POSITIVE DEPENDENCE CONCEPTS FOR ORDINAL CONTINGENCY TABLES

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This paper provides a connection for cross-classified variables between ordinal contingency tables and positive dependence concepts. Various bivariate positive dependence properties are reinterpretted in terms of odd ratios for certain subtables of the two-way array.

1. Introduction. The primary purpose of this paper is to provide a bridge between two literatures: ordinal contingency tables and positive dependence concepts. Both focus on relationships among discrete random variables. Contingency table research has concentrated on statistical analysis of parametric models for cell frequencies or probabilities. Well known examples of such models are Bishop, Fienberg, and Holland's (1975) log-linear models, and Goodman's (1979, 1985) association models. In the modeling of ordinal contingency tables, the ordered structure of the levels of the cross-classifying variables is usually translated into a related ordering constraint on the corresponding model parameters, e.g., Douglas and Fienberg (1990).

The dependence literature, on the other hand, has focused on probabilistic characterizations, which lead to the study of properties associated with the cell probabilities, rather than with the parameters in models. For a discussion of positive dependence properties and their interrelationships, see Barlow and Proschan (1981).

In this paper, we study in depth the connection between ordinal contingency table parametrizations and positive dependence properties of cross-classified variables. The relationships we examine are established through generalized odds ratios in a contingency table. A few of these connections have been considered previously by Agresti (1984), Grove (1984), and Yanagimoto (1972). Throughout our presentation, we pursue the dual goals of promoting potentially new models for contingency table researchers and bringing ordinal statistical models and techniques to researchers in positive dependence.

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To illustrate the connections between the two literatures, we begin by considering a two-way, $I \times J$ ordinal table, with cross-classifying variables X and Y. We suppose that X takes values $\{x_1 < x_2 < \cdots < x_I\}$, and Y takes values $\{y_1 < y_2 < \cdots < y_J\}$. We denote the joint probability $P(X = x_i, Y = y_j)$ by p_{ij}, and we assume that $p_{ij} > 0$, $i = 1, \ldots, I$, $j = 1, \ldots, J$. It is well known that the property requiring the positivity of all log (local) odds ratios:

$$\log \theta_{ij} \equiv \log \frac{p_{ij} p_{i+1,j+1}}{p_{ij+1} p_{i+1,j}} > 0, \tag{1}$$

$i = 1, \ldots, I-1$, $j = 1, \ldots, J-1$, is equivalent to the positive dependence concept that the distribution of (X, Y) is totally positive of order 2 (TP_2) (e.g., see Barlow and Proschan (1981)). Moreover, if the RC model of order 1 (e.g., see Douglas and Fienberg (1990)) holds in this table, i.e., if

$$p_{ij} = \alpha_i \beta_j \exp(\phi \mu_i \nu_j), \tag{2}$$

where μ_i and ν_j are, respectively, the row and column parameters, and the ordering of the row and column parameters is the same as that of the rows and columns, then log local odds ratios are all positive and, hence, the parametrized distribution of expression (2) must be TP_2.

In Section 2 of this paper we explore, in detail, connections between positive dependence concepts and odds ratios, and their suitable generalizations. There is no discussion of the links between parametric models such as in expression (2) and related concepts, but the connections should be kept in mind. In Sections 3 and 4, we present the basic results linking various positive dependence concepts in terms of inequality restrictions on generalized odds ratios.

2. Generalized Odds Ratios.

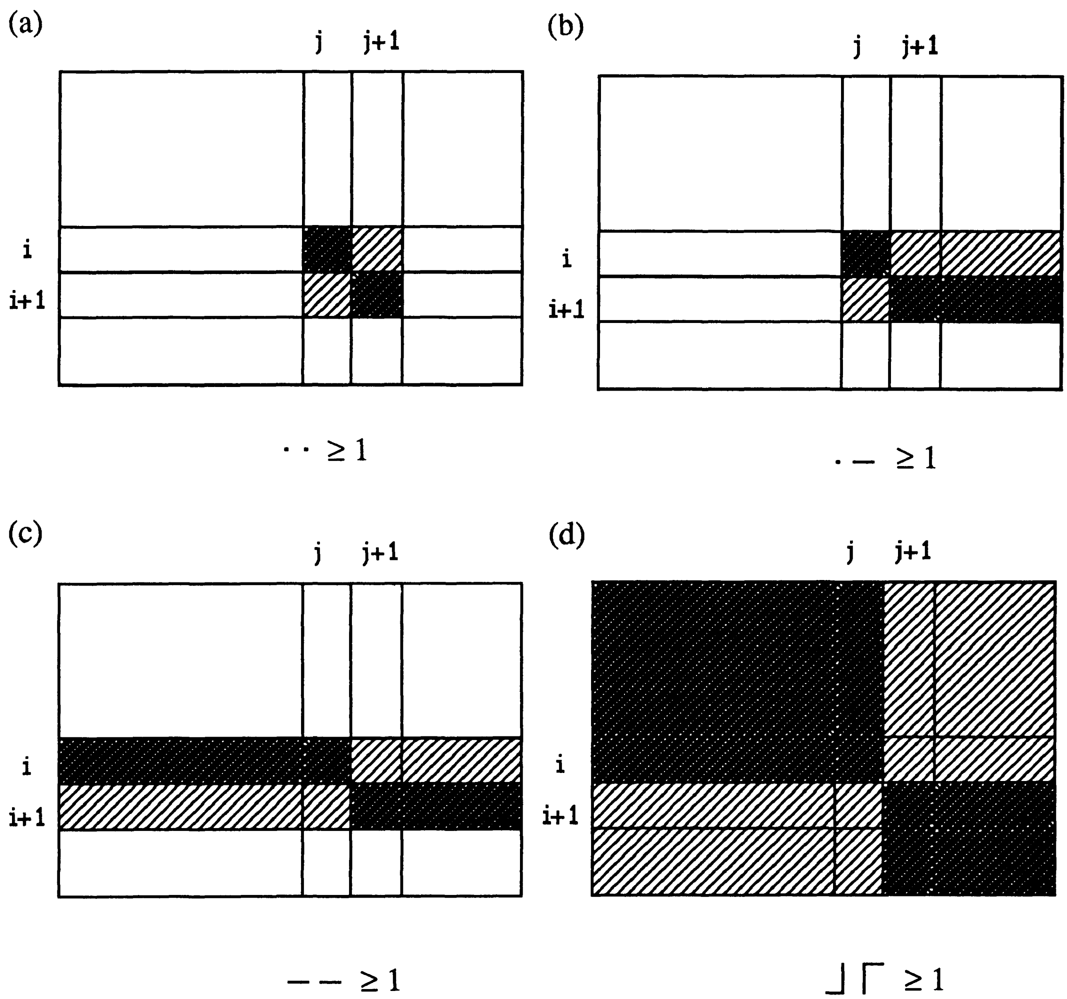
While detailed discussions of the positive dependence properties can be found in many sources, e.g., Barlow and Proschan (1981), Grove (1984), Shaked (1977), or Yanagimoto (1972), virtually every author uses different nomenclature and notation. Here we develop a unified nomenclature and notation that is linked directly to odds ratios for certain subtables of the two-way array. In the next section we apply the nomenclature to establish a hierarchy among various notions of positive dependence.

Agresti (1984) uses the following nomenclature and notation for three basic classes of odds ratios in a two-way table:

1. *Local odds ratios* (see Bishop, Fienberg, and Holland (1975) or Fienberg (1980))

$$\theta_{ij} = \frac{p_{ij} p_{i+1,j+1}}{p_{i+1,j} p_{i,j+1}}, \quad i = 1, \ldots, I-1, \;\; j = 1, \ldots, J-1.$$

Figure 2.1



2. *Local-global odds ratios*

$$\theta^c_{ij} = \frac{(\Sigma_{k\leq j} p_{ik})(\Sigma_{k>j} p_{i+1,k})}{(\Sigma_{k>j} p_{ik})(\Sigma_{k\leq j} p_{i+1,k})}, \quad i = 1,2,\ldots,I-1, \;\; j = 1,2,\ldots,J-1.$$

(These odds ratios are local in the row variable, and global in the column variable as indicated by the superscript c. The notion of global-local odds ratios, θ^r_{ij} can be defined in a similar fashion.).

3. *Global odds ratios*

$$\theta^{rc}_{ij} = \frac{(\Sigma_{k\leq i}\Sigma_{\ell\leq j} p_{k\ell})(\Sigma_{k>i}\Sigma_{\ell>j} p_{k\ell})}{(\Sigma_{k\leq i}\Sigma_{\ell>j} p_{k\ell})(\Sigma_{k>i}\Sigma_{\ell\leq j} p_{k\ell})}, \quad i = 1,2,\ldots,I-1, \;\; j = 1,2,\ldots,J-1.$$

(These odds ratios are global in both the row and column variables, as indicated by the superscript rc.

We illustrate the sets of cells making up three types of odds ratios in Figure 2.1, parts (a), (c), and (d).

The literature of log-linear models is based on local odds ratios, whereas the literature on ordinal categorical variables is more closely linked to local-global or global odds ratios. All three types of quantities can be viewed as special cases of *generalized odds ratios* which are formed by partitioning rectangular subarrays into 4 sets of adjacent cells based on a collapsing of the row and column classifications into dichotomies. For example, the quantity

$$\frac{P\{X = x_i, Y \leq y_j\}P\{X < x_i, Y > y_j\}}{P\{X < x_i, Y \leq y_j\}P\{X = x_i, Y > y_j\}}$$

is a generalized odds ratio for the $i \times J$ subtable that is formed by the partition $\{x_1, x_2, \ldots, x_{i-1}\}$ and $\{x_i\}$ and the partition $\{y_1, y_2, \ldots, y_j\}$ and $\{y_{j+1}, y_{j+2}, \ldots, y_J\}$. This odds ratio is represented diagrammatically in Figure 2.2.

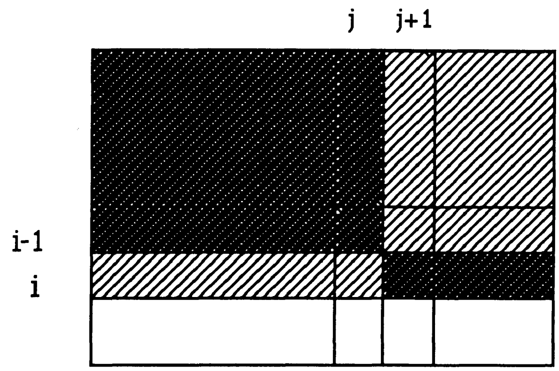
For our purposes here it suffices to restrict attention to those rectangular subarrays that are either 2×2 subtables or that include at least one extremal value of X or Y. The resulting class of odds ratios includes those based on the continuation ratios described by Fienberg (1980), i.e.,

$$\frac{p_{ij}(\Sigma_{k>j} p_{i+1,k})}{(\Sigma_{k>j} p_{ik}) p_{i+1,j}} \text{ for } i = 1,2,\ldots,I-1, \;\; j = 1,2,\ldots,J-1,$$

as depicted in Figure 2.1 (b), in addition to the basic odds ratios of Agresti (1984) described above.

Rather than use these verbal descriptions of different classes of odds ratios, in this paper we adopt a hieroglyphic-like symbolic notation which describes these sets of cells geometrically, and also symbolically gives relationships among concepts. This symbolic notation readily handles generalizations of these odds-ratio concepts.

Figure 2.2.



⌋ − ≥ 1

We will focus on four types of *sets* of cells:

1. individual cells, denoted by ·,
2. horizontal strips, denoted by −,
3. vertical strips, denoted by |,
4. rectangular sections, denoted by ⌋, ⌊, ⌉, and ⌈ for upper-left, upper-right, lower-left, and lower-right corners of the table, respectively.

Our goal is to use inequalities on the $(I-1)(J-1)$ odds ratios of a given type to describe different dependence properties. In general, requiring a collection of odds ratios to be ≥ 1 typically corresponds to some form of positive dependence. Using the hieroglyphic-like notation allows us to express such inequalities in a simple notational form that can be related to relationships among dependence properties. We illustrate by example.

We can describe the property "all local odds ratios are greater than or equal to 1," by the notation "·· $\geq$ 1". The first "·" refers to the fact that the "upper-left hand" set of the local odds ratio is a single cell, and the second "·" to the fact that the "lower-right hand" set is also a single cell. Thus this use of the set notation describes the sets of cells corresponding to the two probabilities in the numerator of the odds ratio and the symbols for the sets of cells corresponding to the two probabilities in the denominator are uniquely defined by implication. Similarly, the property that the "local-global odds ratios are $\geq$ 1" (i.e., odds ratios described by Figure 2.1 (c)) is described by "−− $\geq$ 1", and "⌋⌈$\geq$ 1" means that "all global odds ratios are $\geq$ 1".

Two set symbols are said to be *comparable* if they uniquely describe the cells corresponding to the probabilities that are used to construct a generalized odds ratio for a rectangular subarray. Since the two symbols are used to describe the sets of cells for the probabilities in the numerator of the odds ratios, this means that they must uniquely define the complementary sets of cells for the probabilities in the denominator. Thus $\rfloor$ and $\lceil$ are comparable, but $\rfloor$ and $\lfloor$ are not.

3. Relationships Among Odds Ratio Inequalities. There are 16 possible odds ratios and corresponding dependence properties expressed by appropriate choices of comparable pairs from the four types of sets. To establish the hierarchies among these various positive dependence notions, we employ the following theorem:

THEOREM 3.1.

$$\text{(a) } S\cdot \geq 1 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} S| \geq 1 \\ \\ S- \geq 1 \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} S\lceil \geq 1$$

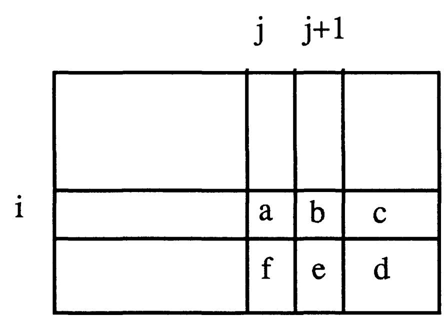
and

$$\text{(b) } \cdot T \geq 1 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} -T \geq 1 \\ \\ |T \geq 1 \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} \rfloor T \geq 1$$

where S and T are any sets which are appropriately comparable.

PROOF OF THEOREM 3.1. A complete proof requires examination of each inequality for each possible type of set S or T. We illustrate the essence of the complete proof by focusing on two inequalities for part (b).

Suppose that $T = \lceil$. The following diagram depicts the relevant structures of the inequalities $\cdot\lceil \geq 1$ for two neighboring points (cells) a and b and a residual horizontal strip c along with the remainder of the table, a lower rectangle d and two vertical strips, e and f:



We shall refer to the probability associated with a set of cells using a capital letter corresponding to lower case letter used to denote the set, e.g., the probability corresponding to the set "a" is denoted by "A."

To prove that $\cdot\lceil \geq 1 \Rightarrow -\lceil \geq 1$, it suffices for us to prove that

$$BD \geq CE \tag{3}$$

and

$$A(E+D) \geq F(B+C) \tag{4}$$

imply

$$(A+B)D \geq (F+E)C, \tag{5}$$

where the capital letter represents the probability of the corresponding lower case letter's region. We consider two possible cases. Suppose that $FB \geq AE$. Then it follows from expression (4) that $AD \geq FC$. Combining this inequality with that of expression (3) yields expression (5). Alternatively, suppose that $FB < AE$. We can rewrite expression (3) as $(CE)^{-1} \geq (BD)^{-1}$. Then

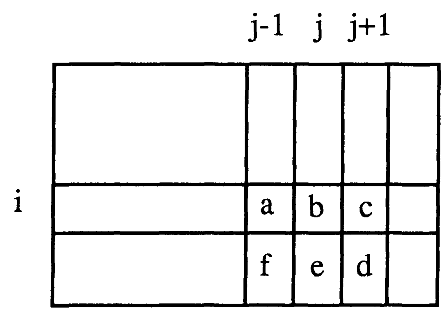
$$AE(CE)^{-1} > FB(BD)^{-1}$$

or

$$AD > FC.$$

Combining this inequality with that of expression (3) yields expression (5) once again. Clearly, the proof of $\cdot\lceil \geq 1 \Rightarrow |\lceil \geq 1$ has exactly the same form.

Next suppose that $T = |$. The following diagram depicts the relevant tabular structure:



To prove that $\cdot| \geq 1 \Rightarrow -| \geq 1$, it suffices for us to prove that

$$BD \geq CE \tag{6}$$

and

$$AE \geq BF \tag{7}$$

imply

$$(A+B)D \geq (E+F)C. \tag{8}$$

We rewrite expression (6) as $(CE)^{-1} \geq (BD)^{-1}$ and combine this through multiplication with expression (7) to get $AD \geq CF$. Adding this inequality to that of expression (6) yields (8). The proof of the remaining components of part (b) of Theorem 3.1 take on one of the two preceding forms.

The proof of part (a) of Theorem 3.1 uses essentially the same form of proof applied to similarly constructed tabular arrays.

An interesting consequence of Theorem 3.1 is that the same relationships hold when the inequalities "≥ 1" are replaced by the equalities "$\equiv 1$." We write this result as:

COROLLARY 3.1.

$$\text{(a) } S\cdot \equiv 1 \begin{array}{ccc} \Nearrow & S| \equiv 1 & \Searrow \\ & & \\ \Searrow & S- \equiv 1 & \Nearrow \end{array} S\lceil \equiv 1.$$

and

$$\text{(b) } \cdot T \equiv 1 \begin{array}{ccc} \Nearrow & -T \equiv 1 & \Searrow \\ & & \\ \Searrow & |T \equiv 1 & \Nearrow \end{array} \rfloor T \equiv 1.$$

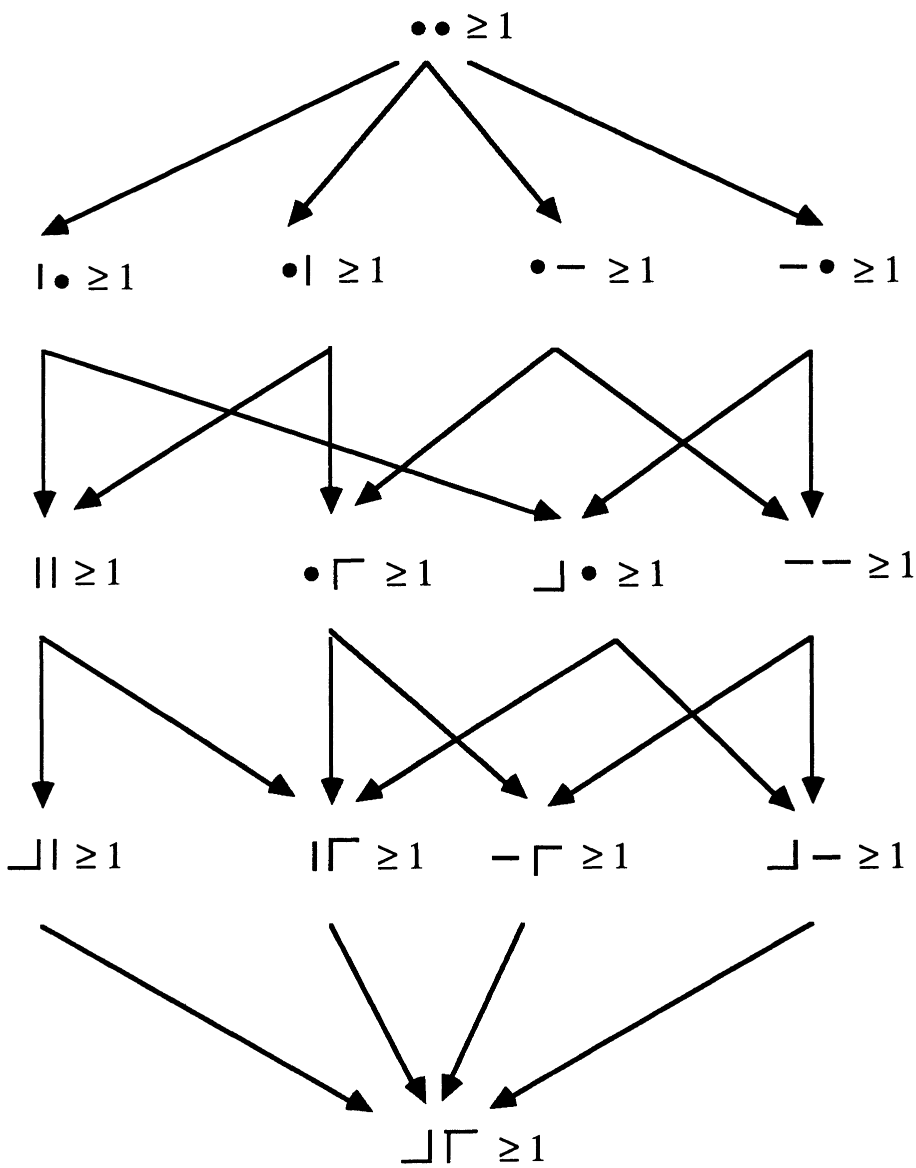
for any sets S and T that are appropriately comparable.

We can display all of the relationships among the 16 different dependence properties in a hierarchical form as in Figure 3.1.

4. Equivalent Dependence Concepts. Thirteen of the 16 concepts in Figure 3.1 have been multiply studied by various previous authors in their specific context. See Shaked (1977) for the definition of DTP; Yanagimoto (1972) for $\mathbf{P}(\cdot,\cdot)$, Barlow and Proschan (1981) for TP_2, SI, RCSI, LTD, RTI, and PQD; and Grove (1984) for his various equations.

Figure 3.1.

Hierarchy Among Bivariate Dependence Properties



THEOREM 4.1. For each subpart below, all dependence concepts are equivalent:

(i) $\cdot\cdot \geq 1$, TP_2, DTP(0,0), **P**(3,3), Grove (1.1).

(ii) $|\cdot \geq 1$, **P**(3,2″).

(iii) $\cdot| \geq 1$, DTP(0,1), **P**(3,2′), Grove (3.1).

(iv) $\cdot - \geq 1$, DTP(1,0), **P**(2′,3), Grove (3.1, x and y interchanged).

(v) $- \cdot \geq 1$, **P**(2″,3).

(vi) $|| \geq 1$, **P**(3,1), SI($Y|X$), Grove (3.2).

(vii) $\cdot\lceil \geq 1$, DTP(1,1), **P**(2′,2′), X and Y are RCSI, Grove (3.3).

(viii) $-- \geq 1$, **P**(1,3), SI($X|Y$), Grove (3.2, x and y interchanged).

(ix) $\rfloor| \geq 1$, **P**(2″,1), LTD($Y|X$).

(x) $|\lceil \geq 1$, **P**(2′,1), RTI($Y|X$), Grove (3.4, x and y interchanged).

(xi) $-\lceil \geq 1$, **P**(1,2′), RTI($X|Y$), Grove (3.4).

(xii) $\rfloor - \geq 1$, **P**(1,2″), LTD($X|Y$).

(xiii) $\rfloor\lceil \geq 1$, **P**(1,1), X and Y are PQD, Grove (3.5).

PROOF. We only indicate how to prove (vii). The proofs of the remaining parts require similar type of manipulations.

Shaked (1977) essentially shows that RCSI <=> DTP(1,1) <=> **P**(2′,2′). Thus, we need only show that **P**(2′,2′) <=> $\cdot\lceil \geq 1$. The condition **P**(2′,2′) requires that

$$\frac{P\{x_\beta < X, y_\beta < Y\}P\{x_\alpha < X \leq x_\beta, y_\alpha < Y \leq y_\beta\}}{P\{x_\beta < X, y_\alpha \leq Y \leq y_\beta\}P\{x_\alpha < X \leq x_\beta, y_\beta < Y\}} \geq 1, \tag{9}$$

for all $x_\alpha < x_\beta$, $y_\alpha < y_\beta$. Fix a point x_i, y_j and note that expression (9) clearly implies $\cdot\lceil \geq 1$ at this point by choosing $x_\beta = x_i$, $x_\alpha = x_{i-1}$, $y_\beta = y_i$, $y_\alpha = y_{i-1}$. Now view expression (9) as X, Y conditional on $X > x_\alpha$, $Y > y_\beta$ having property $\rfloor\lceil \geq 1$ at the point x_i, y_j. We observe that $\cdot\lceil \geq 1$ holding for X, Y at x_i, y_j implies $\cdot\lceil \geq 1$ holds for the preceding conditional random variables at this point. Next, we apply Theorem 3.1 (b) with $T = \lceil$. This shows that $\cdot\lceil \geq 1 \Rightarrow \rfloor\lceil \geq 1$, and, hence, $\cdot\lceil \geq 1 \Rightarrow$ expression (9) holds at x_i, y_j for fixed x_α, y_β. Finally, vary x_α, y_β.

Comments

(i) X and Y being independent implies that $\cdot\cdot \equiv 1$.

(ii) All implications in Figure 3.1 are strict, in that there exist counter-examples that show that the properties are not equivalent.

(iii) No other implications among these dependence notions than those in Figure 3.1 hold for all possible $I \times J$ tables of probabilities.

(iv) ·⌈≥ 1 is equivalent to $P\{X = x_i,\ Y = y_j | X \geq x_i,\ Y \geq y_j\}$ being L-superadditive in i and j. (A real-valued function $h(i,j)$ is L-superadditive in i and j if for all $i < i'$, $j < j'$, $h(i,j) + h(i',j') - h(i,j') - h(i',j) \geq 0$.)

(v) ·| ≥ 1 is equivalent to Prob $\{Y = y_j | X = x_i\}$/Prob $\{Y > y_j | X = x_i\}$ is nonincreasing in x_i for each fixed y_j. Shaked (1977) refers to this property as *conditional hazard rate decreasing* of Y given X (HRD($Y|X$)). The property ·— ≥ 1 is equivalent to HRD($X|Y$).

(vi) |· ≥ 1 is equivalent to Prob $\{Y = y_j | X = x_i\}$/Prob $\{Y < y_j | X = x_i\}$ is nonincreasing in x_i for all y_j. An analogous equivalence exists for —· ≥ 1.

(vii) ⌋· ≥ 1 is equivalent to Prob $\{(X,Y) \leq (x_i,y_j) | (X,Y) \leq (x_i',y_j')\} \geq$ Prob $\{(X,Y) \leq (x_i,y_j) | (X,Y) \leq (x_i^*,y_j^*)\}$ for $x_i' \leq x_i^*$, $y_j' \leq y_j^*$, and for any choice of (x_i,y_j). In the spirit of the terminology for Harris (1970), we refer to this property as *lower corner set decreasing* (LCSD (X,Y)).

(viii) The dependence properties —| ≥ 1 and |— ≥ 1 appear not to have been studied previously.

Comment (i) can be further amplified upon in light of Corollary 3.1.

THEOREM 4.2. *The random variables X and Y are independent if and only if $ST \equiv 1$ holds for any pair of comparable sets, S and T.*

PROOF. Independence is easily shown to be equivalent to ·· $\equiv 1$, and also equivalent to ⌋⌈$\equiv 1$. The result then follows from Corollary 3.1.

One possible interpretation of Theorem 4.2 is that different odds ratios parametrizations provide a variety of positive dependence alternatives to independence by comparing the ratios to 1. This notion has been used in testing by Haberman (1974) who tests $H_0 : X,Y$ are independent (i.e., ·· $\equiv 1$) against $H_A : X,Y$ are strictly TP_2 (i.e., X,Y are ·· ≥ 1, but not ·· $\equiv 1$).

There are various other positive dependence properties leading to interesting ordinal contingency table parametrizations which are obtained by combining two or more dependence properties where there are inequality asymmetries. For instance, the situation where |· ≥ 1 and ·| ≥ 1 both hold is equivalent to P(3,2) of Yanagimoto (1972). Another approach is to combine notions which treat X and Y asymmetrically. For example, Schriever (1983) described the situation where || ≥ 1 and —— ≥ 1 hold together as double regression dependent of order 1.

Other positive dependence properties appear not able to be treated using the notational approach in this paper. For instance, X and Y being *associated* (Esary, Proschan and Walkup (1967)) requires that $P\{(X,Y) \in U | (X,Y) \in V\} \geq P\{(X,Y) \in U\}$ for all possible upper sets U, V (see also Sampson and Whitaker (1989)). We note that each of $\rfloor | \geq 1$, $| \lceil \geq 1$, $- \lceil \geq 1$, and $\rfloor - \geq 1$, implies association which, in turn, implies $\rfloor \lceil \geq 1$.

We underscore the fact that our approach for positive dependence properties permits an easy expression for each of the properties in terms of the parameters $\{p_{ij}\}$. For example, the concept $\cdot| \geq 1$, i.e., HRD$(Y|X)$, is equivalent to

$$p_{ij} \sum_{\ell=j+1}^{J} p_{i+1,\ell} \Big/ [p_{i+1,j} \sum_{\ell=j+1}^{J} p_{i\ell}] \geq 1,$$

for $i = 1, \ldots, I-1$ and $j = 1, \ldots, J-1$; and $\rfloor | \geq 1$ is equivalent to

$$(\sum_{k=1}^{i} \sum_{\ell=1}^{j} p_{k\ell})(\sum_{\ell=j+1}^{J} p_{i+1,\ell}) \Big/ [(\sum_{k=1}^{i} \sum_{\ell=j+1}^{J} p_{k\ell})(\sum_{\ell=l}^{j} p_{i+1,\ell}] \geq 1,$$

for $i = 1, \ldots, I-1$ and $j = 1, \ldots, J-1$. Similarly, our approach permits an easy representation for properties for resulting from intersections of other properties.

5. Discussion. Virtually all of the bivariate concepts that we consider, by reversing inequalities, can be made into negative dependence properties. For example, requiring the local odds ratios to be less than or equal to one, i.e., $\cdot\cdot \leq 1$, is equivalent to reverse regular rule of order 2 (RR_2), and $\rfloor \lceil \leq 1$, is equivalent to negative quadrant dependence. For a further discussion of negative dependence, see Block, Savits, and Shaked (1982).

For at least three of the concepts, requiring the odds ratios all to be equal to the same parameter θ produces an interesting one-parameter family of distributions. For example, $\cdot\cdot = \theta$ corresponds to the uniform association model (Goodman, (1979)), $\rfloor \lceil = \theta$ generates the Plackett family of distributions (Plackett (1965)), and $\cdot \lceil = \theta$ is considered in Clayton (1978). Which, if any of the other positive dependence properties produces interesting one-parameter families of distributions remains an open question.

To date, little attention has been given to the estimation of cell probabilities, p_{ij}, under the various odds ratios ≥ 1, except for models based on local odds ratios (e.g., see Douglas and Fienberg (1990)). Some limited testing results have been considered primarily for the $\cdot\cdot$ and $\rfloor \lceil$ concepts in Haberman (1974), Nguyen and Sampson (1987), and Krishnaiah, Rao, and Subramanyam (1987). Related results have been obtained by Grove (1980), Gilula (1986), and Gilula, Krieger, and Ritov (1988).

We note the interesting but little explored problem of comparing bivariate ordered contingency tables using positive dependence conceptualizations. Results concerning positive dependence orderings are recent and developed primarily for comparing two bivariate distributions. Three of the best known orderings are more positive quadrant dependent (Tchen (1980)), more associated (Schriever (1985)), and more totally positive of order 2 (Kimeldorf and Sampson (1987)).

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