cochran's theorem which require one of the assumptions in (5) — cf. Rao and Mitra (1971, Lemma 9.3.1), Scarowsky (1973, Theorem 5.2), and Khatri (1980, Theorem 7). As observed by Scarowsky (1973, p.70) the conclusion (6) does not follow unless $\boldsymbol{V}$, $\boldsymbol{\mu}$ and the $\boldsymbol{A}_i$ are restricted as in (5).

The original version (Cochran, 1934) of Cochran's Theorem is (1) with $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{V} = \boldsymbol{A} = \boldsymbol{I}_n$, the $n \times n$ identity matrix, and with (d) replaced by

(d_2) in the form of the rank additivity condition (2). Ogasawara and Takahashi (1951) extended Cochran's Theorem by showing that (1) holds when the dispersion matrix $\boldsymbol{V}$ is positive definite and $\boldsymbol{\mu}$ is any $n \times 1$ vector, not necessarily $\mathbf{0}$, and when $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{V}$ is nonnegative definite, possibly singular. The property of inheritance of chi-squaredness was discussed in terms of the rank subtractivity partial ordering by Baksalary and Hauke (1984) and Hartwig and Styan (1986, p. 159; 1987, pp. 363–364). Styan (1970, Theorem 6), Rao and Mitra (1971, Lemma 9.3.1 and Theorem 9.3.3), Scarowsky (1973, Theorem 5.2), Tan (1977, Theorem 4.2), and Khatri (1980, Theorem 7) gave further extensions.

Our second main result is a version of Cochran's theorem for quadratic forms distributed as linear combinations of two independent chi-squared variables. We will use the spectral partial ordering of two real symmetric matrices $\boldsymbol{L}$ and $\boldsymbol{M}$ defined by

$$\boldsymbol{L} \leq_\lambda \boldsymbol{M} \Longleftrightarrow \boldsymbol{L} \leq_{rs} \boldsymbol{M} \quad \text{and} \quad \{\mathrm{ch}(\boldsymbol{L})\} \subseteq \{\mathrm{ch}(\boldsymbol{M})\}, \tag{8}$$

where $\{\mathrm{ch}(\,\cdot\,)\}$ denotes the set of nonzero eigenvalues. This partial ordering was introduced by Baksalary and Hauke (1987) following canonical interpretations of the rank subtractivity partial ordering derived by Hartwig and Styan (1986).

THEOREM 2. *Let $\boldsymbol{A} = \sum_{i=1}^k \boldsymbol{A}_i$, where the $\boldsymbol{A}_i$ are $n \times n$ nonrandom symmetric matrices, $i = 1, \cdots, k$. Furthermore, let the random $n \times 1$ vector $\boldsymbol{X}$ follow a multivariate normal distribution with mean vector $\mathcal{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V} \geq \mathbf{0}$, not necessarily positive definite, and let $\boldsymbol{W}$ denote the $n \times (n+1)$ partitioned matrix $(\boldsymbol{V} : \boldsymbol{\mu})$. Let c_1 and c_2 be given nonzero distinct real numbers. Consider the quadratic forms $Q = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ and $Q_i = \boldsymbol{X}'\boldsymbol{A}_i\boldsymbol{X}, i = 1, \cdots, k$, and the conditions:*

(a) *Q_i is distributed as $c_1\chi^2_{i1} + c_2\chi^2_{i2}$ for all $i = 1, \cdots, k$,*
(b) *$Q_1, \cdots, Q_k$ are mutually independent,*
(c) *Q is distributed as $c_1\chi^2_1 + c_2\chi^2_2$,*
(d) *$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V} \leq_\lambda \boldsymbol{V}\boldsymbol{A}\boldsymbol{V}$ and rank $(\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V}) =$ rank $(\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W})$ for all $i = 1, \cdots, k$,*

where χ^2_{i1} with χ^2_{i2} for each $i = 1, \cdots, k$ in (a), and χ^2_1 with χ^2_2 in (c), are independent chi-squared variables, some of which may have zero degrees of freedom. Then

$$\text{(a)}\ \textit{and}\ \text{(b)} \Longleftrightarrow \text{(c)}\ \textit{and}\ \text{(d)}. \tag{9}$$

It should be pointed out that the second condition in (d) is redundant if any one of the conditions in (5) holds. With $\boldsymbol{V}$ and/or $\boldsymbol{\mu}$ restricted by (5), Theorem 2 is partially comparable with Theorem 8 of Khatri (1980). An interesting particular case is when $c_1 = 1, c_2 = -1$, and $\boldsymbol{V} = \boldsymbol{I}_n$, cf.

Luther (1965); then $\boldsymbol{VA}_i\boldsymbol{V} = \boldsymbol{A}_i$ and $\boldsymbol{VAV} = \boldsymbol{A}$ are tripotent matrices and our Theorem 2 may be compared with Theorem 3.2 in Anderson and Styan (1982).

The following three lemmas give some fundamental results concerning the distribution theory of quadratic forms in normal variables; see also Mathai and Provost (1992).

LEMMA 1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ nonrandom symmetric matrices. Let the random $n \times 1$ vector $\boldsymbol{X}$ follow a multivariate normal distribution with mean vector $\mathcal{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V} \geq 0$, not necessarily positive definite, and let $\boldsymbol{W}$ denote the $n \times (n+1)$ partitioned matrix $(\boldsymbol{V} : \boldsymbol{\mu})$. Then the quadratic forms $\boldsymbol{X}'\boldsymbol{AX}$ and $\boldsymbol{X}'\boldsymbol{BX}$ are independent if and only if $\boldsymbol{W}'\boldsymbol{AVBW} = \boldsymbol{0}$.*

With the dispersion matrix $\boldsymbol{V}$ nonnegative definite and not necessarily positive definite, Lemma 1 was first established by Ogasawara and Takahashi (1951). With $\boldsymbol{\mu} = \boldsymbol{0}$, Lemma 1 was first given by Craig (1943) with $\boldsymbol{V} = \boldsymbol{I}$, the identity matrix, and by Sakamoto (1944, Theorem I, page 5) with $\boldsymbol{V}$ positive definite and not necessarily equal to $\boldsymbol{I}$; for further discussion see Anderson and Styan (1982, page 2; 1990, page 1308).

LEMMA 2. *Let $\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{V}$ and $\boldsymbol{W}$ be defined as in Lemma 1. Then the quadratic form $\boldsymbol{X}'\boldsymbol{AX}$ is distributed as a chi-squared variable if and only if $\boldsymbol{W}'\boldsymbol{AVAW} = \boldsymbol{W}'\boldsymbol{AW}$.*

Lemma 2 was derived, independently, by Ogasawara and Takahashi (1951), Khatri (1963), Rayner and Livingstone (1965), and Mäkeläinen (1966); see also Styan (1970).

LEMMA 3. *Let $\boldsymbol{A}, \boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{V}$ and $\boldsymbol{W}$ be defined as in Lemma 1 and let $c_1, \cdots, c_s$ be distinct nonzero constants. Then the quadratic form $\boldsymbol{X}'\boldsymbol{AX}$ is distributed as a linear combination $\sum_{j=1}^{s} c_j \chi_j^2$ of independent chi-squared variables $\chi_1^2, \cdots, \chi_s^2, s \leq n$, if and only if*

$$\boldsymbol{W}'\boldsymbol{AV}(\boldsymbol{VAV})^{+}\boldsymbol{VAW} = \boldsymbol{W}'\boldsymbol{A}\,\boldsymbol{W} \tag{10}$$

and $c_1, \cdots, c_s$ are eigenvalues of $\boldsymbol{AV}$.

In (10) the matrix $(\boldsymbol{VAV})^{+}$ denotes the Moore-Penrose inverse of $\boldsymbol{VAV}$. Lemma 3 was established by Baldessari (1967) for $\boldsymbol{V}$ positive definite and then extended by Khatri (1977, Theorem 1). The conditions established there were later simplified, and the condition (10) is equivalent to

$$\operatorname{rank}(\boldsymbol{VAW}) = \operatorname{rank}(\boldsymbol{VAV}) \quad \text{and} \quad \boldsymbol{\mu}'\boldsymbol{AV}(\boldsymbol{VAV})^{+}\boldsymbol{VA\mu} = \boldsymbol{\mu}'\boldsymbol{A\mu}, \tag{11}$$

given by Khatri (1980, Theorem 1). An interpretation of the two conditions (9) and (10) follows from a general characterization of the distribution of the quadratic form $X'AX$; cf. Khatri (1977, p.90) and Dik and de Gunst (1985, Remarks 2.2 and 2.3).

Perhaps the first result on quadratic forms in normal variables which involves (implicitly) a partial order of real symmetric matrices is Theorem 6 of Ogasawara and Takahashi (1951), asserting that — with V positive definite and A nonnegative definite — the property of independence of $X'AX$ from $X'BX$ is inherited by every nonnegative definite $X'A_0X$ such that $A_0 \leq_L A$, the Löwner partial ordering defined by

$$A_0 \leq_L A \iff A - A_0 \geq 0, \tag{12}$$

i.e., $A - A_0$ nonnegative definite.

This result was generalized by Baksalary and Hauke (1984) in two directions: first, by deleting the assumption on the rank of the dispersion matrix V, and secondly by replacing the Löwner partial ordering by the column space, or range, preordering,

$$A_0 \leq_C A \iff \mathcal{C}(A_0) \subseteq \mathcal{C}(A). \tag{13}$$

We recall that a preordering is a binary relation which is reflexive and transitive, while a partial ordering is a preordering which is in addition antisymmetric; cf. e.g., Marshall and Olkin (1979, p. 13).

We quote here the generalization by Baksalary and Hauke (1984) in the following:

LEMMA 4. *Let A, A_0 and B be $n \times n$ nonrandom symmetric matrices, no one necessarily nonnegative definite. Let the random $n \times 1$ vector X follow a multivariate normal distribution with mean vector $\mathcal{E}(X) = \mu$ and covariance matrix $V \geq 0$, not necessarily positive definite, and let W denote the $n \times (n+1)$ partitioned matrix $(V : \mu)$. Further, let $X'AX$ be distributed independently of $X'BX$. Then every quadratic form $X'A_0X$, with A_0 satisfying the preordering*

$$W'A_0W \leq_C W'AW, \tag{14}$$

inherits this property, in the sense that $X'A_0X$ is also independent of $X'BX$.

The condition (14) *simplifies to*

$$A_0 \leq_C A, \tag{15}$$

whenever the partitioned matrix $(A : A_0) \leq_C W$.

If $X'AX$ is distributed as a chi-squared variable then every quadratic form $X'A_0X$ with A_0 satisfying the rank subtractivity partial ordering

$$W'A_0W \leq_{rs} W'AW \tag{16}$$

inherits the property of chi-squaredness, as showed by Baksalary and Hauke (1984), who also pointed out there that this is no longer necessarily so when (16) is replaced by the Löwner partial ordering $\boldsymbol{W}'\boldsymbol{A}_0\boldsymbol{W} \leq_L \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}$ or by the column space preordering $\boldsymbol{W}'\boldsymbol{A}_0\boldsymbol{W} \leq_C \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}$.

2. Proofs.

Proof of Theorem 1. In view of Lemmas 1 and 2 and the characterization of the rank subtractivity partial ordering for real symmetric matrices $\boldsymbol{L}$ and $\boldsymbol{M}$

$$\boldsymbol{L} \leq_{rs} \boldsymbol{M} \iff \boldsymbol{M}\boldsymbol{M}^-\boldsymbol{L} = \boldsymbol{L} = \boldsymbol{L}\boldsymbol{M}^-\boldsymbol{L}, \tag{17}$$

where the choice of a generalized inverse $\boldsymbol{M}^-$ is arbitrary, cf. Marsaglia and Styan (1974, Theorem 17), we may write the statements (a) through (d) of Theorem 1 as follows:

$$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{W} = \boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \text{ for all } \quad i = 1, \cdots, k, \tag{18}$$

$$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{V}\boldsymbol{A}_j\boldsymbol{W} = \boldsymbol{0} \text{ for all } i \neq j; \; i, j = 1, \cdots, k, \tag{19}$$

$$\boldsymbol{W}'\boldsymbol{A}\boldsymbol{V}\boldsymbol{A}\boldsymbol{W} = \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}, \tag{20}$$

$$\begin{aligned}\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} &= \boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \\ &= \boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \\ &\qquad \text{for all } i = 1, \cdots, k,\end{aligned} \tag{21}$$

where the choice of a generalized inverse $(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-$ in (21) is arbitrary. It is clear that the pair (18) and (19) implies (20) and

$$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} = \boldsymbol{W}'\boldsymbol{A}\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{W} \qquad \text{for all} \quad i = 1, \cdots, k. \tag{22}$$

Hence $\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \leq_C \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}$ and, consequently,

$$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W} \leq_C \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W} \qquad \text{for all} \quad i = 1, \cdots, k. \tag{23}$$

The preorderings (23) ensure that the equalities in (21) are independent of the choice of $(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-$, cf. Rao and Mitra (1971, p. 21). Reexpressing (20) in the form

$$\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}(\boldsymbol{W}^-\boldsymbol{V}\boldsymbol{W}^{-\prime})\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W} = \boldsymbol{W}'\boldsymbol{A}\boldsymbol{W} \tag{24}$$

shows that $(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-$ may be chosen as $\boldsymbol{W}^-\boldsymbol{V}\boldsymbol{W}^{-\prime}$. In view of (18), (22), and $\boldsymbol{W}\boldsymbol{W}^-\boldsymbol{V} = \boldsymbol{V}$, it is clear that (21) holds with this choice of $(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-$, which completes the proof of the part "(18) and (19) $\Longrightarrow$ (20) and (21)".

Conversely, we note that the conditions (21) are equivalent to the rank additivity property

$$\operatorname{rank}(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W}) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W}), \tag{25}$$

cf. Hartwig (1981, Theorem 1). Hence it follows that

$$\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-\boldsymbol{W}'\boldsymbol{A}_j\boldsymbol{W} = \boldsymbol{0} \qquad \text{for all} \quad i \neq j; \quad i,\ j = 1,\cdots,k, \tag{26}$$

cf. Marsaglia and Styan (1974, Th. 13). Utilizing again the fact that $(\boldsymbol{W}'\boldsymbol{A}\boldsymbol{W})^-$ may be chosen as $\boldsymbol{W}^-\boldsymbol{V}\boldsymbol{W}^{-'}$, the right-hand parts of (21) and (26) yield (18) and (19), respectively. ∎

Proof of Theorem 2. In view of Lemmas 1 and 3, the statement (a) of Theorem 2 is equivalent to the pair of conditions:

$$\{\mathrm{ch}(\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V})\} \subseteq \{c_1,\ c_2\} \qquad \text{for all} \quad i = 1,\cdots,k. \tag{27}$$

and

$$\boldsymbol{Z}'\boldsymbol{A}_i\boldsymbol{V}(\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V})^+\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{Z} = \boldsymbol{Z}'\boldsymbol{A}_i\boldsymbol{Z} \qquad \text{for all} \quad i = 1,\cdots,k; \tag{28}$$

the statement (b) is equivalent to (19); and the statement (c) is equivalent to the pair of conditions:

$$\{\mathrm{ch}(\boldsymbol{V}\boldsymbol{A}\boldsymbol{V})\} \subseteq \{c_1,\ c_2\} \tag{29}$$

and

$$\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{V}(\boldsymbol{V}\boldsymbol{A}\boldsymbol{V})^+\boldsymbol{V}\boldsymbol{A}\boldsymbol{Z} = \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}. \tag{30}$$

Since $\boldsymbol{W} = (\boldsymbol{V} : \boldsymbol{\mu})$, the condition (19) asserts in particular that

$$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V}\boldsymbol{A}_j\boldsymbol{V} = \boldsymbol{0} \qquad \text{for all} \quad i \neq j; \quad i,\ j = 1,\cdots,k. \tag{31}$$

Styan and Takemura (1983, Theorem 4) established that if (31) holds, then the set of nonzero eigenvalues of $\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}$ coincides with the set of all the nonzero eigenvalues of all the $\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V}$, $i = 1,\cdots,k$. It follows at once, therefore, that (27) and (19) imply (29). Further, an immediate consequence of (31) is that

$$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V}\boldsymbol{A}\boldsymbol{V} = (\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V})^2 \qquad \text{for all} \quad i = 1,\cdots,k. \tag{32}$$

This means that we have the star partial ordering (Hartwig and Styan, 1986)

$$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V} \leq_* \boldsymbol{V}\boldsymbol{A}\boldsymbol{V} \qquad \text{for all} \quad i = 1,\cdots,k, \tag{33}$$

which in turn implies the spectral partial ordering

$$\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V} \leq_\lambda \boldsymbol{V}\boldsymbol{A}\boldsymbol{V} \qquad \text{for all} \quad i = 1,\cdots,k, \tag{34}$$

cf. Baksalary and Hauke (1987), and so we see that (31) implies the first part of (d).

Moreover, from (33) it follows that

$$(\boldsymbol{V}\boldsymbol{A}\boldsymbol{V})^+ = \sum_{i=1}^{k}(\boldsymbol{V}\boldsymbol{A}_i\boldsymbol{V})^+, \tag{35}$$

and so (19) leads to

$$(\boldsymbol{VAV})^{+}\boldsymbol{VAW} = \sum_{i=1}^{k}(\boldsymbol{VA}_i\boldsymbol{V})^{+}\boldsymbol{VAW} = \sum_{i=1}^{k}(\boldsymbol{VA}_i\boldsymbol{V})^{+}\boldsymbol{VA}_i\boldsymbol{W}. \tag{36}$$

Consequently, combining (28) with (35) and (36) yields (31). The condition (28) includes the statements

$$\boldsymbol{VA}_i\boldsymbol{\mu} \leq_C \boldsymbol{VA}_i\boldsymbol{V} \qquad \text{for all} \quad i = 1, \cdots, k, \tag{37}$$

and

$$\boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{\mu} = \boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{V}(\boldsymbol{VA}_i\boldsymbol{V})^{+}\boldsymbol{VA}_i\boldsymbol{\mu} \qquad \text{for all} \quad i = 1, \cdots, k. \tag{38}$$

Theorem 19.1 of Marsaglia and Styan (1974) assures that if (37) holds, then

$$\begin{aligned}\operatorname{rank}(\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{W}) =& \operatorname{rank}\begin{pmatrix}\boldsymbol{VA}_i\boldsymbol{V} & \boldsymbol{VA}_i\boldsymbol{\mu} \\ \boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{V} & \boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{\mu}\end{pmatrix} \\ =& \operatorname{rank}(\boldsymbol{VA}_i\boldsymbol{V}) + \operatorname{rank}(\boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{\mu} - \boldsymbol{\mu}'\boldsymbol{A}_i\boldsymbol{V}(\boldsymbol{VA}_i\boldsymbol{V})^{+}\boldsymbol{VA}_i\boldsymbol{\mu}).\end{aligned} \tag{39}$$

Consequently, on account of (38) it follows that (28) implies the second part of (d). This concludes the proof of the part "(a) and (b) $\Longrightarrow$ (c) and (d)."

Conversely, it is clear that (29) and the first part of (d) together imply (27). Further, arguing similarly as in the proof of Theorem 2 in Baksalary and Hauke (1987), it can be shown that, in the particular case specified by (29), the spectral orderings in the first part of (d) are equivalent to the corresponding star partial orderings in (33). In view of Theorem 1 of Hartwig (1981) and Theorem 15 of Marsaglia and Styan (1974), this implies (31). On the other hand, the second part of (d) implies

$$\mathcal{C}(\mathbf{VA}_i\boldsymbol{W}) = \mathcal{C}(\mathbf{VA}_i\boldsymbol{V}) \qquad \text{for all} \quad i = 1, \cdots, k, \tag{40}$$

and thus pre- and post-multiplying (31) by $\boldsymbol{W}'\boldsymbol{A}_i\boldsymbol{V}(\boldsymbol{VA}_i\boldsymbol{V})^{+}$ and $(\boldsymbol{VA}_j\boldsymbol{V})^{+}\boldsymbol{VA}_j\boldsymbol{W}$, respectively, leads to (19). Finally, in view of (39) and the formula (8.7) in Marsaglia and Styan (1974), it follows that if the second part of (d) is satisfied, then (36) and (37) must hold, thus leading to (28). The proof is complete. ∎

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RECTOR: TADEUSZ KOTARBIŃSKI PEDAGOGICAL UNIVERSITY
ALEJA WOJSKA POLSKIEGO 69
PL-65 077 ZIELONA GÓRA, POLAND

DEPT. OF MATHEMATICAL AND STATISTICAL METHODS
AGRICULTURAL UNIVERSITY
UL. WOJSKA POLSKIEGO 28
PL-60 637 POZNAŃ, POLAND

DEPT. OF MATHEMATICS AND STATISTICS
MCGILL UNIVERSITY, BURNSIDE HALL
805 OUEST, RUE SHERBROOKE
MONTRÉAL, QUÉBEC, CANADA H3A 2K6