Stochastic Orders and Decision under Risk
IMS Lecture Notes – Monograph Series (1991)

LORENZ ORDERING OF ORDER STATISTICS

BY BARRY C. ARNOLD AND JOSE A. VILLASEÑOR

University of California, Riverside, and Colegio de Postgraduados, Chapingo

The Lorenz order is a natural tool for comparison of the variability of non-negative random variables. In many contexts (especially reliability) variability comparison of order statistics is of interest. A survey is provided of available results regarding the Lorenz order of order statistics.

1. Introduction. Let $X_{i:n}$ denote the ith order statistic from a sample of size n from the distribution $F_X(i = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$. Yang (1982) and David and Groeneveld (1982) have identified certain situations in which the $X_{i:n}$'s are ordered with respect to variability as measured by the variance. In many contexts, a more basic variability ordering is that provided by the Lorenz order. In a reliability context, the lifetime of a k out of n system is $X_{n-k+1:n}$, the waiting time until less than k components are still functioning. Attention is focussed on controlling not only the mean life but also the variability. Predictable life length is desirable. The engineering may choose how to build his k of n system out of exchangeable components. He is consequently concerned with choosing k and n to reduce variability in the life length, subject to given mean life constraints; a natural scenario for Lorenz order comparisons.

Lorenz ordering relations among uniform order statistics are first developed. Analogous relations are developed involving more general common distributions for the X's. We begin with a brief survey of characterizations of the Lorenz order.

2. The Lorenz Order. A convenient survey of relevant material is available in Arnold (1987).

Let $\mathcal{L}$ denote the class of all non-negative random variables with finite positive expectations. For a random variable X in $\mathcal{L}$ with distribution function

---

AMS 1980 Subject Classification: primary 60E05; secondary 62G30

Key words and phrases: Variability comparisons, density crossings, uniform order statistics, Lorenz curve.

F_X, we define its inverse distribution function F_X^{-1} by

$$F_X^{-1}(y) = \sup\{x : F_X(x) \le y\}. \tag{2.1}$$

The Lorenz curve L_X associated with the random variable X is then defined by

$$L_X(u) = \int_0^u F_X^{-1}(y)dy \Big/ \int_0^1 F_X^{-1}(y)dy, \ u \in [0,1] \tag{2.2}$$

(cf. Gastwirth (1971)). The Lorenz partial order $\le_L$ on $\mathcal{L}$ is defined by

$$X \le_L Y \Leftrightarrow L_X(u) \ge L_Y(u). \ \forall\, u \in [0,1]. \tag{2.3}$$

If $X \le_L Y$ we say that X exhibits no more inequality than Y (in the Lorenz sense).

The following sufficient conditions for Lorenz ordering are useful for our purposes.

THEOREM 2.1. (Strassen (1965)). *Suppose $X, Y \in \mathcal{L}$, $X \le_L Y$ if and only if there exist random variables Y', Z' defined on some probability space such that $Y \stackrel{d}{=} Y'$ and $X \stackrel{d}{=} cE(Y' \mid Z')$ for some $c > 0$. (Here and henceforth $\stackrel{d}{=}$ denotes "has the same distribution as").*

THEOREM 2.2. (Shaked (1980)). *Suppose that X and Y are absolutely continuous members of $\mathcal{L}$ with $E(X) = E(Y)$ and densities f_X and f_Y. A sufficient condition for $X \le_L Y$ is that $f_X(x) - f_Y(x)$ changes sign twice on $(0, \infty)$ and the sequence of signs of $f_X - f_Y$ is $-, +, -$.*

A useful if trite observation is that $X \le_L Y$ implies and is implied by $cX \le_L dY$ for any $c, d \in (0, \infty)$. Rather than compare two random variables X and Y with possibly different expectations, it is sometimes convenient to compare $E(Y)X$ and $E(X)Y$ which necessarily have equal expectations.

THEOREM 2.3. *Suppose that $g : \mathbb{R}^{+2} \to \mathbb{R}^+$ is such that $g(z,x)/x \uparrow$ as $x \uparrow$ for every z. Suppose that X and Z are independent random variables with $X \in \mathcal{L}$ and $Y = g(Z, X) \in \mathcal{L}$, then $X \le_L Y$.*

PROOF. To show that $X \le_L Y$ it is enough to show that $E(\phi(E(Y)X) \le E(\phi(E(X)Y))$ for every convex ϕ. For each fixed $z > 0$, define $Y_z = g(z, X)$. Since $g(z,x)/x \uparrow$ as $x \uparrow$, it follows from Fellman (1976) that $X \le_L Y_z$, and so for every convex ϕ we have $E(\phi(E(X)Y_z) \ge E(\phi(E(Y_z)X))$. We may then argue as follows

$$E(\phi(E(X)Y)) = \int_0^\infty E(\phi(E(X)Y_z))f_Z(z)dz$$

$$
\begin{aligned}
&\geq \int_0^\infty E(\phi(E(Y_z)X))f_Z(z)dz \\
&= \int_0^\infty \int_0^\infty \phi(xE(Y_z))f_X(x)dx f_Z(z)dz \\
&= \int_0^\infty \int_0^\infty \phi(xE(Y_z))f_Z(z)dz f_X(x)dx \\
&\geq \int_0^\infty \phi\left(\int_0^\infty xE(Y_z)f_Z(z)dz\right) f_X(x)dx \text{ (by Jensen's inequality)} \\
&= \int_0^\infty \phi(xE(Y))f_X(x)dx \\
&= E(\phi(E(Y)X)). \qquad \blacksquare
\end{aligned}
$$

3. Lorenz Ordering of Uniform Order Statistics. We will use the notation $U_{i:n}$ to represent the i'th order statistic of a sample of n i.i.d. copies of a random variable U which is uniformly distributed on the interval $(0, 1)$. The following Lorenz order relationships will be shown to hold among $U_{i:n}$'s.

$$U_{i+1:n} \leq_L U_{i:n} \quad \forall\ i, n \tag{3.1}$$

$$U_{i:n} \leq_L U_{i:n+1} \quad \forall\ i, n \tag{3.2}$$

$$U_{n-j+1:n+1} \leq_L U_{n-j:n} \quad \forall\ j, n \tag{3.3}$$

$$U_{n+2:2n+3} \leq_L U_{n+1:2n+1} \quad \forall\ n. \tag{3.4}$$

It is not always easy to intuitively grasp the concept of Lorenz ordering of order statistics. It must be recalled that it is a "measure of variability" that refers to scaled variables. Included in (3.1) to (3.4) are the observations: (i) larger order statistics in a sample exhibit less inequality than smaller ones within the same sample (from (3.1), (ii) Sample minima exhibit more inequality as sample size increases (from (3.2)), (iii) Sample maxima exhibit less inequality as sample size increases and (iv) Sample medians exhibit less inequality as sample size increases. Certainly the last observation has a ring of plausibility, the others are perhaps less compelling. Note that (3.1) is actually a consequence of (3.2) and (3.3).

Relations (3.1) and (3.2) are simple consequences of Strassen's theorem (Theorem 2.1, above). First recall that $U_{i:n} \sim Beta(i, n + 1 - i)$ and that if Z_1, Z_2 are independent Beta random variables with parameters $(a, b - a)$ and $(b, c - b)$ then $Z_1 Z_2 \sim Beta(a, c - a)$. This implies that

$$U_{i:n} \stackrel{d}{=} U_{i:i} U'_{i+1:n} \tag{3.5}$$

and

$$U_{i:n+1} \stackrel{d}{=} U'_{i:n} U_{n+1:n+1}, \tag{3.6}$$

where the r.v.'s on the right hand sides are independent. In (3.5) and (3.6) and henceforth we adopt the convention that if two random variables are labelled identically except that one has a "prime" appended then they are i.i.d. copies of the same random variable (i.e. X and X' are i.i.d. with common distribution function $F_X(x)$). It follows that

$$\left(\frac{i}{i+1}\right) U_{i+1:n} \stackrel{d}{=} E\left(U_{i:n} \mid U'_{i+1:n}\right) \tag{3.7}$$

and

$$\left(\frac{n+1}{n+2}\right) U_{i:n} \stackrel{d}{=} E\left(U_{i:n+1} \mid U'_{i:n}\right) \tag{3.8}$$

Results (3.1) and (3.2) follow from (3.7) and (3.8) by a direct application of Theorem 2.1 (using the fact that the Lorenz order is scale invariant).

Relations (3.3) and (3.4) are not transparently amenable to a proof using Strassen's theorem. We are grateful to a referee for the following Strassen type argument which proves (3.3). Consider n ordered uniform observations $U_{1:n} < U_{2:n} < \ldots < U_{n:n}$ and an additional independent observation U_{n+1}. Combined they form a sample of size $n+1$ whose $n+1-j$th order statistic will be denoted by $U_{n+1-j:n+1}$. Note that the event $\{U_{n+1} > U_{n+1-j:n+1}\}$ depends only on the rank of U_{n+1} and is independent of the vector of order statistics $\{U_{1:n+1}, \ldots, U_{n+1:n+1}\}$. Observe that

$$U_{n-j:n} = \begin{cases} U_{n+1-j:n+1} & \text{if } U_{n+1} < U_{n+1-j:n+1} \\ U_{n-j:n+1} & \text{if } U_{n+1} \geq U_{n+1-j:n+1} \end{cases}$$

and that

$$Pr(U_{n+1} \geq U_{n+1-j:n+1}) = (j+1)/(n+1).$$

Consequently

$$\begin{aligned} &E(U_{n-j:n} \mid U_{n+1-j:n+1}) \\ &\quad = \left(1 - \frac{j+1}{n+1}\right) U_{n+1-j:n+1} + \frac{j+1}{n+1} E(U_{n-j:n+1} \mid U_{n+1-j:n+1}) \\ &\quad = \left(1 - \frac{j+1}{n+1}\right) U_{n+1-j:n+1} + \left(\frac{j+1}{n+1} \cdot \frac{n-j}{n+1-j}\right) U_{n+1-j:n+1} \\ &\quad = \left(\frac{n-j}{n+1} \cdot \frac{n+2}{n+1-j}\right) U_{n+1-j:n+1} \end{aligned}$$

from which (3.3) follows. Relation (3.4) (as well as (3.1) – (3.3)) is verifiable by a straightforward if tedious application of the density crossing theorem

(Theorem 2.2). Details are documented in Arnold and Villaseñor (1984). A key ingredient in the argument is that unscaled Beta densities with differing parameters cross at most twice (a result carefully enunciated in a different context by Boland, Proschan and Tong (1988)).

4. Examples and Counterexamples. Observe that in expressions (3.1) – (3.3) it is always the order statistic with smaller mean which exhibits the most inequality in the sense of Lorenz. One might conjecture that a sufficient condition for $U_{i:n} \leq_L U_{j:m}$ is that $i/(n+1) \geq j/(m+1)$. Such a result cannot be true since, were it to be the case, we would have $U_{1:1} \leq_L U_{2:3} \leq_L U_{1:1}$ which would imply $U_{1:1} \stackrel{d}{=} cU_{2:3}$, which is clearly false.

A second question is whether results (3.1) – (3.4) will hold for order statistics from any parent distribution. For a simple source of counterexamples consider samples from a discrete uniform distribution over the points 1, 2, 3, 4. By direct evaluation of Lorenz curves it may be verified that the Lorenz curves of $X_{1:3}$ and $X_{2:3}$ intersect (so (3.1) fails), the Lorenz curves of $X_{1:2}$ and $X_{1:3}$ intersect (so (3.2) fails) and the Lorenz curves of $X_{2:2}$ and $X_{3:3}$ intersect (so (3.3) fails). Relation (3.4) will fail for many asymmetric parent distributions.

Not only is it possible to have (3.1), (3.2) and (3.3) fail to hold for some non-uniform parent distribution, it is actually possible to have the reverse inequalities hold true. For example if we consider samples drawn from the distribution $F(x) = 1 - x^{-2}$, $x > 1$ (a classical Pareto distribution), then by direct computation of the corresponding Lorenz curves one may verify that

$$X_{1:2} \leq_L X_{1:1} \leq_L X_{2:2}$$

i.e. inequalities (3.1), (3.2) and (3.3) are reversed.

5. Lorenz Ordering of Order Statistics from a Non-Uniform Parent Distribution. From the discussion in Section 4, we cannot expect (3.1) – (3.4) to hold for arbitrary parent distributions. Using Theorems 2.1 – 2.3 it is possible to develop sufficient conditions for some of the inequalities included in (3.1) – (3.4). Sufficient conditions for the reverse inequalities are also sometimes obtainable. First we consider results obtainable using Theorem 2.1.

Observe that if $X = cU^\delta$ where U is uniform $(0,1), c > 0$ and $\delta > -1$ [in order to have $E(X) < \infty$] then

$$X_{i:n} \stackrel{d}{=} c(U_{i:n})^\delta \text{ if } \delta > 0 \tag{5.1}$$

and

$$X_{i:n} \stackrel{d}{=} c(U_{n-i+1:n})^\delta \text{ if } \delta \epsilon (-1, 0) \tag{5.2}$$

(the case $\delta = 0$ is of no interest). We then obtain

THEOREM 5.1. *(a) If X has a power function (γ) distribution [i.e. $F_X(x) = [x/c]^\gamma, 0 \le x \le c, \gamma > 0$] then*

$$X_{i+1:n} \le_L X_{i:n} \quad \forall\, i, n \tag{5.3}$$

and

$$X_{i:n} \le_L X_{i:n+1} \quad \forall\, i, n. \tag{5.4}$$

(b) *If X has a classical Pareto distribution [i.e. $F(x) = 1 - (x/c)^{-\alpha}$, $c \le x < \infty, \alpha > 1$] then*

$$X_{i:n} \le_L X_{i+1:n} \quad \forall\, i, n \tag{5.5}$$

and

$$X_{n-j:n} \le_L X_{n-j+1:n+1} \quad \forall\, j, n. \tag{5.6}$$

PROOF. To verify (5.3), note that (3.5) implies

$$\begin{aligned} X_{i:n} &\stackrel{d}{=} c\left(U_{i:i} U'_{i+1:n}\right)^{1/\gamma} \\ &\stackrel{d}{=} \left(U_{i:i}\right)^{1/\gamma} X'_{i+1:n} \end{aligned}$$

where the random variables on the right hand side are independent. Equation (5.3) is then a direct consequence of Theorem 2.1.

Analogous arguments yield (5.4) – (5.6).

The key ingredient in the proof of Theorem 5.1 was the simple form of the inverse distribution function in the power function and Pareto cases. For more complicated inverse distribution functions, Theorem 2.1 appears to be difficult to apply. In fact, there is a paucity of well known distributions for which F^{-1} has an analytic expression. It is more common to encounter analytic expressions for the corresponding density, so that one might hope for more results using Theorem 2.2. This hope is buttressed by our recollection that Theorem 2.2 provided the most fertile source of results in the case of a uniform parent distribution. In order to apply Theorem 2.2 to order statistics $X_{i:n}, X_{j:m}$ from a general parent distribution we need to evaluate $\mu_{i:n} = E(X_{i:n})$ and $\mu_{j:m} = E(X_{j:m})$ and then consider sign changes in the expression

$$f_{X_{i:n}}(x) - \frac{\mu_{j:m}}{\mu_{i:n}} f_{X_{j:m}}\left(x \frac{\mu_{j:m}}{\mu_{i:n}}\right). \tag{5.7}$$

One case in which (5.7) is relatively easy to study is that in which $\mu_{j:m} = \mu_{i:n}$. If X has a symmetric distribution on $[0, c]$ then all sample medians have the same mean as does X, i.e.

$$\mu_{1:1} = \mu_{n+1:2n+1}, \quad \forall\, n. \tag{5.8}$$

Actually a non-negative random variable has property (5.8) if and only if the parent distribution is symmetric on $[0, c]$ for some finite c. Assume we have such a symmetric parent distribution and we wish to compare (in the Lorenz order) $X_{n+1:2n+1}$ and $X_{n+2:2n+3}$. The following argument was supplied in Arnold and Villaseñor (1986). Since $\mu_{n+1:2n+1} = \mu_{n+2:2n+3}$ we need only consider

$$\frac{f_{n+2:2n+3}(x)}{f_{n+1:2n+1}(x)} = F(x)[1-F(x)]\frac{(4n+6)}{(n+1)} \tag{5.9}$$

Clearly this is > 1 for intermediate values of x and consequently, from Theorem 2.2,

$$X_{n+2:2n+3} \leq_L X_{n+1:2n+1} \quad \forall\ n \tag{5.10}$$

assuming only that the parent distribution is symmetric.

As a consequence of (5.10) we have

$$\operatorname{var}(X_{n+2:2n+3}) \leq \operatorname{var}(X_{n+1:2n+1}). \tag{5.11}$$

Yang (1982) observed that $\operatorname{var}(X_{n+1:2n+1}) \leq \operatorname{var}(X_{1:1})$ for any continuous distribution, symmetric or not. An example in which (5.10) does not hold in the absence of symmetry is provided by the parent distribution with inverse $F^{-1}(y) = y^{.001}$, $0 < y < 1$. In this case, one may verify directly that $X_{2:3} \not\leq_L X_{1:1}$.

The sign change property (5.7) will be useful if we are dealing with cases in which the ratio of the expectations of order statistics is simple and in which the parent distribution function is uncomplicated. Again we fall back on the power function, Pareto and exponential distribution as illustrative examples.

Power function distribution. Here $F(x) = \left(\frac{x}{c}\right)^\gamma$, $0 \leq x \leq c$, $\gamma > 0$; $f(x) = \frac{\gamma}{c}\left(\frac{x}{c}\right)^{\gamma-1}$, $0 < x < c$; $F^{-1}(y) = cy^{\frac{1}{\gamma}}$ and $\frac{\mu_{i+1:n}}{\mu_{i:n}} = \left(1 + \frac{1}{i\gamma}\right)$. In order to verify that $X_{i:n} \geq_L X_{i+1:n}$ we need to consider.

$$\frac{\left(1+\frac{1}{i\gamma}\right) f_{X_{i+1:n}}\left(\left(1+\frac{1}{i\gamma}\right)x\right)}{f_{X_{i:n}}(x)} \quad \text{for } x \in \left(0, \left(1+\frac{1}{i\gamma}\right)^{-1} c\right).$$

Let $\eta = 1 + \frac{1}{i\gamma} > 1$. Then we have

$$\frac{\eta f_{X_{i+1:n}}(\eta x)}{f_{X_{i:n}}(x)} = \frac{\eta\frac{n!}{i!(n-i-1)!}\left(\frac{\eta x}{c}\right)^{\gamma i}\left(1-\left(\frac{\eta x}{c}\right)^\gamma\right)^{n-i-1}\gamma\left(\frac{\eta x}{c}\right)^{\gamma-1}\frac{\eta}{c}}{\frac{n!}{(i-1)!(n-i)!}\left(\frac{x}{c}\right)^{\gamma(i-1)}\left(1-\left(\frac{x}{c}\right)^\gamma\right)^{n-i}\gamma\left(\frac{x}{c}\right)^{\gamma-1}\frac{1}{c}}.$$

This is > 1 for intermediate values of x. We thus confirm (5.3). In similar fashion we may verify that for a power function parent distribution we have

$$X_{i:n} \leq_L X_{i:n+1} \quad \forall\ i, n$$

and

$$X_{n-j+1:n+1} \leq_L X_{n-j:n} \quad \forall\, j, n.$$

Thus (3.1) – (3.3) hold for the power function distribution.

Analogous density crossing arguments may be applied in the case where the X_i's have a Pareto distribution. In this case one finds that (3.1) – (3.3) are reversed.

Consider now the case of an exponential parent distribution. Here $F(x) = 1-e^{-\frac{x}{c}}$, $x > 0$; $f(x) = \frac{1}{c}e^{-\frac{x}{c}}$; $F^{-1}(y) = -c\log(1-y)$ and $\frac{\mu_{i+1:n}}{\mu_{i:n}} = \left(\sum_{j=1}^{i+1} \frac{1}{n-j+1}\right) / \left(\sum_{j=1}^{i} \frac{1}{n-j+1}\right) = \eta$ say. In order to verify that $X_{i+1:n} \leq_L X_{i:n}$ we need to consider

$$\frac{\eta f_{X_{i+1:n}}(\eta x)}{f_{X_{i:n}}(x)} \propto \frac{\left(1-e^{-\eta x/c}\right)^{i} e^{-\eta x(n-i-1)/c} e^{-\eta x/c}}{\left(1-e^{-x/c}\right)^{i-1} e^{-x(n-i)/c} e^{-x/c}}.$$

In general this is a difficult expression to deal with. It is considerably simplified if we consider the special case $n = 2, i = 1$, in which case $\eta = 3$. Here, with $c = 1$ without loss of generality,

$$\frac{3 f_{X_{2:2}}(3x)}{f_{X_{1:2}}(x)} \propto \left(1-e^{-3x}\right) e^{-x}$$

which is clearly > 1 for intermediate values of e^{-x} and hence of x. We thus have in the case of an exponential parent distribution

$$X_{2:2} \leq_L X_{1:2}.$$

Recently Wilfing (1990) has been able to verify that, for exponential variables, $X_{n+1:2n+1} \leq_L X_{n:2n-1}$ using a density crossing argument. But in general density crossing arguments are difficult to implement.

An alternative approach to the exponential problem involves the representation of $X_{i:n}$ as a linear combination of i.i.d. exponential random variables. This technique was used by Karlin and Rinott (1988) to compare exponential means and medians. Subsequently in Arnold and Nagaraja (1990) the technique was used to obtain the following general result.

THEOREM 5.2. *If X has an exponential distribution and $i \leq j$ then the following are equivalent.*

(i) $X_{j:m} \leq_L X_{i:n}$

(ii) $(n-i+1)E(X_{i:n}) \leq (m-j+1)E(X_{j:m})$.

As a consequence of this theorem, one finds that for an exponential parent, (3.3) and (3.4) hold, (3.2) is reversed and (3.1) holds for values of i that are not too large. A sufficient condition for (3.1) to hold (i.e. $X_{i+1:n} \leq_L X_{i:n}$)

is that $i \leq (1 - e^{-1})n$. For example if $n = 4$ one finds $X_{3:4} \leq_L X_{2:4} \leq_L X_{1:4}$ but $X_{3:4}$ and $X_{4:4}$ are not Lorenz comparable.

Finally we inquire into the utility of Theorem 2.3 to verify Lorenz ordering among order statistics. We consider first sufficient conditions for $X_{i+1:n} \leq_L X_{i:n}$. From (3.5) we may conclude that

$$X_{i:n} = F^{-1}\left(U_{i:i} F\left(X'_{i+1:n}\right)\right)$$

where the two random variables on the right are independent. Theorem 2.3 may be applied and a sufficient condition for $X_{i+1:n} \leq_L X_{i:n}$ is that

$$F^{-1}\left(uF(x)\right)/x \uparrow \text{ as } x \uparrow \tag{5.12}$$

for every $u \in (0,1)$. It is readily verified that (5.12) is trivially satisfied when F corresponds to a uniform or power function distribution. A simple sufficient condition for (5.12) is that

$$xF'(x)/F(x) \uparrow \text{ as } x \uparrow . \tag{5.13}$$

An example in which (5.13) holds and hence $X_{i+1:n} \leq_L X_{i:n}$ is provided by the distribution function

$$F(x) = e^x - 1, \quad 0 \leq x \leq \log 2. \tag{5.14}$$

Condition (5.13) is sufficient but not necessary. For example, in the case of an exponential parent distribution, (5.13) fails yet $X_{2:2} \leq_L X_{1:2}$ (as we saw earlier). By analogous arguments using (3.6) we may verify that conditions (5.12) and (5.13) are also sufficient to ensure that $X_{i:n} \leq_L X_{i:n+1}$.

6. Concluding Remarks. The problem of identifying necessary and sufficient conditions for Lorenz ordering among order statistics from a general parent distribution remains open. We conclude with a final example to show how fragile the Lorenz order is. Consider a sample of size 2 from a uniform $(0,1)$ distribution. We know that $U_{2:2} \leq_L U_{1:2}$ yet if we consider $X_i = U_i + \epsilon$, $i = 1,2$ (i.e. uniform $(\epsilon, 1+\epsilon)$ random variables) then one may verify that $X_{1:2}$ and $X_{2:2}$ are not Lorenz comparable: no matter how small ϵ may be.

Acknowledgement. We are grateful for the suggestions of the anonymous referees which have helped to improve and clarify the presentation of these results.

REFERENCES

ARNOLD, B. C. (1987). *Majorization and the Lorenz Order: A Brief Introduction.* Lecture Notes in Statist., Vol. 43, Springer-Verlag, Berlin.

ARNOLD, B. C. and NAGARAJA, H. N. (1990). Lorenz ordering of exponential order statistics. *Statist. Probab. Lett.* To appear.

ARNOLD, B. C. and VILLASEÑOR, J. A. (1984). Lorenz ordering of order statistics. Technical Report 131, Department of Statistics, University of California, Riverside.

ARNOLD, B. C. and VILLASEÑOR, J. A. (1986). Lorenz ordering of means and medians. *Statist. Probab. Lett.* **4**, 47–49.

BOLAND, P. J., PROSCHAN, F. and TONG, Y. L. (1988). Crossing properties of mixture distributions. Technical Report M-796, Department of Statistics, Florida State University, Tallahassee.

DAVID, H. A. and GROENEVELD, R. A. (1982). Measures of local variation in a distribution: expected length of spacings and variances of order statistics. *Biometrika* **69**, 227–232.

FELLMAN, J. (1976). The effect of transformations on Lorenz curves. *Econometrica* **44**, 823–824.

GASTWIRTH, J. L. (1971). A general definition of the Lorenz curve. *Econometrica* **39**, 1037–1039.

KARLIN, S. and RINOTT, Y. (1988). Medians-means comparisons. Paper #24 (private communication).

SHAKED, M. (1980). On mixtures from exponential families. *J. Roy. Statist. Soc., Ser. B* **42**, 192–198.

STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Statist.* **24**, 423–439.

WILFLING, B. (1990). Concepts of Lorenz-ordering with a special regard to the Lorenz ordering of order statistics. Diploma Thesis, Fakultät Statistik, Universität Dortmund, Germany.

YANG, H. (1982). On the variances of median and some other statistics. *Bull. Inst. Math. Acad. Sinica* **10**, 197–204.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
RIVERSIDE, CALIFORNIA 92521

COLEGIO DE POSTGRADUADOS
CHAPINGO, MEXICO