Chapter 4

Stochastic partial differential equations

4.1 Introduction

Beginning with this chapter and throughout the rest of this monograph we will be considering examples from neurophysiology. There will, of course, be examples from other fields of applications. Nevertheless, different stochastic models of neuronal behavior have provided much of the motivation for the theory developed in the later chapters as well as examples of stochastic partial differential equations (SPDE's) that have random field solutions, solutions in Hilbert spaces or solutions in (conuclear) spaces of distributions. These examples will be considered in their proper contexts. It should also be pointed out that SPDE's where the driving processes are Poisson random measures arise naturally in the study of fluctuations of membrane potentials of neurons and can be used (as will be shown later) to derive diffusion approximations in infinite dimensional spaces. A similar approach might yield interesting results in other fields of application such as stochastic models of turbulence.

A brief description of the neurophysiological background may prove useful in understanding how some of the SDE's of this chapter and Chapter 8 are formulated.

In their seminal investigation in the early 1950's, Hodgkin and Huxley [16]) studied the electrical behavior of neuronal membranes and the role of ionic currents. They introduced a mathematical model for the flow of current through the surface membrane of the giant axon from a Loligo Squid. The partial differential equations which bear their name are nonlinear and have been at the center of a deterministic theory.

Although early stochastic models treated the neuron as a "point", in the neurophysiological literature, it has been well recognized that a neuron cell is spatially extended. Thus a realistic description of neuronal activity–such as the evolution of the voltage potential or the potential difference measured across the molecular membrane–would have to take into account synaptic inputs occurring randomly in time and at different sites on the neuron's surface. As we shall see, even the simplest stochastic description of this phenomenon leads to SPDE's or, more generally, to stochastic differential equations in infinite dimensional spaces.

Let $\mathcal{X}$ denote the surface membrane of a spatially extended neuron and let $u(t, x)$ be the fluctuation of the membrane potential as a function of time $t \geq 0$ and location $x \in \mathcal{X}$. In the simplest spatially extended case, $\mathcal{X}$ is taken to be an infinitely thin cylinder, i.e., $\mathcal{X}$ is the interval $[0, b]$ although mathematical models in which $\mathcal{X}$ is a subset of $\mathbf{R}^d$, $d > 1$ lead to more complicated equations.

A deterministic model is available from the so-called core conductor theory according to which, in the absence of external impulses u should solve the **cable equation**

$$\frac{\partial u}{\partial t} = -\alpha u + \beta \Delta u, \quad t > 0, \quad 0 < x < b, \tag{4.1.1}$$

for some positive constants α and β and with initial value

$$u(0, x) = f(x) \quad \forall 0 \leq x \leq b.$$

In (4.1.1), β represents the rate of diffusion within the neuron and α is the rate at which ions leak across the membrane. We assume the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, b) = 0 \quad \forall t > 0.$$

The boundary conditions represent the insulation of the neuron fiber at both ends. From standard methods in the theory of partial differential equations it can be shown that the system (4.1.1) determines a semigroup T_t on the Hilbert space $H = L^2([0, b], dx)$ and the solution to (4.1.1) is given by

$$u(t, x) = (T_t f)(x) \equiv \int_0^b G(t; x, y) f(y) dy \tag{4.1.2}$$

where $G(t; x, y)$ is the Green's function

$$G(t; x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y), \quad t \geq 0, \tag{4.1.3}$$

with

$$\lambda_n = \alpha + \beta \left(\frac{n\pi}{b}\right)^2, \quad \phi_n(x) = \left(\frac{2}{b}\right)^{\frac{1}{2}} \cos \frac{n\pi x}{b} \quad \forall n \geq 1$$

and

$$\lambda_0 = \alpha \qquad \phi_0(x) = \left(\frac{1}{b}\right)^{\frac{1}{2}}.$$

$\{\lambda_n\}$ and $\{\phi_n\}$ are the eigenvalues and orthonormal eigenfunctions respectively, of the generator $-L$ of T_t where $-L \equiv -\alpha I + \beta\Delta$ (here $\Delta = \frac{d^2}{dx^2}$ and I is the identity operator).

In regarding the fluctuation of the voltage potential across the cell membrane as a stochastic phenomenon, the basic assumption is that the membrane potential receives random impulses at time t and site x. Accordingly, the cable equation (4.1.1) is replaced by the **stochastic cable equation**

$$\frac{\partial u}{\partial t} = -\alpha u + \beta \frac{\partial^2 u}{\partial x^2} + \dot{W}_{tx} \tag{4.1.4}$$

where $\dot{W}_{tx}$ is the c.B.m. determined by the Brownian sheet $W(t,x)$. The above equation will be rigorously formulated as a SPDE. Here the "noise" is additive and, from a biological standpoint, it is the "noise" that carries the information to the central nervous system.

A somewhat more complicated model, is one that introduces a nonlinear or quasilinear SPDE and involves reversal potentials which possibly give a more realistic picture of neuronal activity. The idea of reversal (or equilibrium) potentials originated with Hodgkin and Huxley and is best introduced first for a "point" neuron. Increments in the voltage potential of the neuron are due to two kinds of impulses: excitatory and inhibitory. Assume that there are p possible magnitudes a_e^j of the former and q possible magnitudes a_i^ℓ of the latter, arriving in independent Poisson streams N_e^j and N_i^ℓ with rates f_e^j and f_i^ℓ respectively. The reversal potentials are constants V_e^j, V_i^ℓ and the stochastic model for the voltage potential at time t is

$$dV_t = -\gamma V_t dt + \sum_j (V_e^j - V_t) a_e^j dN_e^j(t) + \sum_\ell (V_i^\ell - V_t) a_i^\ell dN_i^\ell(t) \tag{4.1.5}$$

where $V_e^j \geq 0$, $V_i^\ell \leq 0$, a_e^j, a_i^ℓ are positive constants and γ (> 0) is the leakage rate. A glance at equation (4.1.5) shows that the role of the reversal potentials is to act as a brake and to prevent V_t from soaring too far above or below acceptable levels of tolerance. Equation (4.1.5) will be generalized for spatially extended neurons leading to SPDE's driven by Poisson random measures and their diffusion approximations.

4.2 Space-time Ornstein-Uhlenbeck SDE

The simplest version of the Ornstein-Uhlenbeck (O-U) SPDE is given by an evolution equation of the form

$$\frac{\partial u}{\partial t} = -Lu(t,x) + \dot{W}_{tx} \quad 0 < x < b,\ t > 0 \tag{4.2.1}$$

with either of the following boundary conditions (B.C.):

$$u(t,0) = u(t,b) = 0 \qquad \forall t > 0 \text{ (Dirichlet B.C.)} \tag{4.2.2}$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,b) = 0 \qquad \forall t > 0 \text{ (Neumann B.C.)} \tag{4.2.3}$$

and the initial condition

$$u(0,x) = f(x), \quad 0 \leq x \leq b. \tag{4.2.4}$$

Equation (4.2.1) is clearly in a heuristic form but gives a clear idea of how a (deterministic) PDE can be converted into a stochastic equation by the addition of a space-time Gaussian "noise" represented by $\dot{W}_{tx}$. A rigorous formulation of (4.2.1) is the Itô equation

$$du(t,x) = -Lu(t,x)dt + dW_{tx}, \quad t > 0,\ 0 < x < b. \tag{4.2.5}$$

In (4.2.5), W_{tx} is the c.B.m. corresponding to the Brownian sheet $W(t,x)$ ($t > 0$, $0 \leq x \leq b$) and $-L$ is the generator of a contraction semigroup T_t defined on the Hilbert space $H = L^2[0,b]$.

Stochastic cable equation (Fluctuation of neuron potential)

We now give a detailed treatment of the stochastic cable equation (4.1.4), which is a special case of (4.2.5) with $L = \alpha I - \beta \frac{d^2}{dx^2}$ and assuming (4.2.3) and (4.2.4). It is easy to verify directly that

$$u(t,x) \equiv \int_0^b G(t;x,y)f(y)dy + \int_0^t \int_0^b G(t-s;x,y)W(dsdy) \tag{4.2.6}$$

is a solution to (4.1.4). It is convenient for the further discussion to take $b = \pi$ and $\alpha = \beta = 1$. Then we have

$$\phi_0 = \left(\frac{1}{\pi}\right)^{\frac{1}{2}}, \quad \phi_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos nx \qquad (n \geq 1)$$

and

$$\lambda_n = 1 + n^2 \quad (n \geq 0)$$

From (4.1.3) and (4.2.6),

$$\begin{aligned} u(t,x) &= \sum_{n=0}^{\infty} e^{-\lambda_n t} < f, \phi_n > \phi_n(x) \\ &\quad + \int_0^t \int_0^\pi \sum_{n=0}^{\infty} e^{-\lambda_n (t-s)} \phi_n(x)\phi_n(y) W(dyds). \end{aligned}$$

By virtue of the L^2-convergence of the series under the integral, we may interchange the order and obtain

$$\begin{aligned} u(t,x) &= \sum_{n=0}^{\infty} e^{-\lambda_n t} < f, \phi_n > \phi_n(x) \\ &+ \sum_{n=0}^{\infty} \left\{ \int_0^t \int_0^\pi e^{-\lambda_n (t-s)} \phi_n(y) W(dsdy) \right\} \phi_n(x). \end{aligned}$$

We have used here the fact that (ϕ_n) is a CONS in H. Writing

$$W_t^n = \int_0^t \int_0^\pi \phi_n(y) W(dsdy), \quad < f, \phi_n >= \int_0^\pi f(x)\phi_n(x)dx$$

and

$$A_n(t) = e^{-\lambda_n t} < f, \phi_n > + \int_0^t e^{-\lambda_n (t-s)} dW_s^n,$$

we have

$$u(t,x) = \sum_{n=0}^{\infty} A_n(t)\phi_n(x). \tag{4.2.7}$$

Since the W^n's are independent Wiener processes, $A_n(t)$ are independent, one-dimensional Ornstein-Uhlenbeck processes:

$$dA_n(t) = -\lambda_n A_n(t)dt + dW_t^n. \tag{4.2.8}$$

The series

$$\sum_{n=0}^{\infty} E\{A_n(t)\}\phi_n(x)$$

converges absolutely for each (t,x), $(t > 0)$ to $Eu(t,x)$ and the series of the variances

$$\sum_{n=0}^{\infty} Var(A_n(t))\phi_n^2(x) = \sum_{n=0}^{\infty} \frac{1 - e^{-2\lambda_n t}}{2\lambda_n} \phi_n^2(x)$$

converges, again for $t > 0$ and $0 \leq x \leq \pi$. Hence (4.2.7) converges almost surely and we have the following: $u(t,x)$ given by (4.2.6) is a Gaussian random field with

$$E\{u(t,x)\} = \sum_{n=0}^{\infty} e^{-\lambda_n t} < f, \phi_n > \phi_n(x)$$

and

$$Var\{u(t,x)\} = \sum_{n=0}^{\infty} \frac{1 - e^{-2\lambda_n t}}{2\lambda_n} \phi_n^2(x).$$

Thus, $u(t,x)$ may be called a space-time Ornstein-Uhlenbeck random field. For investigating the regularity properties of $u(t,x)$ we need the following bounds obtained by Walsh [57] (except for constants).

Lemma 4.2.1 *(a)*

$$\sum_{n=1}^{\infty} \frac{|\phi_n(x)-\phi_n(y)|^2}{2\lambda_n} \le \frac{6}{\pi}(|x-y| \wedge 2);$$

(b)

$$\sum_{n=0}^{\infty} \frac{1-e^{-\lambda_n t}}{2\lambda_n} \le \frac{5}{2}(1 \wedge \sqrt{t}), \quad t>0.$$

Proof: (a) From $\phi_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos nx$, $(n \ge 1)$

$$[\phi_n(x)-\phi_n(y)]^2 \le \frac{2}{\pi}[4 \wedge n^2|x-y|^2]$$

and $\lambda_n > n^2$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\phi_n(x)-\phi_n(y)|^2}{2\lambda_n} &\le \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{4}{n^2} \wedge |x-y|^2\right) \\
\le\ & \frac{1}{\pi}\left\{(4 \wedge |x-y|^2) + \int_1^\infty \left(\frac{4}{u^2} \wedge |x-y|^2\right) du\right\} \\
\le\ & \frac{1}{\pi}\{2(|x-y| \wedge 2) + 4(|x-y| \wedge 2)\} \\
=\ & \frac{6}{\pi}(|x-y| \wedge 2).
\end{aligned}$$

(b) From the inequality

$$1-e^{-\lambda_n t} \le 1 \wedge (1+n^2)t$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1-e^{-\lambda_n t}}{2\lambda_n} &\le \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{1+n^2} \wedge t\right) \\
&\le \frac{1}{2}(1 \wedge t) + \sum_{n=1}^{\infty}\left(\frac{1}{(1+n)^2} \wedge t\right) \\
&\le \frac{1}{2}(1 \wedge \sqrt{t}) + \int_1^\infty \left(\frac{1}{u^2} \wedge t\right) du \\
&\le \frac{5}{2}(1 \wedge \sqrt{t}).
\end{aligned}$$

■

Theorem 4.2.1 *The solution $u(t,x)$ of the SPDE (4.1.4) is continuous or, more precisely, has a continuous modification.*

Proof: For simplicity, assume $f = 0$. From (4.2.7), (4.2.8), the independence of the processes $A_n(t)$ $(n = 1, 2, \cdots)$ and

$$u(t,x) - u(s,y) = \sum_{n=0}^{\infty}[A_n(t) - A_n(s)]\phi_n(x) + \sum_{n=1}^{\infty} A_n(s)[\phi_n(x) - \phi_n(y)]$$

we have that $u(t,x) - u(s,y)$ (for $s < t$) is a Gaussian random variable with zero mean and variance

$$\begin{aligned}
& E[u(t,x) - u(s,y)]^2 \\
= & \sum_{n=0}^{\infty} \frac{1}{2\lambda_n}\Big\{ e^{-2\lambda_n t}\left(e^{2\lambda_n t} - e^{2\lambda_n s}\right) \\
& + \left(e^{-\lambda_n t} - e^{-\lambda_n s}\right)^2 \left(e^{2\lambda_n s} - 1\right)\Big\}\phi_n^2(x) \\
& + \sum_{n=1}^{\infty} \frac{1 - e^{-\lambda_n s}}{2\lambda_n}|\phi_n(x) - \phi_n(y)|^2 \\
\leq & \frac{2}{\pi}\sum_{n=0}^{\infty}\left\{\frac{1 - e^{-2\lambda_n(t-s)}}{2\lambda_n} + \left(1 - e^{-\lambda_n(t-s)}\right)^2 \frac{1 - e^{-2\lambda_n(t-s)}}{2\lambda_n}\right\} \\
& + \sum_{n=1}^{\infty} \frac{|\phi_n(x) - \phi_n(y)|^2}{2\lambda_n} \\
\leq & \frac{5}{\pi}\left(\sqrt{2(t-s)} + \sqrt{t-s}\right) + \frac{6}{\pi}(|x-y| \wedge 2) \\
\leq & \frac{5(1+\sqrt{2})}{\pi}\left(|t-s|^{\frac{1}{2}} + |x-y|^{\frac{1}{2}}\right).
\end{aligned}$$

From the above we easily have for any integer m

$$E|u(t,x) - u(s,y)|^{2m} \leq C_1(m)\left(|t-s|^{\frac{m}{2}} + |x-y|^{\frac{m}{2}}\right).$$

Taking $m > 4$, we see that Kolmogorov's condition for the existence of a continuous modification is satisfied. (cf. Kunita [34]). ∎

Theorem 4.2.2 *The SPDE (4.1.4) has a unique continuous solution.*

Proof: We have already seen that the solution of (4.1.4) is continuous. If $u'(t,x)$ is any other continuous solution of the SPDE (4.1.4), let $\bar{u}(t,\cdot) = u(t,\cdot) - u'(t,\cdot)$. Then $\bar{u}(t)$, as an H-valued process, satisfies

$$\bar{u}(t) = \int_0^t T_{t-s}\bar{u}(s)ds.$$

It follows that

$$\|\bar{u}(t)\|_H \leq \int_0^t \|\bar{u}(s)\|_H ds \leq \cdots \leq \frac{T^n}{n!}\int_0^T \|\bar{u}(s)\|_H ds \to 0.$$

We then have (restoring the probability parameter in the notation) $\bar{u}(t,x,\omega) = 0$ for $\omega \notin N_{t,x}$, the latter being a P-null set. Since $\bar{u}(t,x,\omega) = u(t,x,\omega) - u'(t,x,\omega)$ is continuous for almost all ω, it follows that

$$P\{\omega : \bar{u}(t,x,\omega) = 0, \ \ \forall(t,x)\} = 1,$$

that is, for almost all ω,

$$u(t,x,\omega) = u'(t,x,\omega) \qquad \forall(t,x)$$

and uniqueness is proved. ∎

Remark 4.2.1 *We can similarly treat the* **Stochastic heat equation***:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{W}_{tx}, \qquad 0 < x < b, \ t > 0$$

with initial and Dirichlet boundary conditions:

$$u(0,x) = f(x), \qquad 0 \leq x \leq b$$

$$u(t,0) = u(t,b) = 0 \qquad \forall t \geq 0.$$

In this case the eigenvalues and eigenfunctions of $L = -\frac{d^2}{dx^2}$ *are given by*

$$\lambda_n = \frac{n^2\pi^2}{b^2} \qquad \textit{and} \qquad \phi_n(x) = \sin\frac{n\pi}{b} \quad n \geq 1.$$

4.3 Stochastic partial differential equations

In this section we study a class of stochastic partial differential equations. We establish the existence and uniqueness of solutions for such equations. Most of the material of this section is taken from Kotelenez [33] (with simplified proof) and is an extension of Walsh's treatment of the nonlinear stochastic cable equation.

Let $\mathcal{O}$ be a bounded open domain in $\mathbf{R}^d$ and $\{L(t) : t \geq 0\}$ be a family of linear operators on $C(\mathcal{O})$. Let F and R be two functions on $[0,T] \times \mathcal{O} \times \mathbf{R}$ and let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a stochastic basis satisfying the usual conditions. Let $W(drdt)$ be the standard white noise random measure on $\mathcal{O} \times [0,T]$ adapted to $\mathcal{F}_t$, i.e., for any $B \in \mathcal{B}(\mathcal{O})$, the Gaussian random variable $W(B \times [0,t])$ is $\mathcal{F}_t$-measurable.

To study the stochastic partial differential equation

$$\begin{aligned} dX(t,r) &= (L(t)X(t,r)+R(t,r,X(t,r)))drdt \\ &\quad +F(t,r,X(t,r))W(drdt) \end{aligned} \tag{4.3.1}$$

with initial condition

$$X(0,r)=\xi(r),$$

we make the following assumption:
(RD1) $\{L(t)\}$ generates a two-parameter evolution semigroup $\{U(t,s):0\le s\le t\}$ on $C(\mathcal{O})$, with kernel function $G(t,s,r,q)$, $0\le s\le t$, $r,q\in\mathcal{O}$, i.e.

$$(U(t,s)f)(r)=\int_{\mathcal{O}}G(t,s,r,q)f(q)dq.$$

Definition 4.3.1 *A random field $\{X(t,r):t\in[0,T],r\in\mathcal{O}\}$ is called a* **mild solution** *of the RDSDE (4.3.1) if for any $t\in[0,T]$, $r\in\mathcal{O}$, we have*

$$\begin{aligned} X(t,r) &= \int_{\mathcal{O}}G(t,0,r,q)\xi(q)dq \\ &\quad +\int_0^t\int_{\mathcal{O}}G(t,s,r,q)R(s,q,X(s,q))dqds \\ &\quad +\int_0^t\int_{\mathcal{O}}G(t,s,r,q)F(s,q,X(s,q))W(dqds) \quad a.s. \end{aligned} \tag{4.3.2}$$

For simplicity, we extend $G(t,s,r,q)$ to $t,s\in[0,T]$, $r,q\in\mathcal{O}$ by defining it to be zero when $t<s$.

To solve (4.3.2), we need some additional conditions:
(RD2) There exist constants $K(T)<\infty$, α_1, $\alpha_2\in(0,1)$ such that
i) For any $t,s\in[0,T]$, $r\in\mathcal{O}$, we have

$$\int_{\mathcal{O}}|G(t,s,r,q)|^2dq\le K(T)(t-s)^{-\alpha_1}. \tag{4.3.3}$$

ii) For any $0\le t_1\le t_2\le T$ and $r_1,r_2\in\mathcal{O}$, we have

$$\int_0^T\int_{\mathcal{O}}|G(t_1,s,r_1,q)-G(t_2,s,r_2,q)|^2dqds\le K(T)\rho((t_1,r_1),(t_2,r_2))^{2\alpha_2}. \tag{4.3.4}$$

where ρ is the Euclidian distance in $[0,T]\times\mathcal{O}\subset\mathbf{R}^{d+1}$.
(RD3) There exists a constant K(R,F,T) such that, for all $x,y\in\mathbf{R}$, $r\in\mathcal{O}$ and $0\le t\le T$,

$$\begin{aligned} &|R(t,r,x)-R(t,r,y)|+|F(t,r,x)-F(t,r,y)| \\ \le\ & K(R,F,T)|x-y| \end{aligned} \tag{4.3.5}$$

and

$$|R(t,r,x)|+|F(t,r,x)| \leq K(R,F,T)(1+|x|). \tag{4.3.6}$$

For $\alpha > 0$, let

$$\mathbf{B}_\alpha = \{\psi \in C(\mathcal{O}) : \|\psi\|_\alpha < \infty\}$$

denote the Banach space with norm

$$\|\psi\|_\alpha = \sup_{r\in\mathcal{O}} |\psi(r)| + \sup_{r,q\in\mathcal{O}} \frac{|\psi(r)-\psi(q)|}{|r-q|^\alpha}.$$

(RD4) For $\alpha > 0$ and $\xi \in \mathbf{B}_\alpha$, we have $\int_\mathcal{O} G(\cdot,0,\cdot,q)\xi(q)dq \in C([0,T],\mathbf{B}_\alpha)$.

Now we proceed to establish the existence of a unique solution to (4.3.2). We first state without proof the following result due to Ibragimov.

Theorem 4.3.1 (Ibragimov) *[17] Let F be a bounded closed subset of $\mathbf{R}^d$ with nonempty interior and $p > 1$. Suppose that ξ is a random function on F such that*

$$\sup_{|x-y|<\delta} E|\xi(x)-\xi(y)|^p \leq \omega(\delta)^p$$

where $\omega(u)$ is a concave modulus of continuity. Then for any $\mu > 0$ there exists a constant K_1 such that

$$E\left\{\sup_{|x-y|<1} \frac{|\xi(x)-\xi(y)|}{|x-y|^\mu}\right\} \leq K_1 \int_0^1 \frac{\omega(u)}{u^{1+\mu+d/p}} du.$$

We shall need the following

Lemma 4.3.1 *Let*

$$Y(t,r) \equiv \int_0^t \int_\mathcal{O} G(t,s,r,q)F(s,q,X(s,q))W(dqds),$$

where X is a random field on $[0,T]\times\mathcal{O}$ such that

$$E \sup_{(t,q)\in[0,T]\times\mathcal{O}} |X(t,q)|^p < \infty, \quad \forall p \geq 2.$$

Then for any $\alpha < \alpha_2$,

$$E\left\{\sup_{\rho((t,r),(s,q))<1} \frac{|Y(t,r)-Y(s,q)|}{\rho((t,r),(s,q))^\alpha}\right\} < \infty. \tag{4.3.7}$$

Proof: It follows from Doob's inequality, Proposition 3.2.3, (RD2) and (RD3) that

$$\begin{aligned}
& E|Y(t+u,r+h)-Y(t,r)|^p \\
\leq\ & K_2 E\left|\int_0^T\int_{\mathcal{O}}|G(t+u,s,r+h,q)-G(t,s,r,q)|^2\right. \\
& \qquad\qquad \left.|F(s,q,X(s,q))|^2 dq ds\right|^{p/2} \\
\leq\ & K_2\left[K(T)(|u|^2+|h|^2)^{\alpha_2}\right]^{p/2} E\left[K(R,F,T)\left(1+\sup_{s\leq T, q\in\mathcal{O}}|X(s,q)|\right)\right]^p \\
\equiv\ & K_3^p(|u|^2+|h|^2)^{\frac{p}{2}\alpha_2},
\end{aligned}$$

where K_2, K_3 are two constants. Therefore, Y satisfies the condition of Ibragimov theorem with $\omega(\delta)=K_3\delta^{\alpha_2}$. Hence

$$E\left\{\sup_{\rho((t,r),(s,q))<1}\frac{|Y(t,r)-Y(s,q)|}{\rho((t,r),(s,q))^\alpha}\right\}\leq K_1\int_0^1\frac{K_3 u^{\alpha_2}}{u^{1+\alpha+(d+1)/p}}du<\infty \quad (4.3.8)$$

by taking p large enough so that $1+\alpha+(d+1)/p-\alpha_2<1$. ∎

Now we present the main theorem of this section.

Theorem 4.3.2 (Kotelenez) *i) Under assumptions (RD1)-(RD4), the RDSDE (4.3.1) has a unique sample continuous mild solution adapted to* $\mathcal{F}_t$*, i.e., for any* $t\geq 0$ *and* $r\in\mathcal{O}$*,* $X(t,r)$ *is* $\mathcal{F}_t$*-measurable.*
ii) Let $0<\alpha<\alpha_2$*. If* $\xi\in\mathbf{B}_\alpha$ *a.s., then, regarded as a stochastic process taking values in function space,* $X\in C([0,T],\mathbf{B}_\alpha)$ *a.s.*

Proof: Let

$$X^0(t,r)=\int_{\mathcal{O}}G(t,0,r,q)\xi(q)dq$$

and let $X^{n+1}(t,r)$ be the right hand side of (4.3.2) with X replaced by $X^n, n=0,1,\cdots$. For any $p\geq 2$, let

$$L_n(t,r)\equiv E|X^{n+1}(t,r)-X^n(t,r)|^p$$

and

$$h_n(t)\equiv\sup\{L_n(t,r):r\in\mathcal{O}\}.$$

By Doob's inequality, there exists a constant K_4 such that

$$\begin{aligned}
& L_n(t,r) \\
\leq\ & K_4 E\left|\int_0^t\int_{\mathcal{O}}G(t,s,r,q)[R(s,q,X^n(s,q))-R(s,q,X^{n-1}(s,q))]dqds\right|^p
\end{aligned}$$

$$+K_4 E\left|\int_0^t \int_{\mathcal{O}} G(t,s,r,q)^2 \right.$$
$$\left.[F(s,q,X^n(s,q)) - F(s,q,X^{n-1}(s,q))]^2 dq ds\right|^{p/2}.$$

Taking $t_2 = t$, $t_1 = 0$ and $r_1 = r_2 = r$ in (RD2)ii), we see that

$$\int_0^T \int_{\mathcal{O}} |G(t,s,r,q)|^2 dq ds \le K(T)T^{2\alpha_2}.$$

Therefore

$$\begin{aligned} L_n(t,r) &\le K_4\left(K(T)T^{2\alpha_2}\right)^{p/2} K(R,F,T)^p (T|\mathcal{O}|)^{\frac{p}{2}-1} \\ &\quad \int_0^t \int_{\mathcal{O}} E|X^n(s,q) - X^{n-1}(s,q)|^p dq ds \\ &\quad + K_4\left(K(T)T^{2\alpha_2}\right)^{p/2-1} \int_0^t \left(\int_{\mathcal{O}} |G(t,s,r,q)|^2 dq\right) h_{n-1}(s) ds \\ &\le K_5 \int_0^t (t-s)^{-\alpha_1} h_{n-1}(s) ds \qquad (4.3.9) \end{aligned}$$

where K_5 is a constant. Hence

$$h_n(t) \le K_5 \int_0^t (t-s)^{-\alpha_1} h_{n-1}(s) ds. \qquad (4.3.10)$$

Applying (4.3.10) to h_{n-1} on the right hand side of (4.3.10), we have

$$\begin{aligned} h_n(t) &\le K_5 \int_0^t (t-s)^{-\alpha_1} K_5 \int_0^s (s-s')^{-\alpha_1} h_{n-2}(s') ds' ds \\ &\le K_5^2 \int_0^t h_{n-2}(s') \left(\int_{s'}^t (t-s)^{-\alpha_1}(s-s')^{-\alpha_1} ds\right) ds' \\ &= 2K_5^2 \int_0^t h_{n-2}(s) \left(\int_s^{(t+s)/2} (t-s')^{-\alpha_1}(s'-s)^{-\alpha_1} ds'\right) ds \\ &\le 2K_5^2 \int_0^t \left(\frac{t-s}{2}\right)^{-\alpha_1} h_{n-2}(s) \left(\int_0^{(t-s)/2} v^{-\alpha_1} dv\right) ds \\ &\le 2K_5^2 \int_0^t h_{n-2}(s) \left(\frac{t-s}{2}\right)^{1-2\alpha_1} ds/(1-\alpha_1). \end{aligned}$$

If $1 - 2\alpha_1 \ge 0$, we stop here. Otherwise, as $1 - 2\alpha_1 > -\alpha_1$, we continue the above calculation and we will find an integer m and a constant K_6 such that

$$h_{n+m}(t) \le K_6 \int_0^t h_n(s) ds, \quad \forall n \ge 0 \text{ and } t \in [0,T]. \qquad (4.3.11)$$

It follows from (4.3.11) and induction that

$$h_{n+mj}(t) \leq K_6^j \int_0^t h_n(s) \frac{(t-s)^{j-1}}{(j-1)!} ds, \quad \forall n \geq 0, j \geq 1, t \in [0,T]. \quad (4.3.12)$$

Let

$$K_7(T) \equiv \sup\{h_n(s) : \ 0 \leq s \leq T, \ n = 0, 1, \cdots, m-1\}.$$

Then

$$\sup_{t \leq T} h_{n+mj}(t) \leq K_6^j K_7(T) \frac{T^j}{j!}.$$

Hence

$$\sup_{t \leq T} \sum_{n=0}^{\infty} h_n(t)^{1/p} \leq \sum_{n=0}^{m-1} \sum_{j=1}^{\infty} \sup_{t \leq T} h_{n+mj}(t)^{1/p} < \infty. \quad (4.3.13)$$

Consequently, there exists an adapted random field $X(t,r)$ such that

$$\sup_{t \leq T, r \in \mathcal{O}} E|X^n(t,r) - X(t,r)|^p \to 0 \quad \text{as } n \to \infty.$$

Let $\tilde{X}$ be given by the right hand side of (4.3.2) and

$$\tilde{h}_n(t) \equiv \sup_{r \in \mathcal{O}} E|X^{n+1}(t,r) - \tilde{X}(t,r)|^p.$$

It follows from the same arguments as in (4.3.10) that

$$\tilde{h}_n(t) \leq K_5 \int_0^t (t-s)^{-\alpha_1} h_{n-1}(s) ds.$$

Then $\tilde{h}_n(t) \to 0$ as $n \to \infty$ and hence, for any $t \in [0,T]$ and $r \in \mathcal{O}$, $X(t,r) = \tilde{X}(t,r)$ a.s. It follows from Lemma 4.3.1 that the third term of the right hand side of (4.3.2) is a sample continuous random field and, regarded as a stochastic process taking values in function space, it is in $C([0,T], \mathbf{B}_\alpha)$ a.s. By similar arguments, the second term on the right hand side of (4.3.2) is in $C([0,T], \mathbf{B}_\alpha)$ a.s. It follows from (RD4) that the same statement holds for the first term. Therefore $\tilde{X}$ is a continuous modification of X and hence $\tilde{X}$ is a mild solution of (4.3.1) such that ii) of the theorem holds.

It remains to prove the uniqueness of solution. Let X_1 and X_2 be two sample continuous mild solutions of (4.3.1). Let

$$h(t) \equiv \sup\{E|X_1(t,r) - X_2(t,r)|^p : r \in \mathcal{O}\}.$$

Then similar to (4.3.11) we have

$$h(t) \leq K_6 \int_0^t h(s) ds, \ \forall t \in [0,T].$$

Therefore $h = 0$ and hence, $X_1 = X_2$ a.s. ∎

Remark 4.3.1 *The assumptions (RD1), (RD2) and (RD4) are satisfied if* $\{L(t)\}$ *is a family of pseudo-differential operators with order* $\gamma > d$ *(defined below) and satisfies the conditions for bounded domain* $\mathcal{O}$ *imposed by Kotelenez [33].*

Definition 4.3.2 *Let* $\gamma \in \mathbf{R}$. *An operator* L *on* $C_0^\infty(\mathcal{O})$, *the collection of smooth functions with compact supports in* $\mathcal{O}$, *is called a pseudo-differential operator of order* γ *if*

$$Lu(x) = (2\pi)^{-d} \int \int e^{i(x-y)v} a(x, y, v) u(y) dy dv, \; u \in C_0^\infty(\mathcal{O}),$$

where a *is a symbol of order* γ *in the following sense:* a *is a complex-valued smooth function on* $\mathcal{O} \times \mathcal{O} \times \mathbf{R}^d$ *with compact support and for any compact set* $C \subset \mathcal{O} \times \mathcal{O}$ *and multi-indices* $\beta_1, \beta_2, \beta_3$, *there exists a constant* $K(C, \beta_1, \beta_2, \beta_3) > 0$ *such that*

$$|\partial_x^{\beta_1} \partial_y^{\beta_2} \partial_v^{\beta_3} a(x, y, v)| \leq K(C, \beta_1, \beta_2, \beta_3)(1 + |v|)^{\gamma - |\beta_3|},$$

for any $(x, y) \in C, v \in \mathbf{R}^d$, *where* ∂ *denotes the derivative.*

For details about pseudo-differential operators, we refer the reader to the book of Stein [51].

Example 4.3.1 *We give here an example of a pseudo-differential operator. Let* γ *be a positive integer and*

$$a(x, y, v) = \sum_{|\beta| \leq \gamma} a_\beta(x) v^\beta, \quad (x, y, v) \in \mathcal{O} \times \mathcal{O} \times \mathbf{R}^d$$

where a_β *is a smooth function on* $\mathcal{O}$ *with compact support,* $\forall |\beta| \leq \gamma$. *Then* a *is a symbol of order* γ *and* L *is a differential operator of order* γ *given by*

$$L = \sum_{|\beta| \leq \gamma} a_\beta(x)(-1)^{|\beta|/2} \partial_x^\beta.$$

In fact, for any $u \in C_0^\infty(\mathcal{O})$ *we have*

$$\begin{aligned} Lu(x) &= (2\pi)^{-d} \int \int e^{i(x-y)v} \sum_{|\beta| \leq \gamma} a_\beta(x) v^\beta u(y) dy dv \\ &= (2\pi)^{-d/2} \int e^{ixv} \sum_{|\beta| \leq \gamma} a_\beta(x)(-1)^{|\beta|/2} \mathcal{F}^{-1}(\partial^\beta u)(v) dv \\ &= \sum_{|\beta| \leq \gamma} a_\beta(x)(-1)^{|\beta|/2} \partial_x^\beta u(x) \end{aligned}$$

where $\mathcal{F}^{-1}$ *denotes the inverse Fourier transformation.*

4.4 Nonlinear stochastic cable equation

As a corollary to the previous result we shall now show that the assumptions on the kernel G are satisfied for the important case of the nonlinear stochastic cable equation. Before we set it up as a special case of equation (4.3.1) we let $d = 1$ and $\mathcal{O}$ be the segment $[0, \pi]$. We consider the nonlinear version of the stochastic cable equation (4.1.4):

$$du(t,x) = -Lu(t,x)dt + F(t,u(t,x))W(dxdt) \quad 0 < x < \pi,\ t > 0 \quad (4.4.1)$$

$$u(0,x) = f(x) \quad 0 \le x \le \pi$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\pi) = 0 \ \ \forall t > 0$$

Here $-Lu = \frac{\partial^2 u}{\partial x^2} - u$ is, as in the linear case, the generator of a contraction semigroup T_t defined on the Hilbert space $L^2[0,\pi]$. The kernel of $-L$ is therefore, the Green function given by

$$G(t;x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y), \quad t \ge 0$$

We may regard (4.4.1) as a special case of (4.3.1) with $R = 0$, $L(t) = -L$. Condition (RD1) is satisfied, since

$$(T_t f)(x) = \int_0^{\pi} G(t;x,y)f(y)dy.$$

Note that (4.3.1) reduces to (4.4.1) when $R = 0$. (There is no need to let $R = 0$; we do this only for convenience). The Lipschitz and growth conditions on F are again assumed to hold.

Theorem 4.4.1 *The SPDE (4.4.1) has a unique solution $u(t,x)$ which is continuous in (t,x); a.s.*

Proof: In view of Theorem 4.3.1 it is sufficient to verify that $G(t;x,y)$ satisfies conditions (RD2) and (RD4).

Note that

$$\begin{aligned}
\int_0^{\pi} G(t,r,q)^2 dq &= \sum_{j=0}^{\infty} e^{-2\lambda_j t}\phi_j(r)^2 \\
&\le \frac{1}{\pi} + \frac{2}{\pi}\sum_{j=1}^{\infty}\exp(-2tj^2) \le \frac{1}{\pi} + \frac{2}{\pi}\int_0^{\infty}\exp(-2tx^2)dx \\
&= \frac{1}{\pi} + \frac{1}{\sqrt{2\pi t}} \le \left(\frac{\sqrt{T}}{\pi} + \frac{1}{\sqrt{2\pi}}\right)t^{-\frac{1}{2}}. \qquad (4.4.2)
\end{aligned}$$

Hence (RD2)i) holds with $\alpha_1 = \frac{1}{2}$. Note that

$$\begin{aligned}
&\int_0^T \int_{\mathcal{O}} |G(t_1 - s, r_1, q) - G(t_2 - s, r_2, q)|^2 dq ds \\
= &\int_0^T \int_0^\pi \left[\sum_{j=0}^\infty \left(e^{-\lambda_j t_1} \phi_j(r_1) 1_{s<t_1} - e^{-\lambda_j t_2} \phi_j(r_2) 1_{s<t_2} \right) e^{\lambda_j s} \phi_j(q) \right]^2 dq ds \\
= &\int_0^T \sum_{j=0}^\infty \left(e^{-\lambda_j t_1} \phi_j(r_1) 1_{s<t_1} - e^{-\lambda_j t_2} \phi_j(r_2) 1_{s<t_2} \right)^2 e^{2\lambda_j s} ds \\
= &\sum_{j=0}^\infty \left[\int_0^{t_1} \left(e^{-\lambda_j t_1} \phi_j(r_1) - e^{-\lambda_j t_2} \phi_j(r_2) \right)^2 e^{2\lambda_j s} ds \right. \\
&\left. + \int_{t_1}^{t_2} e^{-2\lambda_j t_2} \phi_j(r_2)^2 e^{2\lambda_j s} ds \right] \\
\leq &\sum_{j=0}^\infty \left[\frac{1}{2\lambda_j} \left(\phi_j(r_1) - e^{-\lambda_j (t_2 - t_1)} \phi_j(r_2) \right)^2 + \frac{1 - e^{2\lambda_j (t_2 - t_1)}}{2\lambda_j} \frac{2}{\pi} \right] \\
\leq &2 \frac{6}{\pi} (|x - y| \wedge 2) + 2 \frac{2}{\pi} \frac{5}{2} \left(1 \wedge \sqrt{t_2 - t_1} \right) + \frac{2}{\pi} \frac{5}{2} \left(1 \wedge \sqrt{2(t_2 - t_1)} \right) \\
\leq &\frac{10 + 5\sqrt{2}}{\pi} \left(|r_2 - r_1|^{\frac{1}{2}} + |t_2 - t_1|^{\frac{1}{2}} \right). \qquad (4.4.3)
\end{aligned}$$

Using the elementary inequality, $|a + b|^4 \leq 2^3 \left(|a|^4 + |b|^4 \right)$ we have

$$|r_2 - r_1|^{\frac{1}{2}} + |t_2 - t_1|^{\frac{1}{2}} \leq 2^{\frac{3}{4}} \left(|r_2 - r_1|^2 + |t_2 - t_1|^2 \right)^{\frac{1}{4}}.$$

Substituting the above in (4.4.3) we see that (RD2)ii) holds with $\alpha_2 = \frac{1}{4}$.

Finally we verify (RD4). It is easy to verify that $G(t, r, q)$ can be written as

$$\begin{aligned}
G(t, r, q) = &\sum_{j=-\infty}^\infty \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{(q - r + 2k\pi)^2}{2t} \right) \right. \\
&\left. + \exp\left(-\frac{(q + r - 2k\pi)^2}{2t} \right) \right].
\end{aligned}$$

If $\xi \in \mathbf{B}_\alpha$ such that

$$|\xi(r) - \xi(q)| \leq K|r - q|^\alpha \quad \forall r, q \in (0, \pi),$$

extending ξ to $\mathbf{R}$ by

$$\xi(r) = \xi(r + 2k\pi) = \xi(2\pi - r), \quad \forall k \in \mathbf{Z}, r \in (0, \pi),$$

then

$$
\begin{aligned}
&\int_0^\pi G(t,r,q)\xi(q)dq \\
=\ & \int_0^\pi \sum_{j=-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(q-r+2k\pi)^2}{2t}\right)\right. \\
&\qquad\qquad \left. + \exp\left(-\frac{(q+r-2k\pi)^2}{2t}\right)\right]\xi(q)dq \\
=\ & \sum_{j=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\left[\int_{\frac{2k\pi-r}{\sqrt{t}}}^{\frac{(2k+1)\pi-r}{\sqrt{t}}} \xi\left(\sqrt{t}x+r\right)dx + \int_{\frac{r-2k\pi}{\sqrt{t}}}^{\frac{(1-2k)\pi-r}{\sqrt{t}}} \xi\left(\sqrt{t}x-r\right)dx\right] \\
=\ & \frac{1}{\sqrt{2\pi}}\int \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\left(\xi\left(\sqrt{t}x+r\right)+\xi\left(\sqrt{t}x-r\right)\right)dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|\int_0^\pi G(t_1,r_1,q)\xi(q)dq - \int_0^\pi G(t_2,r_2,q)\xi(q)dq\right| \\
\le\ & 2K\frac{1}{\sqrt{2\pi}}\int \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\left(\left|\sqrt{t_1}-\sqrt{t_2}\right|^\alpha |x|^\alpha + |r_1-r_2|^\alpha\right)dx \\
\le\ & K\sqrt{\frac{2}{\pi}}\left(\left|\sqrt{t_1}-\sqrt{t_2}\right|^\alpha + |r_1-r_2|^\alpha\right),
\end{aligned}
$$

and therefore (RD4) holds. ∎

Remark 4.4.1 *If $d>1$, $\mathcal{O}=(0,\pi)^d$ and $L=-\Delta+I$ with Neumann boundary condition, then $L(t)=L$ does not satisfy the conditions of Kotelenez's theorem. In fact, as we shall see in Chapter 8, (4.3.1) has no solution in the ordinary sense.*

Proof: It is easy to see that (RD1) holds with Green function

$$\tilde{G}(t,r,q) = \prod_{i=1}^{d} G(t,r_i,q_i).$$

Note that

$$
\begin{aligned}
\sup_{r_i\in(0,\pi)}\int_0^\pi G(t,r_i,q_i)^2dq_i &\ge \liminf_{r_i\to 0}\frac{1}{\pi}\sum_{j=0}^{\infty}\exp(-2t(j^2+1))\cos^2 jr_i \\
&\ge \frac{1}{\pi}\sum_{j=0}^{\infty}\exp(-2t(j^2+1)) \\
&\ge \frac{e^{-2t}}{\pi}\int_0^\infty \exp(-2tx^2)dx \\
&\ge \frac{e^{-2T}}{2\sqrt{2\pi}}t^{-\frac{1}{2}}. \qquad (4.4.4)
\end{aligned}
$$

Therefore

$$\sup_{r\in\mathcal{O}}\int_{\mathcal{O}}|\tilde{G}(t,0,q)|^2dq \geq \left(\frac{e^{-2T}}{2\sqrt{2\pi}}\right)^d t^{-d/2}$$

and hence (RD2)i) does not hold as $d/2 \geq 1$. ∎

4.5 O-U SDE's driven by Poisson random measures

In Section 4.2 we had introduced the space time Ornstein-Uhlenbeck (O-U) SDE (or SPDE) driven by space-time Gaussian white noise. From the point of view of applications it is equally of interest to consider such an equation driven by a Poisson random measure (r.m.). The Poisson O-U SPDE which we study here is the simplest example of the more general theory developed in Chapter 6. In Chapter 7 we shall investigate in great detail, variants of the Poisson O-U SPDE which occur naturally in certain models of environmental pollution. It is convenient to write our SDE as an integral equation.

Let $\mathcal{X}$ be a bounded region of $\mathbf{R}^d$ and μ a σ-finite measure on the Borel sets of $\mathbf{R}_+ \times \mathcal{X}$. $N(dadxdt)$ be a Poisson random measure on $(\mathbf{R}_+ \times \mathcal{X}) \times \mathbf{R}_+$ with characteristic measure μ and denote by $\tilde{N}$, the compensated r.m.

First consider the simple case with $d = 1$ and $\mathcal{X} = [0,b]$. Let us introduce the relevant notation and assumptions: $H = L^2([0,b], \text{Leb})$, L is a positive definite, self-adjoint operator on H with dense domain $\mathcal{D}(L)$ such that $\mathcal{D}(L)$ contains Φ, the class of all smooth functions ϕ in H. Further, L^{-1} is a trace class operator so that L has a discrete spectrum $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, $\lambda_n \to \infty$ with $\sum_{j=0}^{\infty}\frac{1}{\lambda_j} < \infty$. It is also assumed that the corresponding eigenfunctions ϕ_j belong to Φ.

It is convenient to consider the equation in the form

$$du_t[\phi] = -u_t[L\phi]dt + \int_{\mathbf{R}_+}\int_{\mathcal{X}} a\phi(x)\tilde{N}(dxdadt) \tag{4.5.1}$$

or, as an integral equation

$$u_t[\phi] = u_0[\phi] - \int_0^t u_s[L\phi]ds + M_t[\phi] \tag{4.5.2}$$

where $M_t[\phi] = \int_0^t\int_{\mathbf{R}_+}\int_{\mathcal{X}} a\phi(x)\tilde{N}(dxdads)$ and $\phi \in \Phi$. The solution $u_t[\phi]$ is the average $u_t[\phi] = \int_{\mathcal{X}} u(t,x)\phi(x)dx$ where $u(t,x)$ is the solution process we seek. In general, it may happen that there is no random process or random field $u(t,x)$ such that $u_t[\phi]$ is given by the above integral. In that case $u_t[\phi]$ will be the evaluation at ϕ of a generalized random process or a process taking values in a suitable space of distributions. These ideas will be made precise a bit later and the theory of distribution valued SDE's will be discussed in more detail in Chapter 6.

Let us start with the same deterministic model as in (4.1.1) with $L = \alpha I - \beta \frac{d^2}{dx^2}$. Recall that $\lambda_j = \alpha + \beta \left(\frac{j\pi}{b}\right)^2$ and $\phi_j(x) = \left(\frac{2}{b}\right)^{\frac{1}{2}} \cos \frac{j\pi x}{b}$ for $j \geq 1$; $\lambda_0 = \alpha$, $\phi_0(x) = \left(\frac{1}{b}\right)^{\frac{1}{2}}$. Clearly, L^{-1} is a trace class operator. Observe that, for each ϕ, $M_t[\phi]$ is a square integrable, cadlag martingale with

$$\langle M.[\phi]\rangle_t = tQ(\phi, \phi)$$

where

$$Q(\phi, \psi) = \int_{\mathbf{R}_+} \int_{\mathcal{X}} a^2 \phi(x) \psi(x) \mu(dxda), \tag{4.5.3}$$

assumed finite for each ϕ, $\psi \in \Phi$. The SDE (4.5.1) yields the following infinite system of one dimensional linear SDE's

$$du_t^j = -\lambda_j u_t^j dt + dM_t^j, \qquad j = 0, 1, 2, \cdots$$

where $u_t^j = u_t[\phi_j]$ and $M_t^j = M_t[\phi_j]$. The above equation has the unique solution

$$u_t^j = e^{-\lambda_j t} u_0^j + e^{-\lambda_j t} \int_0^t e^{\lambda_j s} dM_s^j, \qquad t \geq 0. \tag{4.5.4}$$

Let

$$u_N(t, x) \equiv \sum_{j=0}^{N} u_t^j \phi_j(x). \tag{4.5.5}$$

Then for $M < N$, $u_N(t, x) - u_M(t, x) = \sum_{j=M+1}^{N} u_t^j \phi_j(x) \in H$ a.s. for each t. Also

$$\begin{aligned}
& E\|u_N(t, \cdot) - u_M(t, \cdot)\|_H^2 \\
= \ & E \int_0^b \sum_{j,k \in \{M+1, \cdots, N\}} \phi_j(x) \phi_k(x) u_t^j u_t^k dx \\
= \ & \sum_{j=M+1}^{N} E(u_t^j)^2 \\
= \ & \sum_{j=M+1}^{N} e^{-2\lambda_j t} E(u_0^j)^2 + \sum_{j=M+1}^{N} \frac{1 - e^{-2\lambda_j t}}{2\lambda_j} Q(\phi_j, \phi_j) \\
\to \ & 0
\end{aligned}$$

as M, $N \to \infty$ since

$$\sum_{j=0}^{\infty} E(u_t^j)^2 \leq \sum_{j=0}^{\infty} e^{-2\lambda_j t} E(u_0^j)^2 + \frac{1}{2} \sum_{j=0}^{\infty} \frac{Q(\phi_j, \phi_j)}{\lambda_j} < \infty$$

if we impose the following conditions:

$$E\|u_0\|_H^2 = \sum_{j=0}^{\infty} E(u_0^j)^2 < \infty, \tag{4.5.6}$$

$$\sum_{j=0}^{\infty} \frac{Q(\phi_j, \phi_j)}{\lambda_j} < \infty. \tag{4.5.7}$$

Defining

$$u(t, x) = \sum_{j=0}^{\infty} u_t^j \phi_j(x),$$

it is easy to verify that $u_t[\phi]$ is a solution of our SDE and that $E\|u(t,\cdot) - u_N(t,\cdot)\|_H^2 \to 0$.

We will now show that almost surely, $u(t,x)$ belongs to the Skorohod space $D([0,T], H)$. The latter is a complete, separable metric space with the usual Skorohod topology under the metric

$$d(h,k) = \inf_{\lambda_1,\lambda_2 \in \Lambda_T} \max \left\{ \sup_{0 \le t \le T} \|h \circ \lambda_1(t) - k \circ \lambda_2(t)\|_H, \delta_T(\lambda_1, \lambda_2) \right\}$$

where

$$\begin{aligned} \Lambda_T \;=\; & \{\lambda : \lambda(\cdot) \text{ is a continuous and strictly increasing function from } \mathbf{R}_+ \\ & \text{to } \mathbf{R}_+,\ \lambda(0) = 0,\ \lambda(t) \to \infty \text{ as } t \to \infty \text{ and } \lambda(t) = t \text{ for all } t \ge T\} \end{aligned}$$

and

$$\delta_T(\lambda_1, \lambda_2) = \sup_{0 \le s < t \le T} \left| \log \frac{\lambda_1(t) - \lambda_1(s)}{\lambda_2(t) - \lambda_2(s)} \right|$$

(see Kallianpur and Wolpert [28]). For this we need to replace (4.5.7) by the stronger assumption

$$\sum_{j=0}^{\infty} Q(\phi_j, \phi_j) < \infty. \tag{4.5.8}$$

Note that

$$u_N(t,\cdot) - u_M(t,\cdot) \in D([0,T], H) \qquad \text{a.s.}$$

and

$$d(h,k) \le \sup_{0 \le t \le T} \|h(t) - k(t)\|_H. \tag{4.5.9}$$

From the SDE for u_t^j we have, using integration by parts and $M_0 = 0$,

$$u_t^j = e^{-\lambda_j t} u_0^j + M_t^j - \int_0^t \lambda_j e^{-(t-s)\lambda_j} M_s^j ds.$$

Hence

$$\sup_{0\le t\le T}(u_t^j)^2 \le 2(u_0^j)^2 + 8\sup_{0\le t\le T}(M_t^j)^2$$

and we have, using Doob's martingale inequality,

$$\begin{aligned}
&\sum_{j=0}^{\infty} E\sup_{0\le t\le T}(u_t^j)^2 \\
\le\ & 2\sum_{j=0}^{\infty} E(u_0^j)^2 + 8\sum_{j=0}^{\infty} E\sup_{0\le t\le T}(M_t^j)^2 \\
\le\ & 2\sum_{j=0}^{\infty} E(u_0^j)^2 + 32T\sum_{j=0}^{\infty} Q(\phi_j,\phi_j) < \infty.
\end{aligned}$$

Hence

$$\sum_{j=0}^{\infty}\sup_{0\le t\le T}(u_t^j)^2 < \infty \qquad \text{a.s.} \tag{4.5.10}$$

It follows from (4.5.9) and (4.5.10) that

$$\begin{aligned}
d^2(u_M,u_N) &\le \sup_{0\le t\le T}\|u_N(t,\cdot)-u_M(t,\cdot)\|_H^2 \\
&= \sum_{j=M+1}^{N}\sup_{0\le t\le T}|u_t^j|^2 \to 0 \qquad \text{a.s.}
\end{aligned}$$

From the completeness of $D([0,T],H)$ we have $d(u_N,u)\to 0$ a.s., i.e., $u\in D([0,T],H)$ a.s.

We now consider the situation when the spatial dimension d is greater than one. In the SDE (4.5.1) the following assumptions will be made. $\mathcal{X}$ is a bounded domain in $\mathbf{R}^d$, e.g. $\mathcal{X}=[0,b]^d$ and $H=L^2(\mathcal{X},\text{Leb})$, L is a positive definite, self-adjoint operator with domain in H such that $(I+L)^{-r_1}$ is Hilbert-Schmidt for some $r_1>0$. For Φ we take the countably Hilbertian nuclear space so that (Φ,H,L) is a compatible family as defined in Section 1.3.
(A.1) $Q(\phi,\psi)$ is a continuous bilinear form on $\Phi\times\Phi$.
(A.2) For some $r_3>0$, $E\|u_0\|_{-r_3}^2<\infty$.

By the nuclear theorem, (A.1) implies that there exist numbers $r_2\in\mathbf{R}$ and $\theta>0$ such that

$$Q(\phi,\phi)\le\theta\|\phi\|_{r_2}^2 \qquad \text{for all } \phi\in\Phi.$$

Theorem 4.5.1 *Under the above assumptions, the series on the right side of*

$$u_t=\sum_{j=0}^{\infty}u_t^j\phi_j \tag{4.5.11}$$

converges uniformly in $0 \leq t \leq T$ *in the* Φ_{-q} *topology for every* $T > 0$ *and for* $q \geq \max(r_1+r_2, r_3)$ *to a process* $u.$ *whose sample paths lie in the Skorohod space* $D(\mathbf{R}_+, \Phi_{-q})$ *of right continuous* Φ_{-q}*-valued functions on* $\mathbf{R}_+$ *with left limits at each point of* $(0, \infty)$.

The process u_t *defined by (4.5.11) satisfies the equation*

$$u_t[\phi] = u_s[T_{t-s}\phi] + \int_{(s,t]} dM_u[T_{t-u}\phi]$$

where T_t *is the semigroup on* H *determined by the generator* $-L$.

The process u_t *has the following additional properties:*

$$E \sup_{0 \leq t \leq T} \|u_t\|_{-q}^2 \leq C_T \qquad \text{for some constant } C_T < \infty.$$

Let $\mathcal{F}_t = \sigma\{u_0, M_s, 0 \leq s \leq t\}$ *be the smallest* σ*-algebra with respect to which* $u_0[\phi]$ *and* $M_s[\phi]$ *are measurable for all* $s \leq t$ *and* $\phi \in \Phi$. *Then* u_t *has the strict Markov property relative to* $(\mathcal{F}_t)$*; i.e.,* $\mathcal{F}_r$ *is conditionally independent of* $\sigma\{u_s[\phi], s \geq r, \phi \in \Phi\}$ *given* $\sigma\{u_r[\phi], \phi \in \Phi\}$.