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PRESERVATION AND ATTENUATION OF INEQUALITY AS MEASURED BY THE LORENZ ORDER

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The Lorenz order is defined on the class of all non-negative random variables with positive finite expectations. Interest has often focussed on characterization of transformations, defined on the space of such random variables, which either preserve or attenuate inequality. Results of this genre involving deterministic and random transformations are surveyed. In some settings distributions are changed by weightings (in the sense of Rao (1965)) or by mixing, rather than by transformations. Preservation and attenuation results in such scenarios are summarized.

1. Introduction. Let $\mathcal{L}$ denote the class of all non-negative random variables whose expectations exist and are strictly positive. With any random variable X in $\mathcal{L}$ with distribution function F_X, there is associated a Lorenz curve L_X defined by

$$L_X(u) = \int_0^u F_X^{-1}(s)ds \Big/ \int_0^1 F_X^{-1}(s)ds, \quad u \in [0,1] \tag{1.1}$$

where $F_X^{-1}(s) = \sup\{x : F_X(x) \le s\}$. This definition of the Lorenz curve can be traced back explicitly to Gastwirth (1971). The Lorenz partial order on $\mathcal{L}$ denoted by $\le_L$ is defined by

$$X \le_L Y \Longleftrightarrow L_X(u) \ge L_Y(u) \ \forall\ u \in [0,1]. \tag{1.2}$$

If $X \le_L Y$ then we say that X exhibits no more inequality than does Y. We define the strong Lorenz order, $X <_L Y$ by

$$X <_L Y \Longleftrightarrow X \le_L Y \ \text{ and } \ Y \not\le_L X,$$

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in such a case we say that X exhibits less inequality than Y. The Lorenz order is the natural mathematical abstraction of Lorenz's (1905) comparison of income distributions via nested Lorenz curves. Lorenz's original order relating to finite populations can be subsumed within our Lorenz order on non-negative integrable random variables. A finite population with elements $x_1, x_2, \ldots, x_m$ can be identified with a discrete uniform random variable which takes on each of the values $x_1, x_2, \ldots, x_m$ with probability $1/m$. The Lorenz order is closely related to majorization. If two vectors $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$ have the same dimension n and satisfy $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then $\underline{x} \leq_M \underline{y}$ ($\underline{x}$ is majorized by $\underline{y}$) if the corresponding discrete uniform random variables are Lorenz ordered. See Marshall and Olkin (1979) for an extensive survey of the properties and applications of the majorization order.

Since the distribution of an arbitrary member of $\mathcal{L}$ can be represented as a weak limit of a sequence of discrete uniform distributions, it is possible to prove some results for the Lorenz order by limiting arguments applied to analogous majorization results. Generally speaking a direct general proof is more efficient. There are fundamental differences between majorization and the Lorenz order. For example (see Section 3) it is possible to find functions which preserve the Lorenz order but do not preserve majorization.

A useful characterization of the Lorenz order is provided by the following theorem due in essence to Hardy, Littlewood and Polya (1929) and Karamata (1932).

THEOREM 1.1 (HLP-Karamata). *$X \leq_L Y$ if and only if for every continuous convex function g we have*

$$E\left(g\left(\frac{X}{E(X)}\right)\right) \leq E\left(g\left(\frac{Y}{E(Y)}\right)\right).$$

An alternative characterization highlights the role of averaging in the Lorenz order. In a majorization context, where it involves doubly stochastic matrices, it is attributable to Hardy, Littlewood and Polya (1929). In a more abstract setting, the result is usually associated with Strassen (1965).

THEOREM 1.2. (HLP-Strassen). *$X \leq_L Y$ if and only if there exist random variables Y', Z' with $Y \stackrel{d}{=} Y'$ and $X \stackrel{d}{=} cE(Y'|Z')$ for some $c > 0$. (Here $\stackrel{d}{=}$ indicates equality in distribution.)*

The present paper surveys results dealing with preservation and attenuation of inequality under various operations on $\mathcal{L}$ (transformations, mixings and weightings). The results are treated in the abstract setting but are most clearly motivated by envisioning operations on finite populations of income earners.

The result of a taxation or welfare policy on a finite population of incomes is to replace incomes $x_1, x_2, \ldots, x_m$ by new incomes $g(x_1), g(x_2), \ldots, g(x_m)$. Attention is naturally focused on functions g which attenuate inequality and functions which preserve inequality. In the context of the Lorenz order on $\mathcal{L}$, we seek to characterize functions g for which (i) $X \leq_L Y \Rightarrow g(X) \leq_L g(Y)$ (inequality preserving) and (ii) $g(X) \leq_L X \;\; \forall X \in \mathcal{L}$ (inequality attenuating). An obvious inequality preserving transformation is $g(x) = cx$ for some $c > 0$. There are a few others as we shall see in Section 3.

The classic papers on inequality attenuation are Fellman (1976) and Jakobsson (1976) (analogous results in the *-ordering context are to be found in Marshall, Olkin and Proschan (1967)). Attenuation results are covered in Section 4.

Weighted distributions were introduced by Rao (1965). In the present context we can envision situations where the propensity of an individual to file an income tax form may well depend on the size of the individual's income. Thus, rather than observing random variables with density $f(x)$ we actually observe random variables with density proportional to $g(x)f(x)$ where $g(x)$ is a weighting function (the particular case of size biased sampling has received much attention in the literature, here $g(x) \propto x$). It is quite clear that weightings will affect inequality as measured by the Lorenz order. It is then natural to seek characterizations of inequality preserving weightings (Section 5) and inequality attenuating weightings (Section 6).

The third operation on $\mathcal{L}$ that we wish to investigate is that of mixing. What if we combine two finite populations of income earners into one larger population. How does the combined population compare with the component populations with regard to inequality? Some results in this direction are available and are described in Section 7. Interesting partial results are available regarding the effects of differential tax policies within subpopulations.

Random taxation is discussed briefly in Section 8. Certain results on misreporting of income can be subsumed in such a scenario.

2. Two Useful Lemmas. Arnold and Villaseñor (1985) provided the following elementary observations regarding the Lorenz ordering of distributions with common two point support.

LEMMA 2.1. *Suppose $0 < x_1 < x_2$. If we have random variables X and Y defined by*

$$P(X = x_1) = p, \quad P(X = x_2) = 1 - p,$$
$$P(Y = x_1) = p', \quad P(Y = x_2) = 1 - p',$$

then X and Y are not comparable in the Lorenz ordering except in the trivial cases when $p = p', pp' = 0$ or $(1 - p)(1 - p') = 0$.

LEMMA 2.2. *Suppose $x > 0$. If we have two random variables X and Y defined by*

$$P(X = 0) = p, \quad P(X = x) = 1 - p,$$
$$P(Y = 0) = p', \quad P(Y = x) = 1 - p',$$

then $p \leq p' \Rightarrow X \leq_L Y$.

These results are readily verified by sketching the corresponding Lorenz curves.

3. Inequality Preserving Transformations. The basic reference here is Arnold and Villaseñor (1985). Denote by $\mathcal{G}$ the class of all inequality preserving transformations, thus,

$$\mathcal{G} = \{g : X \leq_L Y \Rightarrow g(X) \leq_L g(Y)\}. \tag{3.1}$$

In order for the Lorenz inequality to be well defined, it must relate members of $\mathcal{L}$. Consequently, any member of $\mathcal{G}$ must map $[0, \infty)$ into $[0, \infty)$ and must also have the property that $X \in \mathcal{L} \Rightarrow g(X) \in \mathcal{L}$.

THEOREM 3.1. *The only functions which belong to $\mathcal{G}$ (i.e. which preserve the Lorenz order) are those of one of the three following forms.*

$$\begin{aligned} g_{1,a}(x) &= ax, x \geq 0 \;\; \text{where } \; a \in (0, \infty) \\ g_{2,b}(x) &= b, x \geq 0 \;\; \text{where } \; b \in (0, \infty) \\ g_{3,c}(x) &= 0, x = 0 \\ &= c, x > 0 \;\; \text{where } \; c \in (0, \infty). \end{aligned}$$

PROOF. The details may be found in Arnold and Villaseñor (1985). Repeated use is made of Lemmas 2.1 and 2.2. One interesting step in the argument is that, for $g \in \mathcal{G}$, if $g(0) > 0$ then g must be a constant function while if $g(0) = 0$ then g must be non-decreasing. Thus members of $\mathcal{G}$ are measurable and, in fact, linear on $(0, \infty)$. ∎

If instead one seeks functions which preserve majorization (instead of the Lorenz order), it is readily verified that such functions, if measurable, must be linear (cf. Marshall and Olkin (1979, p. 116)). Knowing that a function preserves the Lorenz order does not imply that it preserves majorization or even weak majorization. Although, there are no anomalous (i.e. non-measurable) functions which preserve the Lorenz order, the possible existence of non-measurable majorization preserving functions is not ruled out by the Marshall and Olkin result. Finally, functions of the form $g_{3,c}$ which are non-linear and measurable do preserve the Lorenz order but do not preserve majorization.

If, as is deemed desirable in many economics contexts, we seek functions which preserve strong Lorenz ordering, the picture is somewhat simpler. Evidently the only functions which preserve the strong Lorenz order are of the form $g_{1,a}(x) = ax, \ x \geq 0$ for $a \in (0, \infty)$.

4. Inequality Attenuating Transformations. Conditions that a tax policy must satisfy in order to guarantee that it will reduce inequality have been of interest for many years. Fellman (1976) and Jakobsson (1976) are names associated with the early characterization of inequality attenuating policies. In short, they must be progressive (condition (ii) below) and incentive preserving (condition (i) below). Early proofs often involved unnecessary regularity conditions or limited areas of applicability. In our general setting, dealing with mappings from $\mathcal{L}$ into $\mathcal{L}$, the result is expressible in the following form

THEOREM 4.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$. The following are equivalent.*

(i) $g(X) \leq_L X$ for every $X \in \mathcal{L}$

(ii) $g(x) > 0$ for every $x > 0, g(x)$ is monotone non-decreasing on $[0, \infty)$ and $g(x)/x$ is monotone non-increasing on $(0, \infty)$.

PROOF. (ii) $\Rightarrow$ (i). Suppose g satisfies (ii), $X \in \mathcal{L}$ and $Y = g(X)$. Since $g(x) > 0$ for $x > 0, E(X) > 0 \Rightarrow E(g(X)) > 0$. Since $g(x)$ is non-decreasing on $[0, \infty)$, we have $g(X) \leq g(1)$ when $X \leq 1$. Since $g(x)/x$ is non-increasing on $(0, \infty)$, we have $g(X)/X \leq g(1)/1$ or $g(X) \leq Xg(1)$ when $X \geq 1$. Thus $g(X) \leq (X + 1)g(1)$ and, hence, $E(g(X)) < \infty$. Thus, $Y = g(X) \in \mathcal{L}$.

Following Fellman, but without any regularity conditions, we can then write, for $u \in [0, 1]$,

$$L_Y(u) - L_X(u) = \int_0^u \left\{ g(F_X^{-1}(v)) - F_X^{-1}(v) \frac{E(Y)}{E(X)} \right\} \frac{dv}{E(Y)}. \tag{4.1}$$

Since $g(x)/x$ is non-increasing on $(0, \infty)$, the integral in (4.1) assumes its smallest value when $u = 1$. However $L_X(1) = L_Y(1) = 1$. Consequently $L_Y(u) \geq L_X(u) \forall \ u \in [0, 1]$, i.e. $g(X) = Y \leq_L X$.

The converse is proved by contradiction making use of Lemmas 2.1 and 2.2. Details are in the Appendix of Arnold and Villaseñor (1985). See also Arnold (1987). ∎

Analogous arguments allow one to characterize inequality accentuating transformations. One finds:

THEOREM 4.2. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$. The following are equivalent.*

(i) *$X \leq_L g(X)$ for every $X \in \mathcal{L}$.*

(ii) *$g(x) > 0$ for every $x > 0$, $g(x)$ is monotone non-decreasing on $[0, \infty)$ and $g(x)/x$ is monotone non-decreasing on $(0, \infty)$.*

Strong Lorenz order versions of Theorems 4.2 and 4.2 may be formulated. For example

THEOREM 4.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$. The following are equivalent.*

(i) $g(X) <_L X$ for every non-degenerate X in $\mathcal{L}$.

(ii) $g(x)$ is strictly increasing on $[0, \infty)$ and $g(x)/x$ is non-increasing on $(0, \infty)$.

Note. Fellman (1976) essentially provided the above sufficient conditions for a transformation to be inequality attenuating. An analogous result was proved earlier by Marshall, Olkin and Proschan (1967) in a majorization context. They proved that if ψ is star shaped on $[0, \infty)$ [i.e. if $\psi(x)/x$ is non-decreasing] then for a vector $\underline{x} \in \mathbb{R}_n^+$ we have

$$\left(\frac{\psi(x_1)}{\sum_{i=1}^n \psi(x_i)}, \ldots, \frac{\psi(x_n)}{\sum_{i=1}^n \psi(x_i)}\right) \leq_M \left(\frac{x_1}{\sum_{i=1}^n x_i}, \ldots, \frac{x_n}{\sum_{i=1}^n x_i}\right) \tag{4.2}$$

where $\leq_M$ denotes majorization. This of course is precisely Lorenz ordering for a discrete uniform random variable with possible values $x_1, x_2, \ldots, x_n$. The sufficiency part of Theorem 4.1 could then be obtained from (4.2) using a limiting argument. Kakwani (1980, p. 163) shows that $X \leq_L (\geq_L) g(X)$ if $g'(x)x/g(x) < (>) 1$ for $x > 0$ and $g'(x) > 0$ for $x \geq 0$. However these are evidently conditions essentially equivalent to those given in Theorems 4.1 and 4.2 above, under the additional regularity condition that g be differentiable. Nygård and Sandström (1981, pp. 176–186) provide an extensive discussion of these and related conditions; again under differentiability assumptions.

Eichhorn, Funke and Richter (1984) provided a careful discussion of inequality attenuation without differentiability assumptions. They credit Jakobsson (1976) with being perhaps the first to formulate the fact that Fellman's conditions were necessary and sufficient, although they note that Jakobsson's arguments did not prove the assertion. Eichhorn et al. (1984) restrict attention to finite populations $x_1, x_2, \ldots, x_n$ but suitable limiting arguments can be used to extend the result.

5. Inequality Preserving Weightings. As described in the introduction, we consider a situation where instead of observing random variables from a density proportional to $f(x)$ we actually observe random variables from a weighted version of the density. We use the following notation.

Suppose that $X \in \mathcal{L}$ and that g is a suitably measurable non-negative function (a weighting function). The g-weighted version of X, denoted X_g, is defined to be a random variable such that

$$P(X_g \le x) = \int_0^x g(y)dF_X(y)/E[g(X)] \tag{5.1}$$

provided $0 < E[g(X)] < \infty$. Note that if $X \in \mathcal{L}$ then in order to have $X_g \in \mathcal{L}$ we will require that $0 < E(g(X)) < \infty$ and $0 < E(Xg(X)) < \infty$.

We will denote $\mathcal{G}_1$, the class of all inequality preserving weightings, thus

$$\mathcal{G}_1 = \{g : \ X \le_L Y \Rightarrow X_g \le_L Y_g\}. \tag{5.2}$$

Note that $\mathcal{G}_1$ is not empty. Trivially the function $g(x) \equiv c$ is a member of $\mathcal{G}_1$.

There is little scope for variation from such homogeneity. In fact, $g \in \mathcal{G}_1$ if and only if g is of the form

$$\begin{aligned} g(0) &= \alpha \\ g(x) &= \beta, \quad x > 0 \end{aligned} \tag{5.3}$$

where $\alpha \ge \beta > 0$.

The proof is detailed in Arnold (1987). It makes repeated use of Lemmas 2.1 and 2.2 to show that any violations of (5.3) will destroy hopes of inequality preservation.

The essential conclusion is that no non-trivial inequality preserving weightings exist.

6. Inequality Attenuating Weightings. Again define X_g using (5.1). Now we focus on the class $\mathcal{G}_2$ of inequality attenuating weightings:

$$\mathcal{G}_2 = \{g : \ X \in \mathcal{L} \Rightarrow X_g \le_L X\}. \tag{6.1}$$

The class of inequality attenuating weightings is non-empty since $g(x) \equiv c$ trivially attenuates inequality in the sense that $X_g \le_L X$ for every $X \in \mathcal{L}$. In order to have $X_g \le_L X$, it must first be true that $X_g \in \mathcal{L}$ for every $X \in \mathcal{L}$. This requires that $g(x) > 0$ for every $x > 0$. Once again repeated use of Lemmas 2.1 and 2.2 (see Arnold (1987)) rules out all but a trivial class of inequality attenuating transformations. One finds that $g \in \mathcal{G}_2$ if and only if g is of the form

$$\begin{aligned} g(0) &= \alpha, \\ g(x) &= \beta, \quad x > 0 \end{aligned} \tag{6.2}$$

where $\beta > 0$ and $0 \le \alpha \le \beta$.

If we wished to have inequality preservation *and* attenuation then g would have to assume the trivial form $g(x) \equiv \alpha$ for $\alpha > 0$.

7. Mixtures. In an income setting involving finite populations interest has focussed on the effects of pooling populations on inequality. In particular we could imagine a scenario in which we have n_1 "native" wage earners with empirical income distribution $F_1(x)$ and n_2 "immigrant" wage earners with corresponding distribution $F_2(x)$. The pooled population will have empirical income distribution

$$\widetilde{F}(x) = \frac{n_1}{n_1 + n_2} F_1(x) + \frac{n_2}{n_1 + n_2} F_2(x).$$

In the context of such finite populations, Lam (1986) addressed the issue of when we might find that the combined population has less inequality than one of the component populations.

Lam's result extends without difficulty to a setting in which we consider mixtures of random variables in $\mathcal{L}$. Let X and Y be arbitrary members of $\mathcal{L}$. For $\alpha \in (0,1)$, an $(\alpha, 1-\alpha)$ mixture of X and Y is a random variable X_α defined by

$$X_\alpha = I_\alpha X + (1 - I_\alpha) Y \tag{7.1}$$

where X and Y are taken to be independent and I_α is a Bernoulli $(0,1)$ random variable independent of X, Y with $P(I_\alpha = 1) = \alpha$.

The question at issue is: Under what circumstances will we have $X_\alpha \leq_L X$? The first result in this direction is

THEOREM 7.1. (Lam, 1986). *Suppose $X, Y \in \mathcal{L}$ and X_α is as defined in (7.1). If $E(X) = E(Y)$ and $Y \leq_L X$ then $X_\alpha \leq_L X$.*

PROOF. Without loss of generality assume $E(X) = E(Y) = 1$ and hence $E(X_\alpha) = 1$. We may use the HLP-Karamata condition for Lorenz ordering (Theorem 1.1). Consider an arbitrary convex continuous function g. We have

$$\begin{aligned} E(g(X_\alpha)) &= \alpha E(g(X)) + (1-\alpha) E(g(Y)) \\ &\leq \alpha E(g(X)) + (1-\alpha) E(g(X)) \text{ (since } Y \leq_L X) \\ &= E(g(X)). \end{aligned}$$

Thus $X_\alpha \leq_L X$. ∎

Lam also observed that the conditions in Theorem 7.1 are almost necessary. He assumed all incomes under discussion were positive. Some such condition is needed. Consider the following example.

Take X and Y with distribution functions

$$\begin{aligned} F_X(x) &= 0, \quad x < 0 \\ &= 2x, \quad 0 \le x \le \frac{1}{2} \\ &= 1, \quad x > \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} F_Y(x) &= 0, \quad x < \frac{1}{2} \\ &= x, \quad \frac{1}{2} \le x < 1 \\ &= 1, \quad x > 1. \end{aligned}$$

Consider a $\left(\frac{1}{3}; \frac{2}{3}\right)$ mixture of X and Y, denoted by $X_{\frac{1}{3}}$. By inspection of the Lorenz curves we find that $X_{\frac{1}{3}} \le_L X$ even though $E(X) \neq E(Y)$.

However, if $F_X^{-1}(0) > 0$ (i.e. if X is almost surely bounded away from 0) then $E(X) \neq E(Y)$ guarantees that $X_\alpha \not\le_L X$ (as Lam observed in finite populations). To see this, first observe that necessarily $F_{X_\alpha}^{-1}(0) \le F_X^{-1}(0)$ and $F_{X_\alpha}^{-1}(1) \ge F_X^{-1}(1)$. Consequently, if $E(X) > E(Y)$ then $E(X) > E(X_\alpha)$ and $L'_X(0) < L'_{X_\alpha}(0)$, whence $X_\alpha \not\le_L X$. If $E(X) < E(Y)$, then $L'_X(1) > L'_{X_\alpha}(1)$ and, again; $X_\alpha \not\le_L X$. If we have $E(X) = E(Y)$ ($= 1$ without loss of generality) and $Y \not\le_L X$ then by the HLP-Karamata theorem (Theorem 1.1) there exists a convex continuous g for which $E(g(Y)) > E(g(X))$. Then for that g we will have $E(g(X_\alpha)) > E(g(X))$ and consequently $X_\alpha \not\le_L X$. Thus we have a partial converse to Theorem 7.1.

THEOREM 7.2. *Suppose $X, Y \in \mathcal{L}$ and X_α is as defined in (7.1). Assume $F_X^{-1}(0) > 0$. If $X_\alpha \le_L X$ then $E(X) = E(Y)$ and $Y \le_L X$.*

In the real world it is not unusual to encounter situations in which different tax schedules are used for different subpopulations. Lambert (1988) provides an interesting introduction to this area. The general impression is that reasonable tax schedules within subpopulations generally lead to overall inequality reduction. He however presents a simple example to show that inequality reduction need not occur. One can have progressive incentive preserving taxes within subpopulations that, in a global sense, increase inequality! We will review Lambert's partial results and reiterate his call for more study of this interesting problem.

In our notation, each subpopulation (for simplicity of discussion we focus on the case of two subpopulations) can be associated with a non-negative random variable. The combined population can be associated with a suitable mixture random variable (as in (7.1)). We envision two tax schedules to be applied within subpopulations that are assumed to be inequality attenuating within subpopulations. They can be associated with two functions g_1 and g_2 satisfying the conditions of Theorem 4.1. The combined pre-tax incomes can then be associated with the mixture random variable

$$X_\alpha = I_\alpha X + (1 - I_\alpha)Y \tag{7.2}$$

while the combined post-tax incomes will be associated with the random variable

$$\widetilde{X}_\alpha = I_\alpha g_1(X) + (1 - I_\alpha)g_2(Y). \tag{7.3}$$

The question to be resolved is: Assuming that both g_1 and g_2 are inequality attenuating, when can we conclude that $\widetilde{X}_\alpha \leq_L X_\alpha$?

It is convenient to define the average tax rate associated with the application of g to the random variable X by

$$T(g, X) = E(X - g(X))/E(X). \tag{7.4}$$

The first result provided by Lambert, then takes the form

THEOREM 7.3. (Lambert, 1988). *If g_1 and g_2 are inequality attenuating and if $T(g_1, X) = T(g_2, Y)$ (equal average tax rates within subpopulations) then $\widetilde{X}_\alpha \leq_L X_\alpha$ (overall inequality attenuation).*

PROOF. Use the HLP-Strassen result (Theorem 1.2).

By directly studying the Lorenz curve of the mixture $\widetilde{X}_\alpha$, Lambert gives two other conditions for inequality attenuation.

THEOREM 7.4. (Lambert, 1988). *If $L_X(F_X(x)) \geq L_Y(F_Y(x))$ $\forall x$ and $T(g_1, X) \geq T(g_2, Y)$ then $\widetilde{X}_\alpha \leq_L X_\alpha$.*

THEOREM 7.5. (Lambert, 1988). If $X \stackrel{d}{=} Y$, $F_X^{-1}(0) > 0$ and g_1 corresponds to a proportional tax (i.e. $g(x) = cx$) while g_2 is inequality attenuating, then $\widetilde{X}_\alpha \leq_L X_\alpha$ if and only if $T(g_1, X) = T(g_2, Y)$.

Theorem 7.4 is probably of only theoretical interest as Lambert observes since it is unlikely that the hypothesis $X \stackrel{d}{=} Y$ (which translates to identical income distributions within subpopulations) will be encountered in practice.

The general problem of delineating interesting sufficient conditions on the distributions of X and Y to ensure that inequality attenuation within subpopulations results in overall attenuation is worthy of further research.

8. "Random" Taxation. It is natural to consider extension of the results of section 4 to cover the case where the transformations have random components. We are thus led to consider a random variable $X \in \mathcal{L}$ (pre-tax income, if you wish) and an associated random variable Y (post-tax income) defined by

$$Y = \psi(X, Z) \tag{8.1}$$

where ψ is deterministic and Z is random. When can we conclude that $\psi(X, Z) \leq_L X$ (attenuation) and when that $\psi(X, Z) \geq_L X$ (accentuation). Accentuation is most probable in the sense that our transformation involves addition of "noise".

It is possible to view Theorem 7.3 as an instance in which random taxation can result in inequality attenuation. Arnold and Villaseñor (1984) provide the following general result identifying a broad class of situations in which inequality accentuation is encountered.

THEOREM 8.1. *Suppose* $\psi : \mathbb{R}_2^+ \to \mathbb{R}^+$ *is such that* $\psi(z, x)$ *and* $\psi(z, x)/x$ *are non-decreasing in* x *for every* z. *Assume that* X *and* Z *are independent random variables with* $X \in \mathcal{L}$ *and* $\psi(X, Z) \in \mathcal{L}$. *It follows that* $X \leq_L \psi(X, Z)$.

PROOF. Condition on Z and use the HLP-Karamata characterization of the Lorenz order (Theorem 1.1). Note that without loss of generality $E(X) = E(\psi(X, Z))$.

A simple example in which Theorem 8.1 can be applied is one involving random misreporting of incomes. The model relating reported income Y to true income X takes the form

$$Y = ZX \tag{8.2}$$

where Z and X are independent and Z is the misreporting (or dishonesty) factor (usually $Z \leq 1$). Theorem 8.1 applies and implies that $X \leq_L Y$, i.e. misreporting accentuates inequality.

In fact in (8.2) full independence is not required. A direct argument using Jensen's inequality and the HLP-Karamata theorem can be used instead of Theorem 8.1 to conclude that a sufficient condition for $X \leq_L Y$ in (8.2) is that $E(Z|X) = c$.

Misreporting can thwart inequality attenuating efforts. We might pick an attenuating function g and apply it to reported income (rather than true income). In that setting, true post-tax income will be given by

$$Y = X - ZX + g(ZX).$$

We cannot be sure that $Y \leq_L X$ here. A trivial counterexample involves a degenerate X but non-trivial examples also exist.

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