SOME COMMENTS ON POSITIVE QUADRANT DEPENDENCE IN THREE DIMENSIONS

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An extreme point analysis has been performed on two natural definitions of positive quadrant dependence of three random variables. This analysis helps us to understand how much these two notions of dependence are different. In the case of two random variables these two notions of dependence are equivalent.

1. Introduction. Let X and Y be two random variables with some joint probability distribution function F. X and Y (or F) are said to be positively quadrant dependent (PQD) if

$$\Pr(X \leq x, Y \leq y) \geq \Pr(X \leq x)\Pr(Y \leq y) \tag{1}$$

for all real numbers x and y. The condition (1) is equivalent to

$$\Pr(X \geq x, Y \geq y) \geq \Pr(X \geq x)\Pr(Y \geq y) \tag{2}$$

for all x and y. See Lehmann (1966, p. 1138).

One faces problems if one wishes to extend the notion of positive quadrant dependence to more than two random variables. If X, Y, and Z are three random variables, one could say that X, Y, and Z are PQD by adapting either of the conditions (1) or (2) in a natural way. To be more precise, we say that X, Y, and Z are positively lower orthant dependent (PLOD) if

$$\Pr(X \leq x, Y \leq y, Z \leq z) \geq \Pr(X \leq x)\Pr(Y \leq y)\Pr(Z \leq z) \tag{3}$$

for all x, y, and z; and we say that X, Y, and Z are positively upper orthant dependent (PUOD) if

$$\Pr(X \geq x, Y \geq y, Z \geq z) \geq \Pr(X \geq x)\Pr(Y \geq y)\Pr(Z \geq z) \tag{4}$$

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for all x, y, and z.

These two concepts have been examined by Ahmed, Langberg, Léon and Proschan (1978) and by several authors cited in that paper. See also Block and Ting (1981), and Chhetry, Kimeldorf and Sampson (1989).

In this paper, we discuss the ramifications of the definitions of PLOD and PUOD. These two notions of PLOD and PUOD are not equivalent. Ahmed, Langberg, Léon and Proschan (1978) gave an example of a trivariate distribution which is PUOD, but not PLOD.

The main goal of this paper is to examine how different are these two notions of dependence. More precisely, we want to perform extreme point analysis on these two notions of dependence. In some special cases, extreme point analysis helps us to characterize all trivariate distributions which are both PLOD and PUOD.

2. Extreme Point Analysis. We consider the case where each of X, Y, and Z assumes only two values 1 and 2, say. Let $P_{ijk} = \Pr(X = i, Y = j, Z = k)$, $i = 1,2; j = 1,2,; k = 1,2$. The joint probability law of X, Y, and Z is written, for convenience,

$$P = \begin{bmatrix} P_{111} & P_{112} & P_{121} & P_{122} \\ P_{211} & P_{212} & P_{221} & P_{222} \end{bmatrix}$$

In terms of this new notation, P is PLOD if

$$P_{111} \geq p_1 q_1 r_1 \tag{5}$$
$$P_{111} + P_{112} \geq p_1 q_1 \tag{6}$$
$$P_{111} + P_{121} \geq p_1 r_1 \tag{7}$$
$$P_{111} + P_{211} \geq q_1 r_1 \tag{8}$$

and P is PUOD IS

$$P_{222} \geq p_2 q_2 r_2 \tag{9}$$
$$P_{222} + P_{221} \geq p_2 q_2 \tag{10}$$
$$P_{222} + P_{212} \geq p_2 r_2 \tag{11}$$
$$P_{222} + P_{122} \geq q_2 r_2 \tag{12}$$

where $p_1 = \Pr(X = 1)$; $q_1 = \Pr(Y = 1)$; $r_1 = \Pr(Z = 1)$; $p_2 = 1 - p_1$; $q_2 = 1 - q_1$; and $r_2 = 1 - r_1$.

Let $0 < p_1 < 1$, $0 < q_1 < 1$, and $0 < r_1 < 1$ be three fixed numbers. Let $M_{\text{PLOD}}(p_1, q_1, r_1)$ be the collection of all trivariate distributions $P = (P_{ijk})$ with support contained in $\{(i, j, k); i = 1,2, j = 1,2, \text{and } k = 1,2\}$ such that P is PLOD, and the marginal distributions of X, Y, and Z under P are $p_1, 1 - p_1$; $q_1, 1 - q_1$; and $r_1, 1 - r_1$, respectively. The set $M_{\text{PUOD}}(p_1, q_1, r_1)$ is defined analogously. The following result is obvious.

THEOREM 1. *The sets $M_{PLOD}(p_1, q_1, r_1)$ and $M_{PUOD}(p_1, q_1, r_1)$ are compact and convex. More strongly, they are simplexes, i.e., each of these sets is bounded and a finite intersection of hyperplanes.*

Nguyen and Sampson (1985) have looked into properties of sets of the above type for bivariate distributions with fixed marginals. Subramanyam and Bhaskara Rao (1986) have developed an algebraic method for identifying the extreme points of sets of the above type in the context of bivariate distributions.

Being simplexes, the sets $M_{\text{PLOD}}(p_1, q_1, r_1)$ and $M_{\text{PUOD}}(p_1, q_1, r_1)$ have each a finite number of extreme points. Once we identify the extreme points of the set $M_{\text{PLOD}}(p_1, q_1, r_1)$ say, we can express every member of $M_{\text{PLOD}}(p_1, q_1, r_1)$ as a convex combination of its extreme points. We describe now a method of identifying the extreme points of $M_{\text{PLOD}}(p_1, q_1, r_1)$ as well as $M_{\text{PUOD}}(p_1, q_1, r_1)$. First, we take up the case of $M_{\text{PLOD}}(p_1, q_1, r_1)$. Any $P = (P_{ijk}) \varepsilon M_{\text{PLOD}}(p_1, q_1, r_1)$ must satisfy the inequalities (5), (6), (7), and (8). Also, due to marginality restrictions, we should have

$$\begin{aligned}
&(13) \qquad & P_{111} + P_{112} + P_{121} &\leq p_1 \\
&(14) \qquad & P_{111} + P_{112} + P_{211} &\leq q_1 \\
&(15) \qquad & P_{111} + P_{121} + P_{211} &\leq r_1.
\end{aligned}$$

The following are the natural nonnegativity conditions.

$$\begin{aligned}
&(16) \qquad & P_{112} &\geq 0 \\
&(17) \qquad & P_{121} &\geq 0 \\
&(18) \qquad & P_{211} &\geq 0
\end{aligned}$$

All these inequalities (5) to (8) and (13) to (18) involve $P_{111}, P_{112}, P_{121}, P_{211}$ only. If some four numbers $P_{111}, P_{112}, P_{121}, P_{211}$ satisfy the inequalities (5) to (8) and (13) to (18), then one could define

$$\begin{aligned}
&(19) \qquad & P_{122} &= p_1 - (P_{111} + P_{112} + P_{121}), \\
&(20) \qquad & P_{212} &= q_1 - (P_{111} + P_{112} + P_{211}), \\
&(21) \qquad & P_{221} &= r_1 - (P_{111} + P_{121} + P_{211}), \\
& & \text{and} & \\
&(22) \qquad & P_{222} &= 1 - p_1 - q_1 - r_1 + P_{111} + \\
& & & \qquad (P_{111} + P_{112} + P_{121} + P_{211}).
\end{aligned}$$

The numbers P_{122}, P_{212}, and P_{211} will be nonnegative. If $P_{222} \geq 0$, then

$$P = (P_{ijk}) \,\varepsilon\, M_{\text{PLOD}}(p_1, q_1, r_1).$$

A standard method of identifying the extreme points of $M_{\text{PLOD}}(p_1, q_1, r_1)$ is as follows. Select 4 inequalities from (5) to (8) and (13) to (18). Replace the inequality signs by equality signs. Solve the resultant system of 4 linear equations in 4 unknowns P_{111}, P_{112}, P_{121}, and P_{211}. If there is a solution, and this solution satisfies the remaining inequalities, determine P_{122}, P_{212}, P_{221}, and P_{222} as per the equations (19), (20), (21), and (22). If $P_{222} \geq 0$, then

$$P = (P_{ijk}) \,\varepsilon\, M_{\text{PLOD}}(p_1, q_1, r_1)$$

It is easy to check that this P is an extreme point of $M_{\text{PLOD}}(p_1, q_1, r_1)$, and every extreme point of $M_{\text{PLOD}}(p_1, q_1, r_1)$ arises this way. For ideas concerning this approach, one may refer to Subramanyam and Bhaskara Rao (1986). A computer program is easy to write which will identify the extreme points of $M_{\text{PLOD}}(p_1, q_1, r_1)$.

In this context, define the joint distribution function

$$F_U(x, y, z) = F_1(x) \wedge F_2(y) \wedge F_3(z)$$

for all x, y, and z, where $F_1(x) = 0$ if $x < 1$, $= p_1$ if $1 \leq x < 2$, and $= 1$ if $x \geq 2$; $F_2(y) = 0$ if $y < 1$, $= q_1$ if $1 \leq y < 2$, and $= 1$ if $y \geq 2$; and $F_3(z) = 0$ if $z < 1$, $= r_1$ if $1 \leq z < 2$, and $= 1$ if $z \geq 2$; and for any two numbers u, v, $u \wedge v$ stands for the minimum of the numbers u and v. $F_U(x, y, z)$ is the upper Fréchet bound with marginals F_1, F_2, and F_3. An explicit computation shows that the corresponding distribution P_U has the following entries

$$\begin{aligned}
P_{111} &= p_1 \wedge q_1 \wedge r_1; P_{112} = p_1 \wedge q_1 - P_{111}; P_{121} = p_1 \wedge r_1 - P_{111}; \\
P_{211} &= q_1 \wedge r_1 - P_{111}; P_{221} = r_1 - P_{211} - P_{121} - P_{111}; \\
P_{212} &= q_1 - P_{112} - P_{211} - P_{111}; P_{122} = p_1 - P_{121} - P_{112} - P_{111}; \\
P_{222} &= 1 - P_{111} - P_{112} - P_{121} - P_{211} - P_{122} - P_{212} - P_{221}
\end{aligned}$$

It can be verified that the bound is PLOD, as well as PUOD. Furthermore, it is an extreme point.

Pursuing the above approach, we have isolated the extreme points of $M_{\text{PLOD}}(p_1, q_1, r_1)$ and $M_{\text{PUOD}}(p_1, q_1, r_1)$ when $p_1 = q_1 = r_1 = 1/2$, given in Table 1. The above extreme point analyses of the sets $M_{\text{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M_{\text{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ reveal the following insights.

1. The extreme points of $M_{\text{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $M_{\text{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ fall into three distinct categories. The first five extreme points are common to both the sets. Observe that

$$\begin{aligned}
P_6 &= \frac{1}{2}P_4 + \frac{1}{2}P_{15} \\
P_8 &= \frac{1}{2}P_2 + \frac{1}{2}P_{15}
\end{aligned}$$

Table 1. Extreme Points of $M_{\text{PLOD}}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $M_{\text{PUOD}}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

Serial No.	$M_{\text{PLOD}}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$M_{\text{PUOD}}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$
1.	$P_1 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$P_1 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
2.	$P_2 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$	$P_2 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$
3.	$P_3 = \frac{1}{8}\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$	$P_3 = \frac{1}{8}\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$
4.	$P_4 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix}$	$P_4 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix}$
5.	$P_5 = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$	$P_5 = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$
6.	$P_6 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$	$P_7 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
7.	$P_8 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$	$P_9 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$
8.	$P_{10} = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$	$P_{11} = \frac{1}{8}\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$
9.	$P_{12} = \frac{1}{8}\begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & 0 & 0 & \frac{5}{2} \end{bmatrix}$	$P_{13} = \frac{1}{8}\begin{bmatrix} \frac{5}{2} & 0 & 0 & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{bmatrix}$
10.	$P_{14} = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$	$P_{15} = \frac{1}{8}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

$$P_{10} = \frac{1}{2}P_3 + \frac{1}{2}P_{15}$$
$$P_{12} = \frac{1}{4}P_5 + \frac{3}{4}P_{15}$$

Consequently, $P_6, P_8, P_{10}, P_{12} \varepsilon M_{\mathrm{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Also observe that

$$P_7 = \frac{1}{2}P_4 + \frac{1}{2}P_{14}$$
$$P_9 = \frac{1}{2}P_2 + \frac{1}{2}P_{14}$$
$$P_{11} = \frac{1}{2}P_3 + \frac{1}{2}P_{14}$$
$$P_{13} = \frac{1}{4}P_5 + \frac{3}{4}P_{14}$$

Consequently, $P_7, P_9, P_{11}, P_{13} \varepsilon M_{\mathrm{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $P_i \varepsilon M_{\mathrm{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cap M_{\mathrm{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = M_{\mathrm{POD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for $i = 1, 2, \ldots, 12, 13$. The extreme point trivariate distribution P_{14} of $M_{\mathrm{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is not PUOD, because of (9). The extreme point trivariate distribution P_{15} of $M_{\mathrm{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is not PLOD, because of (5).

2. Because of the symmetry present in the probabilities $p_1 = \frac{1}{2} = p_2$, $q_1 = \frac{1}{2} = q_2$, and $r_1 = \frac{1}{2} = r_2$, the extreme points of $M_{\mathrm{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ can be obtained from those of $M_{\mathrm{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ by flipping 1 and 2 among the indices of P_{ijk}'s of P_i's, $i = 1, 2, 3, 4, 5, 6, 8, 10, 12, 14$.

3. The distributions P_i's, $i = 1, 2, \ldots, 12, 13$ are extreme points of $M_{\mathrm{PLOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cap M_{\mathrm{PUOD}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

4. If one wishes to construct a trivariate distribution P which is PLOD but not PUOD, one could use P_{14} as a building block. Look for convex combinations of P_{14} and some or all of P_1,P_2,P_3,P_4,P_5,P_6,P_8,P_{10},P_{12}. For instance, any convex combination $\lambda P_1 + (1 - \lambda)P_{14}$ with $0 \leq \lambda < 1$ is PLOD but not PUOD, because of (9).

5. Note that the joint distribution P_5 is the upper Fréchet bound.

3. Concluding Remarks. The extreme point analysis of two natural definitions of positive quadrant dependence in three dimensions reveals that these two notions of dependence are not violently different in this $2 \times 2 \times 2$ case. Extreme point analysis is useful in evaluating the power function of any test proposed for testing independence of X, Y, and Z against strict positive quadrant dependence of X, Y, and Z. For details, in the case of 2 dimensions, see Subramanyam and Bhaskara Rao (1986). Also, certain measures of dependence can be shown to be affine functions over the sets M_{PLOD} and M_{PUOD}. This affine function property is useful to evaluate asymptotic power of tests based on these measures of dependence. All these ideas and an algebraic method for isolating extreme points of the sets M_{PLOD} and M_{PUOD} will be the subject matter of a forthcoming report.

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