Sequential Selection of an Increasing Subsequence From a Random Sample with Geometrically Distributed Sample-Size

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Following a long-standing suggestion by Samuels and Steele we study the problem of sequential selection of an increasing subsequence from a random sample of size N, where N is geometrically distributed with parameter p. The maximum expected length of a subsequence which can be selected by a nonanticipating policy is shown to be asymptotic to $p^{-1/2}$, as $p \to 0$.

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1. Introduction. There has been a great deal of interest in long increasing subsequences of a random sequence (see surveys [10], [1]). A central result in this vein says that the expected length of the longest increasing subsequence in a sequence of n random items is asymptotic to $2n^{1/2}$ (see [8],[11]).

Samuels and Steele [9] studied selection of a long increasing subsequence as a dynamical decision problem. In their model, n i.i.d. items with known continuous distribution are inspected in strict succession. Each item can be selected or rejected only at the time it is inspected: once rejected, the item cannot be recalled and if accepted cannot be discarded. The selected sequence must increase. The objective it to maximize the expected length of the selected sequence. Samuels and Steele showed that for n large the maximum expected length is $v \sim (2n)^{1/2}$ and demonstrated a policy which attains this value asymptotically. Comparing v with the length of the longest increasing subsequence, Samuels and Steele interpreted the ratio $2 : 2^{1/2}$ as the long-run advantage of a prophet with complete foresight of the sequence over an intelligent but non-clairvoyant gambler, who observes the items in time and must exploit nonanticipaiting policies.

In the same paper, Samuels and Steele introduced an analogous model where the gambler does not know the number of observations, which is a random variable N with given distribution. This selection problem is more difficult than its fixed-n counterpart, because the uncertainty over N adds a new component to the risk.

For concentrated distributions like Poisson ($\lambda \to \infty$) or binomial ($k \to \infty$, p fixed) the generic value of N does not deviate much from its mean, and EN is therefore a good estimate for the number of observations. For such distributions the prophet's value is about $2(EN)^{1/2}$, the gambler's value is about $(2EN)^{1/2}$ and the prophet-to-gambler ratio is still $2^{1/2}$ (see [9], [5]).

The problem with N geometrically distributed (parameter p) was introduced in [9] as the next most complex case. Motivated by their fixed-n result Samuels and Steele put forth the conjecture that for $p \to 0$

$$v \sim cp^{-1/2}$$

with some positive constant $c < 2^{1/2}$.

Stating the conjecture Samuels and Steele kept an eye on the quantity $(2EN)^{1/2}$ which is relevant to concentrated distributions and amounts to $\sim 2^{1/2}p^{-1/2}$ in the geometric case. However, in general $E(2N)^{1/2}$ provides a much better asymptotic upper bound on v. To justify this bound we just need to apply the fixed-n result to the *informed* version of the problem where the gambler is told the value of N prior to the observation. In the geometric case the new benchmark says that c cannot be larger than $(\pi/2)^{1/2}$.

Although the conjecture is settled trivially as it was stated, the substantial question implicit in the Samuels–Steele paper remains: what is the gambler's price for the ignorance about the value of N? In particular, is the prophet-to-gambler ratio higher than $2^{1/2}$?

In this note we show that the Samuels–Steele constant is $c = 1$. Because the prophet's value is asymptotic to

$$E(2N^{1/2}) \sim \pi^{1/2}p^{-1/2},$$

the prophet-to-gambler ratio increases from $2^{1/2}$ in the informed version of the problem to $\pi^{1/2}$.

In [5] the problem with arbitrary random N was solved in full generality and the main result of this note follows from the asymptotics for v proved there. A good reason to present the geometric case separately is that it can be treated by direct analysis of the optimality equation, along the lines initiated in [9].

2. Setup. To set the problem formally, let $X_1, X_2, \ldots$ be i.i.d. random points sampled from a continuous distribution. A specific form of the distribution does not matter, so we will assume that it is uniform $[0,1]$. Let N

be independent of the X_j's, with geometric distribution

$$P(N = j) = p(1-p)^j, \quad j = 0, 1, \ldots$$

We consider decision policies which are adapted to $X_1, X_2, \ldots$ and, at each stage j, prescribe either to select or to reject X_j. To assure that the selected sequence is ascending we require that X_j could be accepted only if X_j is greater than all items accepted so far.

More precisely, a policy is defined to be an infinite sequence of (finite) stopping times $\tau = (\tau_1, \tau_2, \ldots)$, which are adapted to the X_j's (but not to N) and satisfy

$$1 \leq \tau_1 < \tau_2 < \ldots, \qquad X_{\tau_1} < X_{\tau_2} < \ldots$$

Let

$$L_\tau = \#\{i : \tau_i \leq N\},$$

be the length of the subsequence selected by τ from $X_1, \ldots, X_N$. The performance of τ is measured by the expected value EL_τ. The basic quantity of interest is the maximum expected length

$$v(p) = \sup_\tau EL_\tau\,.$$

Note: although τ selects infinitely many items we count only those which fall within $X_1, \ldots, X_N$ (which is an empty sequence if $N = 0$). In case $\tau_i = j$ the ith item selected by τ is X_j and this event depends solely on $X_1, \ldots, X_j$.

To understand better the problem think first of the *greedy* policy which selects all consecutive records. In particular, the greedy policy always selects X_1. The value of this policy is $\log p^{-1}$, that is the expected number of records in $X_1, \ldots, X_N$. For p large there is a high probability for $N \leq 1$ and the greedy policy does not perform wrong when selecting the first item whatever its size.

In contrast to this, for small p the typical value of N is large. If X_1 is large the greedy policy cannot be optimal because accepting X_1 blocks selecting of many future items of smaller sizes. Quite naturally, the gambler should set a limit on the size of the first item in order to optimize.

3. Asymptotic solution. A characteristic feature of the geometric case is a kind of self-similarity of the selection problem. Namely, given that at stage j the last selected item is of size x, further selection is possible only from those forthcoming items which are larger than x. The number of such items is geometrically distributed with parameter

$$s = \frac{p}{p + (1-x)(1-p)} \tag{1}$$

while their sizes are uniformly distributed on $(x, 1]$. Upon obvious identifications the selection from $X_{j+1}, X_{j+2}, \dots$ becomes equivalent to the original problem with the modified parameter.

Applying this idea to the first observation and evaluating two possible decisions we arrive at a dynamic programming equation

$$(2) \qquad v(p) = (1-p) \int_0^1 \max\left\{v(p), 1 + v\left(\frac{p}{p + (1-x)(1-p)}\right)\right\} \mathrm{d}x,$$

(This is a corrected version of the equation appearing in Samuels-Steele [9]. Also, note that the geometric distribution on top of their p. 946 should start with $k = 0$.)

From the equation or directly from the definition of $v(p)$ as a supremum one sees readily that $v(p)$

(a) is continuous on $(0, 1]$,

(b) is strictly decreasing,

(c) goes to infinity as $p \downarrow 0$,

(d) has the boundary value $v(1) = 0$.

Upon plugging (1) equation (2) becomes

$$(3) \qquad v(p) = \int_p^1 (v(s) + 1 - v(p))_+ s^{-2} \mathrm{d}s$$

where $+$ denotes the positive part.

Define $p(1)$ by $v(p(1)) = 1$. For $p > p(1)$ taking the positive part is redundant and (3) is easily solved as

$$(4) \qquad v(p) = \log p^{-1},$$

which does not look unexpectedly since the greedy policy is optimal for sufficiently large p. It follows that $p(1) = e^{-1}$.

For other values of p there is no simple formula. One explanation for this is that v is only piecewise analytical. Defining $p(k)$ by $v(p(k)) = k$ for $k = 0, 1, \dots$ we have

(d) $v(p)$ is analytical on each interval $[p(k+1), p(k)]$,

(e) the $(k+1)$th derivative at $p(k)$ has a jump, as it will be clear from what follows.

For $p > p(1)$ set $s^*(p) = 1$ and for $p < p(1)$ define $s^*(p)$ to be a single solution to

$$v(s) + 1 - v(p) = 0. \tag{5}$$

Let $p(v)$ be the inverse function to $v(p)$.

For $p < p(1)$ the integral term in (3) can be transformed as

$$\begin{aligned}\int_p^{s^*(p)} (v(s)+1-v(p))s^{-2}\mathrm{d}s &= \int_p^{s^*(p)} s^{-2}\mathrm{d}s \int_{v(p)-1}^{v(s)} \mathrm{d}u \\ &= \int_{v(p)-1}^{v(p)} \mathrm{d}u \int_{p(v)}^{p(u)} s^{-2}\mathrm{d}s \\ &= \int_{v(p)-1}^{v(p)} \left(\frac{1}{p(v)} - \frac{1}{p(u)}\right)\mathrm{d}u\end{aligned}$$

so that the inverse function satisfies

$$v = \frac{1}{p(v)} - \int_{v-1}^{v} \frac{\mathrm{d}u}{p(u)} .$$

Substituting the reciprocal function

$$q(v) = \frac{1}{p(v)}$$

and differentiating we obtain for $v > 1$ a retarded differential equation

$$q'(v) = 1 + q(v) - q(v-1)\,, \tag{6}$$

which should be complemented for $v < 1$ by

$$q(v) = e^v \tag{7}$$

in accord with (4). Now it is seen that q'' has a jump at $v = 1$, whence the properties (d) and (e) follow by induction in k.

Decomposing $q(v)$ as

$$q(v) = e^v f(v)$$

leads to the retarded Cauchy problem for $f(v)$

$$f'(v) = e^{-v} - e^{-1}f(v-1), \qquad v > 1$$

with the initial germ $f(v) = 1$, $v \in [0,1]$. The equation is solved recursively, by integrating over consecutive intervals with integer endpoints: for $v \in [k, k+1]$ we have

$$f(v) = e^{-k} - e^{-v} + f(k) + e^{-1}\int_k^v f(u-1)\mathrm{d}u.$$

Computing for $k = 1, 2$ by this formula we find

$$f(v) = 1 + \frac{2}{e} - e^{-v} - \frac{v}{e}, \qquad v \in [1,2]$$

$$f(v) = 1 + \frac{5}{e^2} + \frac{2}{e} - 2e^{-v} - \frac{(3+e)v}{e^2} + \frac{v^2}{2e^2}, \qquad v \in [2,3].$$

The expressions become quickly involved as k grows, not to say about $v(p)$ which must be obtained by inverting $p(v) = 1/(e^v f(v))$.

To get around consider the Laplace transform

$$\hat{q}(\lambda) = \int_0^\infty q(v) e^{-\lambda v} dv.$$

Multiplying (6) by $e^{-\lambda}$ and integrating in $\lambda \in [1, \infty)$ yields

$$\hat{q}(\lambda) = \frac{\lambda e^\lambda + 1}{\lambda^2 e^\lambda - \lambda e^\lambda + \lambda} \,.$$

This cannot be simply inverted, but the asymptotic analysis of $q(v)$ is now easy via the behaviour of the Laplace transform at 0. Expanding at zero we get

$$\hat{q}(\lambda) = \frac{2}{\lambda^3} + \frac{2}{3\lambda^2} + \frac{19}{18\lambda} + \cdots$$

where the rest is an entire function. Applying familiar Tauberian arguments (see [4], p. 151) we obtain for $v \to \infty$

$$q(v) = v^2 + \frac{2}{3} v + \frac{19}{18} + o(1), \tag{8}$$

where the o term goes to zero faster than any inverse power of v.

Computing $q(v)$ numerically shows that (8) gives an approximation of supreme quality even for moderate values of v. For example, the numerical value 19.7(2) computed by (8) for $v = 4$ coincides with the true value up to four decimal points.

Inverting (8) leads to

Theorem 1 *For* $p \to 0$

$$v(p) = \left(p^{-1} - \frac{17}{18}\right)^{1/2} - \frac{1}{3} + \cdots$$

where the remainder goes to zero faster than any power of p.

which is our main result. We see that the conjectured constant is $c = 1$.

3. Selection policies. The optimal selection policy amounts to the next rule. Suppose at stage j the size of the last item selected is y and the observed item $X_j = x$ is larger than y. Then X_j should be selected if and only if

$$\frac{p}{p+(1-x)(1-p)} \leq s^*\left(\frac{p}{p+(1-y)(1-p)}\right). \tag{9}$$

Stating it differently, the gambler should set certain thresholds. Let $\delta^*(y)$ be the critical value of x which turns (9) into equality. If the size of the last item selected so far is y the next selected item should be of size between y and $y+\delta^*(y)$.

Asymptotics of $v(p)$ implies

$$s^*(p) = p + 2p^{3/2} + \cdots$$

whence

$$\delta^*(y) \sim 2(1-y)^{1/2}p^{1/2}$$

uniformly in $y \in [0,1]$. We will prove next that taking the threshold suggested by this formula results in a policy which is close to optimality for large n.

Let $\hat{\tau}$ be a policy with a threshold function satisfying

$$\delta(y) \sim 2(1-y)^{1/2}p^{1/2}. \tag{10}$$

Theorem 2 *Policy $\hat{\tau}$ is asymptotically optimal, that is*

$$EL_{\hat{\tau}} \sim p^{-1/2}, \quad \text{as } p \to 0.$$

Proof. Define Y_j by setting $Y_0 = 0$ and, recursively,

$$Y_{j+1} = \begin{cases} X_{j+1}, & \text{if } Y_j < X_{j+1} \leq Y_j + \delta(Y_j) \\ Y_j, & \text{otherwise.} \end{cases}$$

The Markov sequence $\{Y_j\}$ describes the selection process by $\hat{\tau}$, with Y_j being the size of the last item selected from $X_1, \ldots, X_j$. Clearly, $\{Y_j\}$ is weakly increasing: a jump occurs at index j exactly when $\hat{\tau}$ selects X_j. Let $M(y)$ be the number of jumps and $N(y)$ the number of observations until $\{Y_j\}$ exceeds y.

We write $\approx$ for 'with probability tending to one as $p \to 0$'. Because the threshold function is smooth, the law of large numbers implies $M(y) \approx \mu(y)p^{-1/2}$ for some smooth μ with $\mu(0) = 0$. To determine μ take a small increment Δ and note that

$$M(y) - M(y-\Delta) \approx 2\Delta/\delta(y) + o(\Delta),$$

because the (conditional) expected height of the first jump which follows $Y_j = y$ is $\delta(y)/2$. Letting $\Delta \to 0$ and integrating we get $\mu(y) = 2-2(1-y)^{1/2}$.

Similarly, noting that the periods between successive jumps are geometrically distributed we have

$$N(y) - N(y - \Delta) \approx 2\Delta/(\delta(y))^2 + o(\Delta),$$

whence $N(y) \approx \nu(y)p^{-1}$ with $\nu(y) = -\log(1-y)^{1/2}$.

For a generic j of order p^{-1}, Y_j is close to $y = 1 - e^{-2jp}$, as it follows by inverting $N(y)$. The number of jumps to this time is then $M(y) \approx p^{-1/2}(2 - 2e^{-jp})$. The expected number of choices from N observations is obtained by averaging the number of jumps over the geometric distribution. Finally, using the exponential approximation gives

$$EL_{\hat{\tau}} \sim p^{-1/2} \sum_{j=0}^{\infty} p(1-p)^j (2 - 2e^{-jp}) \sim$$

$$p^{-1/2} \int_0^{\infty} e^{-t}(2 - 2e^{-t})\mathrm{d}t = p^{-1/2}. \qquad \square$$

4. Selected sequence. The optimal policy is stationary in the sense that the threshold which applies to X_j depends only on the size Y_j of the last item accepted so far, but does not depend on j explicitly. In [5] another asymptotically optimal policy is constructed, which has a threshold depending only on index j, and not on Y_j.

In other words, there are two very different almost optimal policies; the first policy has time-independent thresholds, and the second policy has state-independent thresholds. An explanation of this phenomenon lies in the concentration pattern which appeared in the proof of Theorem 2. That is to say, the optimal sequence converges in a sense to a non-random process, so that a deterministic dependence between state and space variables appears in the limit.

To state it expicitly, let $\phi_p(t)$, $t \geq 0$, be the piecewise linear function which interpolates the random function $pj \mapsto Y_j$ defined on the lattice $\{pj\}_{j=1,2,\ldots}$. Set $\phi(t) = 1 - e^{-t}$.

Theorem 3 *Suppose the threshold function satisfies* (10). *Then as* $p \to 0$

$$\sup_t |\phi_p(t) - \phi(t)| \to 0$$

in probability. Furthermore, the length of the selected subsequence satisfies

$$L_{\hat{\tau}} \sim 2p^{-1/2}\phi(T)$$

(in probability), where T *is a standard exponential variable.*

From a broader perspective this theorem is an instance of approximating the path of a Markov chain by an integral curve of a differential equation, as studied in [7]. Concentration resaults for increasing subsequences in the fixed-n case are found in [3]. We refer to [5], [2], [6] for further results on sequential selection of increasing subsequences.

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