THE REASSEMBLING OF SHATTERED BROWNIAN SHEET

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If $D_n : n \geq 1$ is a given countable collection of Borel subsets of the unit square $[0,1]^2 = I^2$ and Z is a standard Brownian sheet over I^2, is it possible to reconstruct Z from the knowledge of all of the patches $Z - c_n$ over transformed domains $\tau_n(D_n)$ for unknown constants c_n and unknown rotation-translation transformations τ_n? We show that the answer to this question is yes under fairly natural restrictions on the sets D_n. The main property of Brownian sheet that leads to this possibility is that the local behaviour of Z around a point $\mathbf{t}$ actually determines $\mathbf{t}$. In this sense, a Brownian sheet carries with it its own location coordinates.

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1 Introduction

It is a privilege to be able to contribute to this volume in honor of Professor van Zwet on the occasion of his 65$^{\text{th}}$ birthday. I have known Bill for about half of those 65 years, giving me many opportunities to observe and benefit from his warm hospitality, clever insights, wise counsel and keen enthusiasm evidenced throughout his numerous theoretical and professional undertakings.

In this written contribution I consider a specific question about Brownian sheet, one that might imply that if Humpty-Dumpty were to have had a 'Brownian complexion' then the ending to the popular nursery rhyme may have concluded with "All the king's men and all the king's horses *could* put Humpty together again."

I believe it is fair to say that most mathematical researchers work primarily on problems that are interrelated and part of long term programs. However, most of us also enjoy the challenging diversions that come along in the form of easily stated, fairly specific, open questions, especially when they do not easily yield to available theory and techniques. When pursued, such problems can have the beneficial result of leading the pursuers to new methods and theory. Such was the nature of the question mentioned by R. M. Dudley at a 1976 Oberwolfach meeting: "Does the Kakutani interval-splitting procedure (in which at each stage the largest spacing is uniformly divided) yield an asymptotically uniform partitioning of the unit interval?" This intriguing problem, based on a conjecture by S. Kakutani, got Bill's attention, and his solution, introducing a clever re-indexing allowing him to utilize martingale theory, appears in van Zwet (1978) in the form of a Glivenko-Cantelli theorem for the division points generated by the Kakutani procedure. Following a subsequent proof of a Glivenko-Cantelli result for the spacings generated by the Kakutani procedure in Pyke (1980), Bill and I commenced a collaboration into the weak convergence of the empirical processes associated with the partition points and the spacings. The study of this problem has been an interesting thread throughout many of our communications over the past 15 years or so; cf. Pyke and van Zwet (2000).

In this paper, I look at another quite specific problem whose most general formulation may also present some interesting challenges. Many puzzles outside of mathematics involve manipulating or assembling pieces in order to obtain specified shapes or pictures. In this paper, I pose a problem of this type, namely, that of reassembling a shattered Brownian surface.

In the following section, several formulations of the problem are illustrated in the one-dimensional case of Brownian motion. These are then studied for the main context of the random surface known as Brownian sheet in Section 3.

2 A Brownian motion illustration

Let $B = \{B_t : 0 \leq t \leq 1\}$ be standard Brownian motion. Pretend that the graph of B, say $G = \{(t, B_t) : 0 \leq t \leq 1\}$, is made of breakable thread which, upon being dropped, shatters into a possibly countable number of fragments. Suppose these fragments can be described as graph portions,

$$G_n := \{(t, B_t) : t \in D_n\}, \mathcal{G} = \{G_n : n \geq 1\},$$

for a countable family $\mathcal{D} = \{D_n : n \geq 1\}$ of disjoint Borel subsets of $[0,1]$. Assume there exists an interior point $t_n \in D_n$ for each n. Furthermore, although we do not wish to assume that $\mathcal{D}$ is a partition of $[0,1]$, we would want $\bigcup_n D_n$ to be nearly all of $[0,1]$. This will be discussed more later when studying the main case of Brownian sheet. For now, however, to give some concreteness to the problem, assume that the closure $\overline{\bigcup_n D_n} = [0,1]$ so that in some sense the fragments comprise almost all of the original graph. The D_n's represent the bases of the fragmented pieces of the graph.

Problem 1. Given $\mathcal{G}$, reconstruct B.

This is, of course, completely trivial in view of the continuity of B.

Now, suppose, more realistically, that upon shattering, the shattered pieces of the graph fall, so that instead of G_n one is given $G_n - (0, d_n)$ for some *unknown* d_n. Thus, the fragments of the Brownian curve that one has to reassemble are fragments that have fallen (or risen) *vertically*. Hence, one still knows the bases of the fragments but one does not know their vertical placement. This leads to

Problem 2. Given $\{G_n - (0, d_n) : n \geq 1\}$ for *unknown* constants d_n, reconstruct B.

The problem now is non-trivial, as may be illustrated by the Cantor functions. Suppose that the D_n's are the "middle thirds" of $[0,1]$ as used in constructing the standard Cantor function, f, say. Each of the fragments, $G_n - (0, d_n)$ are then flat and indistinguishable from those of the zero-function's graph over D_n. There is now no way of recapturing the original Cantor-like function, since there are uncountably many ways of transforming the vertical displacements of the flat segments without affecting the function's continuity. E.g., if $\varphi : [0,1] \to \mathbb{R}^1$ is any $1-1$ continuous mapping, $g : \varphi \circ f$ would have fallen fragments that are indistinguishable from those of f. Thus, in this case, one would need to use some property of f stronger than continuity in order to have a chance to identify f from its fragments. We discuss this further in Section 3.

Continuing to use Brownian motion to illustrate the problem of interest, suppose now that the fragments do not just fall vertically. Suppose instead that each fragment after the shattering is representable by $G_n - (h_n, d_n)$ for unknown real constants h_n, d_n, allowing therefore for arbitrary translations.

Problem 3. Given $\{G_n - (h_n, d_n) : n \geq 1\}$ for unknown constants h_n, d_n, reconstruct B.

This problem has, of course, no solution for Brownian motion in view of the stationarity of its increments.

Let us allow now for the turning of the graphs' fragments as they fall. If R_θ is used to denote the (counterclockwise) rotation of $\mathbb{R}^2$ about the origin through an angle θ, then suppose that one is given the fragments in the form $R_{\theta_n}(G_n) - (d_n, h_n)$ for unknown constants d_n, h_n and θ_n. That is, each fragment is allowed to rotate about the origin as well as being translated. The associated problem then becomes:

Problem 4. Given $\{R_{\theta_n}(G_n) - (d_n, h_n) : n \geq 1\}$ for unknown constants d_n, h_n and θ_n, reconstruct B.

Although our previous discussion shows that for Brownian motion the unknown translations (d_n, h_n) prevent the identifiability of B (unless the D_n's themselves have some special structure), observe that the rotation angles can be determined, within a multiple of π. This follows in particular from the law of the iterated logarithm (LIL) which implies that with probability 1, there exists for each fragment exactly one family of parallel lines that intersect the fragment at at most one point (thereby making the fragment the graph of a *function.*) This uses the assumption that each D_n contains an interior point (though an accumulation point should suffice). To be precise, if there were some rotation of the graph of $B_t : t \in D_n$, other than by a multiple of π, that was also that of a function, then there would be a family of parallel lines, not parallel to the y-axis, with each of them intersecting the graph G_n at no more than one point. Take the particular line in this family that passes through (t_n, B_{t_n}). The LIL implies that, almost surely, for each $\epsilon > 0$, both of the inequalities $B_{t_n+u} - B_{t_n} > \sqrt{u}$ and $B_{t_n+u} - B_{t_n} < -\sqrt{u}$ occur infinitely often in $u \in (0, \epsilon)$. Thus for *all* slopes c, $|c|u < \sqrt{u}$ for u sufficiently small, and so $B_{t_n+u} - B_{t_n} = cu$ infinitely often in $u \in (0, \epsilon)$, contradicting the possibility that the line of slope c through (t_n, B_{t_n}) intersects the graph of B_t only once.

It would be natural in these discussions to assume further that the bases D_n are connected, so that the fragments themselves are connected. If this assumption were to be made, the D_n's would then be intervals in this one-dimensional case. There would therefore be two choices of an end point to serve as the eventual left hand end point of the resultant curve. Through that end point the LIL then implies that there is precisely one supporting line for the curve that is therefore parallel to the 'y-axis' for the original Brownian motion. Identification then of the vertical direction is possible for each of the two choices for the left hand end point of the fragment.

It is of interest to consider how quadratic variation may be used in determining the rotation angle (up to a multiple of π) when the D_n's are (or at least contain) intervals. Suppose that points $P_1, P_2, \ldots, P_k$ are chosen on the n-th curve so that $P_j \equiv R_{\theta_n}(s_j, B_{s_j}) - (d_n, h_n)$ for some ordered index times, s_j. Include the end points P and P^* of the curve, using continuity if needed to define them. The quadratic variation for these points is

$$\sum_{j=1}^{k+1} |P_j - P_{j-1}|^2 \equiv \sum_{j=1}^{k+1} \{(s_j - s_{j-1})^2 + (B_{s_j} - B_{s_{j-1}})^2\}.$$

If for example, these points came from a single dense sequence of points, so that the resulting k-point partitions become refinements, then one has

$$\lim_{k\to\infty} \sum_{j=1}^{k+1} |P_j - P_{j-1}|^2 = \lim_{k\to\infty} \sum_{j=1}^{k+1} (B_{s_j} - B_{s_{j-1}})^2 = |D_n|$$

by Lévy (1940), where $|D_n|$, the Lebesgue measure, is also the length of the interval D_n in this case. If one then draws the circle of radius $r = |D_n|$ that is centered at one of the end points of the curve, exactly one of the two lines through the other end point that are tangent to this circle, must with probability one be parallel to the original 'y-axis'. It could again be chosen as the one that is part of a supporting line for the curve.

3 Reassembling shattered Brownian sheet

Let $Z = \{Z(v,w) : v \geq 0, w \geq 0\}$ be standard Brownian sheet, namely, a zero mean Gaussian process on the positive orthant $\mathbb{R}^2_+$ with covariance function

$$R(\mathbf{s},\mathbf{t}) = EZ(\mathbf{s})Z(\mathbf{t}) = (s_1 \wedge t_1)(s_2 \wedge t_2) \tag{1}$$

in which $\mathbf{s} = (s_1, s_2)$ and $\mathbf{t} = (t_1, t_2)$. The structure of a Brownian sheet as a generalization of Brownian motion is best seen by viewing it as a set-indexed process, in which (1) simply reflects the stationary and independent increments of the process, namely, whenever defined, $Z(A)$ and $Z(B)$ are independent $N(0, |A|)$ and $N(0, |B|)$ r.v.s if A and B are disjoint. Thus, with $Z(\mathbf{s}) \equiv Z([\mathbf{0}, \mathbf{s}])$, (1) states the consequence of this that

$$R(\mathbf{s},\mathbf{t}) = EZ([\mathbf{0},\mathbf{s}])Z([\mathbf{0},\mathbf{t}]) = |[\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}]|.$$

Direct calculation gives, for any $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^2_+$

$$\begin{aligned} E[Z(\mathbf{a}) - Z(\mathbf{b})]^2 = (a_1 \wedge b_1)|a_2 - b_2| \quad &+ \quad (a_2 \wedge b_2)|a_1 - b_1| \\ &+ \quad ((a_2 - b_2)(a_1 - b_1))^+ \end{aligned} \tag{2}$$

so that for $\mathbf{a}$, $\mathbf{b} \in B_r(\mathbf{1})$ and $|\mathbf{a} - \mathbf{b}| < 1$ with $0 < r < 1$,

$$(1 - r)|\mathbf{a} - \mathbf{b}| \leq E[Z(\mathbf{a}) - Z(\mathbf{b})]^2 \leq 4|\mathbf{a} - \mathbf{b}|. \tag{3}$$

The graph of Z over $I \equiv [0,1]^2$ is a random surface, namely, $G = \{(\mathbf{s}, Z(\mathbf{s}) : \mathbf{s} \in I^2\}$. Suppose that this surface is shattered into a collection $\mathcal{G}$ of surface fragments

$$G_n := \{(\mathbf{s}, Z(\mathbf{s})) : \mathbf{s} \in D_n\} \text{ with } \mathcal{G} = \{G_n : n \geq 1\}$$

in which each D_n is a Borel subset of I^2 that possesses an interior point, $\mathbf{t}_n$, say. Thus each of the fragments contains a patch with a circular base, and so, for each n, there exists a radius r_n such that the ball, $B_n := B_{r_n}(\mathbf{t}_n) \subset D_n$, and the sub-fragment

$$G_n^* := \{(\mathbf{s}, Z(\mathbf{s})) : \mathbf{s} \in B_n\} \subset G_n.$$

The assumption that each base contains an interior point not only insures that each fragment represents a tangible part of the surface, but also determines precisely the location of each fragment's base! This is because Brownian sheet surfaces carry with them built-in coordinates! Before giving the theorem that justifies this statement, it may be helpful to the reader to envisage the following description of the problem. Suppose that out of your sight a 3-dimensional random solid (paperweight?) is made from the Brownian sheet by filling in the vertical space beneath the surface $Z(\cdot)$ and above the random constant surface $f(\cdot) = C$, with the constant C chosen so that $Z > C$ throughout the base $I^2 = [0,1]^2$. Suppose further that a vertical core with a circular base is cut out from the interior of this solid. If this cylindrical column (with flat base and Brownian upper surface) is given to you, together with a drawing of I^2, could you use the information in its Brownian upper surface to decide precisely where the column should be placed on I^2 *and* how much to rotate it so that it will be oriented exactly as it was before being drilled out? The following theorem states that the answer is "yes". The only aspect that can not be determined is the core's vertical displacement that is implicit in $C = C(\omega)$.

Theorem 3.1 *If* $\mathbf{t} \in \mathbb{R}_+^2$, $\theta \in [0, 2\pi]$ *and* N *is an open ball centered at* $\mathbf{0} \in \mathbb{R}^2$ *for which* $N + \mathbf{t} \subset \mathbb{R}_+^2$, *then with probability 1, knowledge of* $\{Z(\mathbf{t} + \mathbf{h}) - Z(\mathbf{t}) : \mathbf{h} \in R_\theta(N)\}$ *determines* $\mathbf{t}$ *and the rotation* R_θ.

Proof Suppose N is the ball, $B_r(0)$ of radius $r > 0$. For any unit vector $\mathbf{w} = (w_1, w_2)$ and $\mathbf{t} \in \mathbb{R}_+^2$, define

$$Z_{\mathbf{t},\mathbf{w}}(u) = Z(\mathbf{t} + ur\mathbf{w}) - Z(\mathbf{t}), \qquad 0 \le u \le 1,$$

to be the increment process along the line segment of length r through $\mathbf{t}$ in the direction $\mathbf{w}$; we observe that $\mathbf{t} + r\mathbf{w} \in \mathbb{R}_+^2$. Let $\gamma_{\mathbf{w},r}$ denote this line segment; that is, $\gamma_{\mathbf{w},r}(u) = \mathbf{t} + ur\mathbf{w}$ for $0 \le u \le 1$. We will compute the quadratic variation of this process for every direction $\mathbf{w}$, and then show that these values determine $\mathbf{t}$ and the orientation of the axes.

In the following, we use the notation and results of Adler and Pyke (1993). For any $n \ge 1$ and partition $\pi_n \equiv \{0 \equiv u_{n,0} < u_{n,1} < \cdots < u_{n,k_n} \equiv 1\}$ of $I = [0,1]$, define the n-th level quadratic variation of a process W along a curve $\gamma : I \to \mathbb{R}_2^+$ by

$$Q_n(\gamma) = \sum_{j=1}^{k_n} \{W(\gamma(u_{nj})) - W(\gamma(u_{n,j-1}))\}^2.$$

The limit of $Q_n(\gamma)$ as $n \to +\infty$, when it exists, is denoted by $q(\gamma)$. Under assumptions including uniform Lipshitz continuity of the curves γ and uniform second order continuous differentiability of the covariance functions of $W(\gamma)$, it is known (beginning with Baxter (1956); see Adler and Pyke (1993) for recent results and a full statement of assumptions and their application to Brownian sheet) that the limit exists almost surely, and can be represented by

$$q(\gamma) = \int_0^1 g_\gamma(u)du \tag{4}$$

in which $g_\gamma(u) = D_\gamma^-(u) - D_\gamma^+(u)$ and the D's here are the left and right hand derivatives at diagonal points of the process' covariance function $R_\gamma(u,v) = EW(\gamma(u))W(\gamma(v))$. That is,

$$D_\gamma^-(u) = R_\gamma^{(1,0)}(u-,u), D_\gamma^+(u) = R_\gamma^{(0,1)}(u,u+)$$

with $R_\gamma^{(ij)}(u,v) = (\partial^{i+j}/\partial u^i \partial v^j) R_\gamma(u,v)$.

The special Brownian sheet case described in the first paragraph of this proof, is worked out as Example 4.2 in Adler and Pyke (1993), where it is shown that for $\gamma = (\gamma_1, \gamma_2)$, the integrand in (4) is of the form

$$g_\gamma(u) = |\dot\gamma_2(u)|\gamma_1(u) + |\dot\gamma_1(u)|\gamma_2(u). \tag{5}$$

Thus, for the rays $\gamma_{\mathbf{w},r}(u) = (t_1 + urw_1, t_2 + urw_2)$, $0 \le u \le 1$,

$$q(\gamma_{\mathbf{w},r}) = r(|w_2|t_1 + |w_1|t_2) + \frac{r^2}{2}(|w_2|w_1 + |w_1|w_2). \tag{6}$$

Although for our purposes here it suffices to know $q(\gamma_{\mathbf{w},r})$ for a countably dense set of directions $\mathbf{w}$, and so the result of Baxter (1956) would suffice, we use the uniformity result of Adler and Pyke (1993) for convenience since it permits one to speak of obtaining with probability one from the given patch of Brownian sheet all of the quadratic variations $q(\gamma_{\mathbf{w},r})$ at once. In particular, knowledge of the $q(\gamma_{\mathbf{w},r})$'s yields

$$\lim_{r\to 0+} r^{-1} g_{\gamma_{\mathbf{w},r}}(u) = |w_2|t_1 + |w_1|t_2. \tag{7}$$

But this, the scalar projection of $\mathbf{t}$ onto $(|w_2|, |w_1|)$, is the length of the projection of $\mathbf{t}$ onto $\mathbf{w}^\perp \equiv (w_2, -w_1)$ when $w_2 \ge 0$, $w_1 \le 0$, which is in turn the width of the rectangle $[\mathbf{0}, \mathbf{t}]$ in the direction $\mathbf{w}$. In fact, one may check that as $\mathbf{w}$ varies around the unit circle in the counterclockwise direction, starting at $(1,0)$, the 'width' function $W(\mathbf{w}) := |w_2|t_1 + |w_1|t_2$ varies alternatively between relative minima at each of the axis directions $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ and relative maxima (each equal to the lengths of the diagonals of $[\mathbf{0},\mathbf{t}]$) at the four directions perpendicular to the diagonals $\mathbf{t}$ and $(t_1, -t_2)$. Thus, the axis directions and the dimensions of the rectangle are determined. This implies that given the rotated fragment $G^* = \{(\mathbf{t}+\mathbf{h}, Z(\mathbf{t}+\mathbf{h}) - Z(\mathbf{t})) : \mathbf{h} \in R_\theta(N)\}$, one is able to determine the set of coordinates $\{t_1, t_2\}$ and the orientation θ of the fragment up to a multiple of $\pi/2$. The reader may check that there are four possible positionings of the fragment in each of the two cases, $t_1 < t_2$ or $t_1 > t_2$. That is, for $a = t_1 \wedge t_2$ and $b = t_1 \vee t_2$, the fragment may be placed either at (a,b) with an orientation for which the larger of the two relative minima occurs for horizontal directions, or at (b,a) with an orientation that yields the larger minima along vertical directions. In either case, there are four possibilities for the location of the true positive quadrant.

To complete the determination of the correct location and orientation, one may repeat the above for the new point $\mathbf{t}^*$ which is the selected location of (a, b) or (b, a) shifted by $(h, 2h)$, where h is positive and sufficiently small to insure that $(h, 2h) \in N$. Depending upon the chosen orientation, this shift may in reality be any one of the four values, $\pm(h, 2h)$ and $\pm(2h, -h)$. Since only one of these will result in a set of larger values, either $\{a + h, b + 2h\}$ or $\{a + 2h, b + h\}$, the correct orientation is thereby identified. Moreover, t_1 (and hence $\mathbf{t}$) is also determined: It is the unique value in $\{a, b\}$ that increased by h.

Alternatively, the determination could be completed by using (6) and observing that

$$q(\gamma_{\mathbf{w},r}) - rW(\mathbf{w}) = \frac{r^2}{2}(|w_2|w_1 + |w_1|w_2)$$

as a function of r equals zero whenever $w_1 w_2 \leq 0$, is increasing in r when $w_1 > 0$, $w_2 > 0$ and is decreasing in r when $w_1 < 0$, $w_2 < 0$. Thus, after $W(\cdot)$ is found, it follows that one is able to determine which of the four quadrants in the previous paragraph is the positive one. Consequently, t_1 and t_2 are determined: t_2 is the minimum value of $W(\mathbf{w})$ that obtains when $\mathbf{w} = (0, 1)$. ∎

Consider now for Brownian sheet, the problems discussed in Section 2. Here, $\mathcal{D}$ is a countable family of disjoint Borel subsets D_n of I^2, with each $D_n \in \mathcal{D}$ having an interior point $\mathbf{t}_n$ and hence being of positive Lebesgue measure. Also, the graph over the base D_n is now

$$G_n = \{(\mathbf{t}, Z_{\mathbf{t}}) : \mathbf{t} \in D_n \subset I^2\}$$

The analog of Problem 1 is again addressed by the continuity of Z. The analog of Problem 3 now has an affirmative answer for recovering the translations of the bases in view of Theorem 3.1. However, Theorem 3.1 permits us to consider an extension of Problem 3 in which rotations of the bases D_n are allowed. (In the one-dimensional context of Section 2, the only rotation of this type that could have been considered, would have been the flip of the direction of the t-axis, but this could not be inferred from Brownian motion.) Let $T_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denote the transformation that rotates $\mathbb{R}^3$ around its z-axis; namely, $T_\theta((x, y, z)) = (R_\theta(x, y), z)$. The extended Problem 3 may then be stated as

Problem 5. Given $\{T_{\theta_n}(G_n - (x_n, y_n, z_n)) : n \geq 1\}$ for unknown constants θ_n, x_n, y_n, z_n, reconstruct Z.

An application of Theorem 3.1 to each of the moved fragments shows that with probability one, the partial translations (x_n, y_n) and the bases' rotations θ_n can be uniquely determined. Thus each such transformed fragment G_n of the graph of Brownian sheet may be relocated correctly in all respects except for its vertical placement.

If the fragments of the Brownian sheet's graphs are allowed to rotate in $\mathbb{R}^3$ (and not just around the z-axis as for Problem 5) as they fall, one is led to the following analog of Problem 4 in which the two-dimensional rotation is replaced by a three-dimensional one:

Problem 4*. . Given $\{\tau_n(G_n) - (x_n, y_n, z_n)\}$ for unknown rotations τ_n and constants x_n, y_n, z_n, reconstruct B.

The main question to be addressed here is: Given a rotated fragment, can one determine which way is up? Certainly, if there were only one positioning of the fragment (together with its upside-down reflection) that makes it the graph of a function, the problem would be solved. We show that this is the case. Without loss of generality, we focus on the surface in the neighborhood of a fixed point $\mathbf{t} = \mathbf{1}$, and show that for every non-vertical line in $\mathbb{R}^3$ that passes through $(1, 1, Z(\mathbf{1}))$, there exists a parallel line passing through a neighborhood of that point that intersects the graph of Z more than once. The reader should note that it is a straightforward consequence of known LIL's that for a *fixed* direction $\mathbf{w}$, the probability is one that every non-vertical line through $(1, 1, Z(\mathbf{1}))$, whose projection onto the (x, y) plane is parallel to $\mathbf{w}$, intersects $Z_{\mathbf{t},\mathbf{w}}(\cdot)$ more than once. But this is not enough. One has to know this holds with probabilty one for *all* directions $\mathbf{w}$.

Observe first that for a given continuous real function f that passes through the origin, and a given slope $\beta \in \mathbb{R}^1$, if f has both an "upper chord" (one whose slope exceeds β) and a "lower chord" (one whose slope is less than β), there must exist a line of slope β that intersects f at least twice. It therefore suffices for our purposes to establish that along every ray through $\mathbf{1} \in \mathbb{R}^2$ of the form $\mathbf{1} + t\mathbf{u}$ for $0 \le t \le r < 1$ and direction vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ satisfying $|\mathbf{u}| = 1$, the process $\{Z(\mathbf{1} + t\mathbf{u}) : 0 \le t \le r\}$ has both arbitrarily steep increasing and decreasing chords. It is convenient to consider a large increasing (decreasing) chord to be one satisfying

$$(8) \qquad Z(\mathbf{a}) - Z(\mathbf{b}) > |\mathbf{a} - \mathbf{b}|^{1/2}; \quad (Z(\mathbf{a}) - Z(\mathbf{b}) < -|\mathbf{a} - \mathbf{b}|^{1/2}),$$

respectively. These choices make for simple calculations and clearly imply the existence of chords with arbitrary large and small slopes. We will show that with probability one, each of the inequalities in (8) hold for some pair of points along every ray of the form $\mathbf{1} + (\cdot)\mathbf{u}$. This then implies, by the above comment, the existence of a line in $\mathbb{R}^3$ that is parallel to any line through $(1, 1, Z(\mathbf{1}))$ and which intersects the surface of Z more than once.

The approach used here is to establish the following result; other approaches are mentioned in Remark 3 below.

Lemma 3.1 *For all* $r \in (0,1)$

$$P\left[\inf_{\mathbf{u}:|\mathbf{u}|=1}\ \sup_{0<s<t\le r} \frac{Z(\mathbf{1}+t\mathbf{u}) - Z(\mathbf{1}+s\mathbf{u})}{\sqrt{t-s}} > 1\right] = 1$$

Proof First of all, for any $\mathbf{a}, \mathbf{b} \in B_r(\mathbf{1})$ with $0 < r < 1$, it follows from the left hand inequality of (2) that

$$\begin{aligned} P[Z(\mathbf{a}) - Z(\mathbf{b}) > |\mathbf{a}-\mathbf{b}|^{\frac{1}{2}}] &\ge P[N(0,1) > (1-r)^{-1/2}] \\ &\ge r\sqrt{1-r}(1/\sqrt{2\pi})e^{-1/2(1-r)} := c_r > 0 \end{aligned} \tag{9}$$

by the standard Mills ratio bounds for normal tails (cf. Feller (1968), Lemma VII.2). Since the lower bound in (9) is independent of $\mathbf{a}$ and $\mathbf{b}$, it follows that whenever $u_1u_2 \ge 0$, for any integer $n \ge 1$,

$$\begin{aligned} P[Z(\mathbf{1}+\frac{k}{n}r\mathbf{u}) - Z(\mathbf{1}+\frac{k-1}{n}r\mathbf{u}) &> (r/n)^{1/2} \text{ for some } k=1,\dots,n] \\ &\ge 1-(1-c_r)^n \end{aligned} \tag{10}$$

in view of the independence of the increments when $u_1u_2 \ge 0$. By letting $n \to \infty$, it then follows that for each $r \in (0,1)$ and each direction $\mathbf{u}$ for which $u_1u_2 \ge 0$, there exists with probability one an increment that exceeds the square-root of its base's length as desired.

When $u_1u_2 < 0$, the increments appearing in (10) are no longer independent. Assume w.l.o.g. that $u_1 < 0 < u_2$. Viewing Brownian sheet as being defined on all rectangles, introduce

$$Z_r^*(\mathbf{t}) = Z([0,\mathbf{t}] \cap L_{r,\mathbf{u}}) \tag{11}$$

where $L_{r,\mathbf{u}}$ is the L-shaped region

$$L_{r,\mathbf{u}} = \{(x,y) \in \mathbb{R}^2 : x \le 1 + ru_1 \text{ or } y \le 1\}$$

that is the complement of the upper right orthant with vertex at $(1+ru_1, 1)$. Now, if one repeats the argument for (10), but using Z_r^* in place of Z, one still has independent increments with the same bound and therefore the same conclusion about a.s. existence of large and small chords obtains.

It remains to handle the differences between the desired increments for Z and those of Z_r^*. To do this, introduce the Gaussian difference rv's

$$W_{r,n,k} = \sqrt{n}\{(Z_r^* - Z)(\mathbf{1}+\frac{k}{n}r\mathbf{u}) - (Z_r^* - Z)(\mathbf{1}+\frac{k-1}{n}r\mathbf{u})\}. \tag{12}$$

The supremum of the variances of these rv's is

$$\sup_{\substack{1\le k\le n\\ n\ge 1}} E[W_{r,n,k}]^2 \le \sup_{\substack{1\le k\le n\\ n\ge 1}} \left\{ \frac{n-k}{n} r|u_1|\frac{r}{n}|u_2| + \frac{k-1}{n} r|u_2|\frac{r}{n}|u_1| \right\} n$$

$$\le r^2|u_1u_2| \le \frac{r^2}{2}. \tag{13}$$

Thus by Borell's inequality for bounded Gaussian processes (Adler (1990), Chap. II.1 and Corollary 4.15)

$$P[\sup_{\substack{1\le k\le n\\ n\ge 1}} |W_{r,n,k}| > r^{3/4}] \le \text{const.}e^{-1/\sqrt{r}} \tag{14}$$

a bound that may be made arbitrarily small as $r \to 0$. One is using here the smallness of the process' metric entropy (the index set is essentially $[0, 2\pi) \times [0, 1]$) and hence the finiteness of the expectation of the sup-norm of the process.]

Let A_r denote the event, described following (10), that chords of desired slope exist for Z along $\mathbf{u}$ within $B_r(\mathbf{1})$. Let A_r^* denote the same event, but for Z_r^* rather than Z and in which the slopes are *double* those for A_r. The discussion following (11) shows that $P(A_r^*) = 1$ for each r. Moreover, the above shows that for any $\epsilon > 0$, there is an $r_\epsilon > 0$ for which $P(B_r) > 1 - \epsilon$ for $r < r_\epsilon$ where B_r is the complement of the event in (14). Since $r < 1$, $A_r^* \cap B_r \subset A_r$ so that then $P(A_r) > 1 - \epsilon$. But by definition of the events A_r it follows that $A_r \subset A_{r'}$ for $r < r'$. Together, this implies that $P(A_r) = 1$ for every $0 < r < 1$.

To this point, we have shown that for each fixed direction $\mathbf{u}$ and each $r \in (0, 1)$, the probability is one that any non-vertical line in $\mathbb{R}^3$ through $(1, 1, Z(\mathbf{1}))$ whose projection on $\mathbb{R}^2$ is parallel to $\mathbf{u}$ has a parallel line that intersects the graph of Z over $B_r(\mathbf{1})$ more than once. The same is therefore true for a countably dense set of directions $\mathbf{u}$. It remains to show that the statement holds with probability one for all directions simultaneously.

Fix $\epsilon > 0$. Define

$$(\Delta Z)_{r,n,k,\mathbf{u}} = Z(\mathbf{1} + \frac{k}{n} r\mathbf{u}) - Z(\mathbf{1} + \frac{k-1}{n} r\mathbf{u}) \tag{15}$$

and

$$M_{\epsilon,r,\mathbf{u}^*} = \sup_{\substack{|\mathbf{u}-\mathbf{u}^*|\le\epsilon\\ |\mathbf{u}|=1}} \sup_{\substack{1\le k\le n\\ n\ge 1}} \sqrt{n}|(\Delta Z)_{r,n,k,\mathbf{u}} - (\Delta Z)_{r,n,k,\mathbf{u}^*}|. \tag{16}$$

From (2) and (3) above and Lemma 2.1 of Adler and Pyke (1993), the variances of the Gaussian rv's in the definition of $M_{\epsilon,r,\mathbf{u}^*}$ satisfy

$$nE\left[(\Delta Z)_{r,n,k,\mathbf{u}} - (\Delta Z)_{r,n,k,\mathbf{u}^*}\right]^2 \le nC\left\{\left[\frac{r}{n}(|\mathbf{u}| \vee |\mathbf{u}^*|)\right] \wedge\right.$$

$$\left[\left(\frac{k}{n}r|\mathbf{u}-\mathbf{u}^*|\right) \vee \left(\frac{k-1}{n}r|\mathbf{u}-\mathbf{u}^*|\right]\right\}$$

$$(17) \qquad = C\left\{r \wedge \left(\frac{k}{n}r|\mathbf{u}-\mathbf{u}^*|\right)\right\} \leq Cr\epsilon,$$

for some constant C. Thus by Borell's inequality one may again deduce that there is a constant C_1 for which

$$P[M_{\epsilon,r,\mathbf{u}^*} > z] \leq C_1 \exp\{-z^2/2Cr\epsilon\}.$$

In particular,

$$(18) \qquad P[M_{\epsilon,r,\mathbf{u}^*} \leq \sqrt{r}\,] \geq 1 - C_1 e^{-1/2C\epsilon}.$$

Now, the inequalities $(\Delta Z)_{r,n,k,\mathbf{u}^*} > 2\sqrt{r}$ and $M_{\epsilon,r,\mathbf{u}^*} \leq \sqrt{r}$ imply that $(\Delta Z)_{r,n,k,\mathbf{u}} > \sqrt{r}$. But with probability 1, the first of these inequalities has been shown to occur for some n and k for each $\mathbf{u}^*$ in a countable dense set of directions. By (18), the probability equals $1 - O(\epsilon^{-1}\exp(-1/2C\epsilon))$ that the second inequality holds for all $\mathbf{u}^*$ in an ϵ-net of directions. Since this probability may be made arbitrarily close to 1 by an appropriate choice of ϵ, it follows that with probability one, the third inequality holds for some n and k for every $\mathbf{u}$. The situation concerning arbitrarily steep decreasing chords follows immediately since $Z \overset{L}{=} -Z$, thereby completing the proof. ∎

The above result then permits one to reposition the transformed fragment almost surely into either the correct orientation or the upside down position. In either case, once one utilizes Theorem 3.1 above to identify the correct location coordinates of at least two points, one is able to determine whether the surface is right side up or not. This would then complete a solution (up to the determination of vertical displacement) of Problem 4*.

The following natural extension of Problem 2 is now the key remaining question, namely,

Problem 2*. Given $\{G_n - (0,0,d_n) : n \geq 1\}$ for unknown constants d_n, reconstruct B. In order for this problem to be solvable it is necessary for the bases in $\mathcal{D}$ of these resituated fragments to comprise 'nearly all' of I^2. That is, some condition on the size of $\bigcup D_n$ is necessary. Even then, however, the standard Cantor function over I^2 shows that it would not be enough to know its vertically displaced patches over the countable disjoint middle-third rectangles whose union has I^2 as its closure. All functions that are constant over each D_n would be indistinguishable from it. One needs to make use of some additional property of the surface beyond continuity. Thus, the question before us here is whether or not Z is (uniquely) determinable a.s. from the function ζ defined by

$$(19) \qquad \zeta(\mathbf{t}) = \sum_{n=1}^{\infty}\{Z(\mathbf{t}) - Z(\mathbf{t}_n)\}1_{D_n}(\mathbf{t}),$$

the function that locates each fragment's base properly but sets its height (w.l.o.g.) at the specified interior point t_n to be zero.

Suppose $\mathcal{D}$ satisfies the following assumption:

$$\text{For each } n \geq 1,\ D_n \text{ is the closure of its interior } D_n^0, \quad \text{the closure of } \bigcup_n D_n \text{ is } I^2 \text{ and } |\textstyle\bigcup_n D_n^0| = 1. \tag{20}$$

Thus, the bases of the fragments have relatively nice boundaries and the excess 'dust' generated by the shattering, namely the part of the graph over the complement of $\bigcup_n D_n$, is 'negligible'. This assumption (20) also insures that for each n and each $\mathbf{s} \in I^2$ a set-indexed value $Z(D_n^0 \cap [\mathbf{0}, \mathbf{s}])$ is determined uniquely by the incremental information contained in the patch G_n. Thus for every $\mathbf{s} \in I^2$, define

$$Z^*(\mathbf{s}) = \sum_{n=1}^{\infty} Z(D_n^0 \cap [\mathbf{0}, \mathbf{s}]),$$

a mean zero normal r.v. with

$$\operatorname{var}(Z^*(\mathbf{s})) = \sum_{n=1}^{\infty} |D_n^0 \cap [\mathbf{0}, \mathbf{s}]| = |\bigcup_n D_n^0 \cap [\mathbf{0}, \mathbf{s}]| = |[\mathbf{0}, \mathbf{s}]|$$

by (20). Moreover, $Z^*(\mathbf{s}) = Z(\mathbf{s})$ a.s. In particular, this determines with probability 1 the heights $Z(\mathbf{t}_n)$ for every n as required to solve Problem 2*. The values of Z on the boundaries $\partial D_n = D_n \setminus D_n^0$ and on the complement of $\bigcup_n D_n$ are determined by the continuity of Z; note that (20) implies that the closure of $\bigcup_n D_n^0$ is also I^2.

Remarks. 1. Although it is possible to have bases D_n that by their particular shapes can be fit together into I^2 in only one way, the discussions here do not consider such possible information. Our emphasis is upon utilizing only the information in the Brownian surface so that the procedures apply to such uninformative shapes as rectangles and balls.

2. Let us comment further on the extendability of the definition of the Brownian sheet Z from points $\mathbf{t} \in I^2$ (or, equivalently, rectangles $[\mathbf{0}, \mathbf{t}] \subset I^2$) to the family of sets

$$\mathcal{A} = \{D \cap (\mathbf{s}, \mathbf{t}] : D \in \mathcal{D}, \mathbf{s}, \mathbf{t} \in I^2\}$$

used in the above derivation. Under assumption (20), each D in the countable family $\mathcal{D}$ is a closed subset of I^2 that satisfies $|\partial(D)| = |D \setminus D^0| = 0$ and the associated family of sets $\mathcal{A}_D = \{D \cap (\mathbf{s}, \mathbf{t}] : \mathbf{s}, \mathbf{t} \in I^2\}$ indexed by I^2 is, for example, of zero metric entropy under inclusion with respect to the Hausdorff or symmetric-difference metrics. Thus a continuous extension of Z

on $\mathcal{A}_0$, and hence on $\mathcal{A}$, exists that agrees with the given $Z((\mathbf{s},\mathbf{t}])$ whenever $(\mathbf{s},\mathbf{t}] \subset D$.

3. It should be pointed out that the existence of chords of arbitrarily large and small slopes in the trajectories of Z along fixed rays through $\mathbf{1}$ could also be shown by means of LIL's or Hölder results for the particular Gaussian processes involved. For directions $\mathbf{u}$ with $u_1u_2 \geq 0$, such results are easily deduced since along such rays Z is Brownian motion with a quadratically, rather than linearly, growing variance. In the other directions the dependent increments require results for more general Gaussian processes. It would be of interest to obtain general uniform LIL and uniform Hölder results for Gaussian processes analogous to the uniform quadratic variation result of Adler and Pyke (1993). In this regard, the reader should note the powerful results in Dalang and Mountford (1996). In particular, Theorem 2 of their paper, in the equivalent form given prior to their equation (2), implies that with probability one, it is true that for every $\theta \in [0, 2\pi)$ and $\mathbf{t}$ in a small rectangle about $\mathbf{1}$, it is true that $|Z(\mathbf{s}) - Z(\mathbf{t})| > c|\mathbf{s}-\mathbf{t}|$ for infinitely many $\mathbf{s}$ for which $\mathbf{s}-\mathbf{t}$ has direction θ, regardless of the value of $c > 0$. (Much more is proved in Dalang and Mountford (1996) since this statement about the increments is shown to hold along all Jordan curves constrained within wedges about the lines of direction θ, and not just along the straightline segments needed here.) A modification of this result that would resolve Problem 4* is that in which the absolute values of the increments is replaced by the positive and negative parts, $(Z(\mathbf{s}) - Z(\mathbf{t}))^+$ and $(Z(\mathbf{s}) - Z(\mathbf{t}))^-$. It would then be possible to utilize these results in the same way that the LIL was used to resolve Problem 4 in Section 2. However, this extension of the Dalang and Mountford result is not immediate; in particular, the representation in their Lemma 1(b) does not hold. (The author is grateful to Robert Dalang for his assistance on this point.)

4. In Arató (1997), the problem of estimating μ when $Z + \mu$ is observed over a domain $D \subset I^2$ is considered, and its maximum likelihood estimator derived. The results of this paper permit the context to be modified so that the base D may be treated as being unknown.

5. In this paper, only distance preserving transformations have been considered. It would be of interest to allow for other perturbations of the bases, or even of the graphs, that are recoverable from the coordinate information inherent in Brownian sheet.

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