A SURVEY OF SOME INFERENCE PROBLEMS FOR DEPENDENT SYSTEMS

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In this paper a survey of some inference problems relating to dependent systems are considered. A number of multivariate models, both parametric and nonparametric, are given and related tests of dependence and tests of exponentiality are considered.

1. Introduction. The univariate exponential distribution with density function

$$f(x) = \lambda \exp(-\lambda x), \quad x \geq 0,\ \lambda > 0$$

and distribution function

$$F(x) = 1 - \exp(-\lambda x), \quad x \geq 0$$

is well known as the most important model in reliability theory. Here the survival function is given by

$$\bar{F}(x) = 1 - F(x) = \exp(-\lambda x), \tag{1}$$

and the failure rate function

$$r(x) = f(x)/\bar{F}(x)$$

for $F(x) < 1$, is λ, a constant. A random variable X with survival function (1) will be denoted by $X \sim e(\lambda)$.

The exponential distribution has a number of interesting properties. Some of these are given below.

P1. $F(x)$ is absolutely continuous.

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P2. $F(x)$ possesses the loss (or lack) of memory property (LMP). That is $\bar{F}(x+t) = \bar{F}(x)\bar{F}(t)$ for all $x, t \geq 0$.

P3. The failure rate $r(x)$ is a constant.

P4. Let X_1 and X_2 be independently distributed with $X_i \sim e(\lambda_i), i = 1, 2$. Then $\min(X_1, X_2) \sim e(\lambda_1 + \lambda_2)$.

Because of the usefulness of the univariate exponential distribution, it is natural to consider multivariate exponential distributions as models for multicomponent systems. However, unlike the normal distribution, there is no unique natural extension and a number of multivariate exponential distributions have been proposed. For a survey of useful multivariate exponential distributions, see Basu (1988). Section 2 describes a few of these multivariate distributions and related tests of independence are given in Section 3. Finally, some tests for multivariate exponentiality against distributions with specific types of multivariate failure rates (to be defined later) are given in Section 4.

2. Dependent Models.

A. *Multivariate exponential distributions.* A number of multivariate distributions have been derived where multivariate analogues of some of the properties for the univariate distribution have been considered. For simplicity, and without loss of much generality, we shall primarily consider bivariate exponential distributions. For example, Marshall and Olkin (1967) consider the following bivariate analogue of property P2:

$$(2) \qquad \bar{F}(x_1 + y_1, x_2 + y_2) = \bar{F}(x_1, x_2)\bar{F}(y_1, y_2), \text{ for all } x_1, x_2, y_1, y_2 \geq 0.$$

Here $\bar{F}(x, y) = P(X > x, Y > y)$, is the bivariate survival function. Marshall and Olkin showed that the only solution of (2) with univariate exponential marginals is

$$(3) \qquad \bar{F}(x, y) = \exp\{-\theta_1 x - \theta_2 y\},$$

for some $\theta_1, \theta_2 > 0$. That is, (2) implies that X and Y have independent exponential distributions.

By relaxing (2) we obtain the following definition of bivariate loss of memory property (BLMP).

DEFINITION 2.1. The random vector (X, Y) is said to have the BLMP if

$$(4) \qquad \bar{F}(x_1 + y, x_2 + y) = \bar{F}(x_1, x_2)\bar{F}(y, y), \text{ for } x_1, x_2, y \geq 0.$$

Assuming (4) and exponential marginals Marshall and Olkin (1967) obtain the following class of distributions, to be denoted by the BVE.

(5) $\bar{F}(x,y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\}, \quad \lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0, x, y \geq 0.$

Note that $\bar{F}(x,y)$ is not absolutely continuous.

Similarly Basu (1971) considered an extension of property P_3 and defined the bivariate failure rate as

$$r(x,y) = f(x,y)/\bar{F}(x,y). \tag{6}$$

It is shown in Basu (1971) that, except for the case of independence, there does not exist any absolutely continuous bivariate exponential distribution with constant bivariate failure rate and marginal exponential distributions. Brindley and Thompson (1972) consider a more general definition of bivariate failure rate and obtained the BVE as the class of distributions with constant bivariate failure rate and marginal exponential distributions.

The BVE has a number of interesting properties. Let $(X,Y) \sim$ BVE with parameters λ_1, λ_2, and λ_{12}. Denote this by $(X,Y) \sim \text{BVE}(\lambda_1, \lambda_2, \lambda_{12})$. Then $X \sim e(\lambda_1 + \lambda_{12}), Y \sim e(\lambda_2 + \lambda_{12})$, the BLMP is satisfied, $\min(X,Y) \sim e(\lambda_1 + \lambda_2 + \lambda_{12})$. Note that $P(X = Y) \neq 0$, and the BVE is not absolutely continuous. The correlation coefficient $\rho = \rho_{XY} = \lambda_{12}/\lambda$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Thus $\lambda_{12} = 0(\rho_{XY} = 0)$ implies independence. If X and Y denote the lifetimes of a two component series system then it follows that the system lifetime will follow the exponential distribution if (X,Y) follows the BVE.

A related model is proposed by Block and Basu (1974). We shall denote this by the ACBVE. Here the survival function is given by

$$\begin{aligned} \bar{F}(x,y) &= \frac{\lambda}{\lambda_1 + \lambda_2}\exp[-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)] \\ &\quad -\frac{\lambda_{12}}{\lambda_1 + \lambda_2}\exp[-\lambda\max(x,y)], \\ &\quad \lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0, x, y \geq 0, \end{aligned} \tag{7}$$

where, as before, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. The ACBVE is absolutely continuous, satisfies the BLMP and here also $\min(X,Y) \sim e(\lambda)$. However, the marginals are not univariate exponential distributions. The ACBVE is the absolutely continuous part of the BVE. However, it is not a special case of the BVE since the BVE distributions are not absolutely continuous. As in the case of the BVE $\lambda_{12} = 0(\rho_{xy} = 0)$ implies independence.

Multivariate extensions of the BVE and the ACBVE have been considered by Marshall and Olkin (1967) and Block (1975) respectively.

Note that min $(X,Y) \sim e(\lambda)$ for both BVE and ACBVE. Esary and Marshall (1974) consider the general class of distributions with exponential minima. As a special case, consider the following class of bivariate distributions, to be called the EM, with survival function

(8) $\bar{F}(x,y) = \exp[-\lambda_1 x - \lambda_2 y - \max(\lambda_3 x, \lambda_4 y)], \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0, \ x, y \geq 0$

The EM reduces to the BVE if $\lambda_3 = \lambda_4$.

B. *Multivariate distributions based on the notions of aging.* Although the exponential distribution is the most useful model, other models have also been found useful. Because of the nonrobustness of inference procedures based on the exponential distribution, a number of classes of nonparametric distributions, based on the notions of aging, have been proposed as models. The most commonly studied classes of life distributions in the univariate case, based on the notions of aging, are the following:

1) Increasing failure rate class (IFR);

2) Decreasing failure rate class (DFR);

3) Increasing failure rate in average class (IFRA);

4) Decreasing failure rate in average class (DFRA);

5) New better that used class (NBU);

6) New worse than used class (NWU);

7) Harmonic new better than used in expectation class (HNBUE);

8) Harmonic new worse than used in expectation class (HNWUE).

See Barlow and Proschan (1981) and Basu and Ebrahimi (1986a) for a description of these classes.

Multivariate versions of these and other classes have been defined and their properties have been developed by Basu and Ebrahimi (1988), Basu, Ebrahimi, and Klefsjö (1983), Block and Savits (1980, 1981), Buchanan and Singpurwalla (1977), Ghosh and Ebrahimi (1981) and others. An important problem is to see if a class of distributions is closed under convolution. That is, let $\underline{X} = (X_1, \ldots, X_p)$ and $\underline{Y} = (Y_1, \ldots, Y_p)$ both belong to the class of distributions G and let $\underline{X}$ and $\underline{Y}$ be independent. Then, under what condition will $\underline{X} + \underline{Y} \sim G$?

Block and Savits (1980) proved the closure under convolution for the class of multivariate IFRA distributions defined by them. El-Neweihi (1984) and El-Neweihi and Savits (1987) have proved the closure property of a multivariate NBU distribution under convolution. Similarly, Basu and Ebrahimi (1986b, 1988) have proved closure under convolution of the class of multivariate NBUE and the class multivariate HNBUE distributions.

Other properties have been discussed in the papers mentioned above.

3. Tests for Independence. Since independently distributed component lifetimes are easier to analyze, a number of tests for independence have been considered. For the BVE and the ACBVE, testing for independence is equivalent to testing the null hypothesis

$$H_0 : \lambda_{12} = 0, \tag{9}$$

against the alternative hypothesis

$$H_1 : \lambda_{12} > 0.$$

For the BVE, Bemis et al. (1972) derive the UMP test for the above hypothesis when λ_1 and λ_2 are known; and Bhattacharyya and Johnson (1973) derive the UMP test when $\lambda_1 = \lambda_2$ but the common value is unknown. Similarly for the ACBVE, Gupta, Mehrotra, and Michalek (1984) derive a test for (9) when the marginal distributions are equal.

Let $P_{\text{MO}}(P_{\text{BB}})$ denote the power of a test T when the BVE (ACBVE) is the underlying model. Since, $P(X = Y) = \lambda_{12}/\lambda = 0$, under the null hypothesis (9) for the BVE, H_0 is rejected if $X_i = Y_i$ for some i. Weier and Basu (1981) show that, based on a random sample $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$,

$$P_{\text{MO}} = 1 - (1-\rho)^n + (1-\rho)^n P_{\text{BB}}. \tag{10}$$

It is thus enough to consider tests for the ACBVE, which are easier to derive because of absolute continuity. Assuming $\lambda_1 = \lambda_2$, Weier and Basu (1981) have also considered nonparametric tests of independence of Kendall and Spearman. For a definition of Kendall's test for independence and that due to Spearman see, for example, Lehmann (1975). Assume $\lambda_1 = \lambda_2$. Let U denote the UMP test, M denote the test based on MLE of λ_{12}, T = Kendall's tau, R = Spearman's ρ based on a random sample of size n. Then the following result concerning Pitman asymptotic relative efficiency (ARE) is obtained by Weier and Basu (1981).

$$\begin{gathered} \text{ARE(T,U)} = \text{ARE(R,U)} \\ = \text{ARE (T,M)} = \text{ARE(R,M)} = .5, \\ \text{ARE(M,U)} = 1. \end{gathered}$$

The above ARE results hold for the BVE also. Weier and Basu (1980) have also considered tests for independence for a special trivariate distribution with exponential marginals. This model is an extension of the BVE. A similar trivariate extension of the ACBVE is also given. However, like the bivariate case, here the marginals are not exponential distributions. This ACBVE extension is a special case of Block's (1975) more general extension. The case for general multivariate exponential distribution is open.

4. Tests for Multivariate Exponentiality. In this section we consider tests for multivariate exponentiality. Without much loss of generality, we consider the bivariate case.

4.1. A Test for Bivariate Exponentiality Against BNBU. Basu and Ebrahimi (1984) have considered statistical procedures to test whether a bivariate distribution follows the Marshall-Olkin bivariate exponential distribution (BVE) against the alternative that it is nonexponential bivariate new better than used (BNBU). Hollander and Proschan (1972) have considered the univariate case. Throughout we assume $F(0,0) = 0$.

DEFINITION 4.1. F is said to be BNBU-I if

$$\bar{F}(x+t, y+t) \leq \bar{F}(x,y)\bar{F}(t,t), x, y \geq 0, t \geq 0, \tag{11}$$

and similar inequalities hold for both marginal survival functions.

DEFINITION 4.2. F is said to be BNBU-II if $\bar{F}(x+t, x+t) \leq \bar{F}(x,x)\bar{F}(t,t)$, for all $x, t \geq 0$, and similar inequalities hold for the marginal survival functions.

The boundary members of BNBU-I obtained by insisting on equalities in (11), are the family of Marshall-Olkin bivariate exponential distributions (BVE).

Basu and Ebrahimi (1984) have considered testing

$$H_0 : \bar{F}(x,y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)), x, y, \lambda_1, \lambda_2 > 0, \lambda_{12} \geq 0 \tag{12}$$

versus

$$H_1 : F \text{ is BNBU-I and not BVE,} \tag{13}$$

on the basis of a random sample $(X_1, Y_1), (X_2, Y_2) \ldots (X_n, Y_n)$ from F. Consider the functionals

$$\Delta(\bar{F}) = \int\int\int \{\bar{F}(x,y)\bar{F}(t,t) - \bar{F}(x+t, y+t)\} dF(x,y) dF(t,t)$$

and

$$\Gamma(\bar{F}) = \int\int\int \{\bar{F}(x,y)\bar{F}(t,t) - \bar{F}(x+t, y+t)\} dx dy dt$$

Under $H_0 : \Delta(\bar{F}) = 0$ and $\Gamma(\bar{F}) = 0$. If the H_0 is not true, both $\Delta(\bar{F})$ and $\Gamma(\bar{F})$ will be large. Estimates of $\Delta(\bar{F})$ and $\Gamma(\bar{F})$ will also be expected to be large under H_1. Thus estimates of $\Gamma(\bar{F})$ and $\Delta(\bar{F})$ or, quantities which are asymptotically equivalent to these estimates, could be used as test statistics.

Two test statistics have been proposed for testing the above hypothesis. These are given below.

a) Reject H_0 in favor of H_1 if $\Delta(\bar{F}_n)$ is too large. Here

$$\Delta(\bar{F}_n) = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} t(X_j, X_i)t(Y_j, Y_i) - \frac{1}{n^3} \tag{14}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\{t(X_i, X_j + Z_j)t(Y_i, Y_j + Z_k)\}$$

where $Z_k = \min(X_k, Y_k)$ and

$$t(a,b) = \begin{cases} 1, & \text{if } a > b, \\ 0, & \text{otherwise.} \end{cases}$$

b) Reject H_0 in favor of H_1 if $\hat{\Gamma}(\bar{F})$ is too large, where

$$\hat{\Gamma}(\bar{F}) = \frac{1}{n^2}\{\sum_{i=1}^{n} X_i Y_i\}\{\sum_{i=1}^{n} Z_i\} + \{\frac{1}{2n}\sum_{i=1}^{n}(X_i + Y_i)Z_i^2\} - \tag{15}$$

$$\{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i Z_i + \frac{1}{3n}\sum_{i=1}^{n} Z_i^3\}$$

Properties of $\Delta(\bar{F}_n)$ and $\hat{\Gamma}(\bar{F})$ are given in Basu and Ebrahimi (1984). In particular both $\Delta(\bar{F}_n)$ and $\hat{\Gamma}(\bar{F})$ are consistent and asymptotically unbiased estimates of $\Delta(\bar{F})$ and $\hat{\Gamma}(\bar{F})$ respectively and are asymptotically normally distributed.

Note that the above tests can be considered bivariate extensions of the univariate Hollander-Proschan test (1972).

4.2. Testing for Bivariate Increasing Failure Rate Average (BIFRA). Basu and Habibullah (1987) have considered a test for bivariate exponentiality against the BIFRA alternative. Esary and Marshall (1979) and Block and Savits (1980) have studied properties of BIFRA distributions.

DEFINITION 4.3. (X, Y) is said to have a BIFRA distribution if and only if

$$\bar{F}^{\alpha}(x, y) \le \bar{F}(\alpha x, \alpha y) \tag{16}$$

for all $x, y > 0$ and all $\alpha, 0 < \alpha < 1$.

Note that equality in (16) implies a bivariate distribution with exponential minimum. An example of a bivariate distribution with exponential minimum, which includes the BVE as a special case, is given by (8).

Note that the BIFRA distribution is also the BNBU-I. We consider testing the null hypothesis

(17) $$H_0 : \bar{F}^\alpha(x, y) = \bar{F}(\alpha x, \alpha y) \text{ for all } \alpha\epsilon(0,1) \text{ and all}, x > 0,\ y > 0$$

against the alternative hypothesis

(18) $$H_1 : \bar{F}^\alpha(x, y) \leq \bar{F}(\alpha x, \alpha y), \alpha\epsilon(0,1), x > 0,\ y > 0,$$

where the inequality holds for some $(x, y)\epsilon R_2^+$ and for at least some α. Define

$$\Delta_\alpha(F) = \int_0^\infty \int_0^\infty [\bar{F}^{1/\alpha}(\alpha x, \alpha y) - \bar{F}(x, y)]dF(x, y), (0 < \alpha < 1).$$

Under $H_0, \Delta_\alpha(F) = 0$ and under $H_1, \Delta_\alpha(F) > 0$. Thus $\Delta_\alpha(F)$ meaures the deviation from H_0. Basu and Habibullah (1987) propose a test for the above hypothesis based on an estimator of $\Delta_\alpha(F)$ where α is a fixed constant.

The following lemma due to Basu and Habibullah (1987) shows that if equality (17) holds for a particular $\alpha_0\epsilon(0,1)$ then it will imply that F has a bivariate exponential distribution with exponential minimum. To this end it is sufficient to prove that the distribution of $Z = \min(X, Y)$ is univariate exponential.

LEMMA 4.1. *Let* $\bar{F}^\alpha 0(t,t) = \bar{F}(\alpha_0 t, \alpha_0 t)$ *where* α_0 *is fixed,* $0 < \alpha_0 < 1, t > 0$ *and* $\bar{F}(t,t) = P(Z > t)$. *Then* $\bar{F}(t,t) = e^{-\lambda t}$ *for some* $\lambda > 0$.

From Lemma 4.1 it is clear that H_0 does not hold if (17) is not true for some fixed α. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample from F. For testing the above hypothesis Basu and Habibullah (1987) propose rejecting H_0 in favor of H_1 if the statistic

$$J_{.5}^{(n)} = U_1 - U_2$$

is large. Note that $\alpha = .5$ is taken to simplify computations. Here

$$U_1 = \frac{2}{n(n-1)\binom{n-2}{1}} \sum_c h_1\{(X_{\alpha_1}, Y_{\alpha_1}), (X_{\alpha_2}, Y_{\alpha_2}); (\frac{1}{2}X_{\alpha_3}, \frac{1}{2}Y_{\alpha_3})\},$$

and the sum $\sum_c$ extends over all combinations $1 \leq \alpha_i \leq n,\ i = 1, 2, 3,\ \alpha_1 < \alpha_2$, $\alpha_1 \neq \alpha_3,\ \alpha_2 \neq \alpha_3$, and

$$h_1\{(a_1, b_1), (a_2, b_2); (a_3, b_3)\} = \begin{cases} 1 & \text{if } a_1, a_2 > a_3 \text{ and } b_1, b_2 > b_3 \\ 0, & \text{otherwise.} \end{cases}$$

Also

$$U_2 = \frac{1}{\binom{n}{2}} \sum_{\alpha_1 \neq \alpha_2} h_2\{(X_{\alpha_1}, Y_{\alpha_1}), (X_{\alpha_2}, Y_{\alpha_2})\},$$

where

$$h_2\{(a_1, b_1), (a_2, b_2)\} = \begin{cases} 1 & \text{if } (a_1 > a_2, b_1 > b_2) \\ 0, & \text{otherwise.} \end{cases}$$

$J_{.5}^{(n)}$ is a difference of two U-statistics and therefore is asymptotically normally distributed, details are given in Basu and Habibullah (1987).

5. Concluding Remarks. In this paper we present a survey of some inference problems relating to dependent systems. Some other related problems namely, the problem of competing risks and that of converting dependent models to independent ones have been described in Basu and Klein (1981).

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