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A TECHNIQUE OF CONSTRUCTING PLANAR HARMONIC MAPPINGS AND THEIR PROPERTIES

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Abstract

The analytic part of a planar harmonic mapping plays a vital role in shaping its geometric properties. For a normalized analytic function f defined in the unit disk, define an operator $\Phi[f](z) = f(z) + \overline{f(z) - z}$. In this paper, necessary and sufficient conditions on f are determined for the harmonic function $\Phi[f]$ to be univalent and convex in one direction. Similar results are obtained for $\Phi[f]$ to be starlike and convex in the unit disk. This results in the coefficient estimates, growth results and convolution properties of $\Phi[f]$. In addition, various radii constants associated with $\Phi[f]$ have been computed.

1. Introduction

Let $\mathscr{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$ and $\mathscr{S}$ be its subclass consisting of univalent functions. Let $\mathscr{H}$ denote the class of all harmonic functions $f = h + \bar{g}$, where h and g are analytic in $\mathbf{D}$ and normalized so that $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. Therefore, if $f = h + \bar{g} \in \mathscr{H}$, then

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbf{D}. \tag{1.1}$$

The functions h and g are called analytic and co-analytic parts of f respectively. By Lewy's theorem [9], we know that the Jacobian of a locally univalent harmonic function does not vanish. Thus the Jacobian of a locally univalent function $f \in \mathscr{H}$ is, in view of $|f_z(0)|^2 - |f_{\bar{z}}(0)|^2 = |h'(0)|^2 - |g'(0)|^2 = 1 > 0$, positive in $\mathbf{D}$, and so f is sense-preserving in $\mathbf{D}$. Let $\mathscr{S}_H^0$ be the subclass of $\mathscr{H}$ consisting of sense-preserving univalent functions. Finally, let $\mathscr{S}_H^{*0}$, $\mathscr{K}_H^0$ and $\mathscr{C}_H^0$ be the subclasses of $\mathscr{S}_H^0$ consisting of functions mapping $\mathbf{D}$ onto starlike, convex and

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close-to-convex domains, respectively, just as $\mathscr{S}^*$, $\mathscr{K}$ and $\mathscr{C}$ are the subclasses of $\mathscr{S}$ mapping $\mathbf{D}$ onto their respective domains.

The analytic part h of a harmonic mapping $f = h + \bar{g}$ plays a crucial role in shaping the geometric properties of f (for instance, see [5, Theorem 5.17, p. 20] and [3, Theorem 1, p. 768]). Consequently, univalent harmonic mappings can be constructed in such a manner that the co-analytic part is a slight modification of its analytic part. Motivated by these ideas and Shear Construction Theorem [5, Theorem 5.3, p. 14], we define an operator $\Phi : \mathscr{A} \to \mathscr{H}$ by

$$\Phi[f](z) = f(z) + \overline{f(z) - z}, \quad z \in \mathbf{D},\ f \in \mathscr{A}.$$

If $f \in \mathscr{A}$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbf{D}) \tag{1.2}$$

then

$$\Phi[f](z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} a_n z^n}, \quad z \in \mathbf{D}.$$

In this paper, we study the geometric properties of the operator Φ. In Section 2, necessary and sufficient conditions are obtained for $\Phi[f]$ to be univalent and convex in one direction. As a consequence, coefficient bounds and convolution properties are investigated. In the last section of the paper, the radius of convexity and other related radii constants are determined corresponding to the function $\Phi[f]$. The following lemma will be needed in our investigation which determines a sufficient coefficient condition for functions of the form $f = h + \bar{g} \in \mathscr{H}$ to be in the classes $\mathscr{S}_H^{*0}$ and $\mathscr{K}_H^0$. It is worth to note that these conditions in fact yield the sufficient conditions for functions to be fully starlike and fully convex in $\mathbf{D}$ (see [1, 4, 16]).

Lemma 1.1 ([2]). *Let $f = h + \bar{g} \in \mathscr{H}$ where h and g are given by (1.1). If $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1$, then $f \in \mathscr{S}_H^{*0}$ and if $\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \le 1$, then $f \in \mathscr{K}_H^0$. Moreover, if $a_n \le 0$ and $b_n \ge 0$ for $n \ge 2$, then these conditions are also necessary for f to be in $\mathscr{S}_H^{*0}$ and $\mathscr{K}_H^0$.*

2. Properties of the operator Φ

If we consider the Koebe function $k(z) = z/(1-z)^2 \in \mathscr{S}$, then it is easy to see that the harmonic function $\Phi[k]$ is not univalent in $\mathbf{D}$, since its Jacobian vanishes inside $\mathbf{D}$. In particular, this shows that $\Phi[\mathscr{S}] \not\subset \mathscr{S}_H^0$, $\Phi[S^*] \not\subset \mathscr{S}_H^{*0}$ and $\Phi[\mathscr{C}] \not\subset \mathscr{C}_H^0$. Similarly, if $l(z) = z/(1-z) \in \mathscr{K}$, then the Jacobian of the function $\Phi[l](z) = z/(1-z) + \bar{z}^2/(1-\bar{z})$ vanishes at $z = 1 - \sqrt{2}$ and hence $\Phi[\mathscr{K}] \not\subset \mathscr{K}_H^0$. The following theorem determines a subclass of $\mathscr{S}$ which is mapped into $\mathscr{C}_H^0 \subset \mathscr{S}_H^0$ by the operator Φ.

Theorem 2.1. *Let* $f \in \mathcal{A}$. *Then we have the following*:
(i) $\Phi[f]$ *is sense-preserving in* $\mathbf{D}$ *if and only if* $\operatorname{Re} f'(z) > 1/2$ *for all* $z \in \mathbf{D}$.
(ii) *If* $\operatorname{Re} f' > 1/2$ *in* $\mathbf{D}$, *then* $\Phi[f] \in \mathcal{S}_H^0$ *and is convex in the direction of real axis. In particular*, $\Phi[f]$ *is close-to-convex in* $\mathbf{D}$.

Proof. (i) Write $\Phi[f] = h + \bar{g}$, where $h(z) = f(z)$ and $g(z) = f(z) - z$ are analytic functions in $\mathbf{D}$. Then $\Phi[f]$ is sense-preserving in $\mathbf{D} \Leftrightarrow |h'(z)| > |g'(z)| \Leftrightarrow |f'(z)| > |f'(z) - 1| \Leftrightarrow \operatorname{Re} f'(z) > 1/2$ for all $z \in \mathbf{D}$.

(ii) If $\operatorname{Re} f' > 1/2$ in $\mathbf{D}$, then $\Phi[f] = h + \bar{g}$ is sense-preserving in $\mathbf{D}$ by part (i). Also, $h(z) - g(z) = z$ is univalent and convex in the direction of real axis. Therefore, by Shear Construction Theorem [5, Theorem 5.3, p. 14], $\Phi[f]$ is univalent and is convex in the direction of real axis. □

Corollary 2.2. *If* $f \in \mathcal{A}$ *is given by* (1.2) *and* $\Phi[f] \in \mathcal{S}_H^0$, *then* $|a_n| \le 1/n$ *for all* $n = 2, 3, \ldots$. *The bound* $1/n$ *is best possible. Moreover, the sharp inequality* $|\Phi[f](z)| \le -|z| - 2\log(1 - |z|)$ *holds for all* $z \in \mathbf{D}$.

Proof. By Theorem 2.1(i), $\operatorname{Re} f' > 1/2$ in $\mathbf{D}$ which gives $|a_n| \le 1/n$ for $n \ge 1$ and

$$|\Phi[f](z)| \le |z| + 2\sum_{n=2}^{\infty} |a_n|\,|z|^n \le |z| + 2\sum_{n=2}^{\infty} \frac{1}{n}|z|^n = -|z| - 2\log(1 - |z|)$$

for all $z \in \mathbf{D}$.

Since the analytic function $f_0(z) = -\log(1 - z)$ satisfies $\operatorname{Re} f_0'(z) > 1/2$ for all $z \in \mathbf{D}$, therefore the harmonic function

$$\Phi[f_0](z) = -2\log|1 - z| - \bar{z} = z + \sum_{n=2}^{\infty} \frac{z^n}{n} + \sum_{n=2}^{\infty} \frac{\bar{z}^n}{n}, \quad z \in \mathbf{D} \tag{2.1}$$

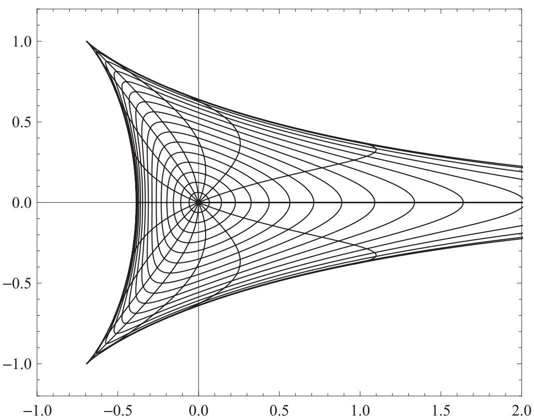

Figure 1. Image of the unit disk under $\Phi[f_0](z) = -2\log|1 - z| - \bar{z}$

belongs to the class $\mathcal{S}_H^0$. Figure 1 illustrates that the image domain $\Phi[f_0](\mathbf{D})$ is convex in the direction of real axis. □

If $f \in \mathcal{A}$ is given by (1.2), then it is easily seen that if $\sum_{n=2}^{\infty} n|a_n| \le 1/2$, then $\Phi[f] \in \mathcal{S}_H^{*0}$ and if $\sum_{n=2}^{\infty} n^2|a_n| \le 1/2$, then $\Phi[f] \in \mathcal{K}_H^0$ by Lemma 1.1. For the special case $f(z) = z + a_2 z^2 \in \mathcal{A}$, the following theorem determines the necessary and sufficient coefficient conditions for the function $\Phi[f]$ to belong to the classes $\mathcal{S}_H^0$, $\mathcal{S}_H^{*0}$, $\mathcal{K}_H^0$ and $\mathcal{C}_H^0$.

THEOREM 2.3. *Let* $f(z) = z + a_2 z^2 \in \mathcal{A}$. *Then*
(a) $\Phi[f] \in \mathcal{S}_H^0 \Leftrightarrow |a_2| \le 1/4$;
(b) $\Phi[f] \in \mathcal{S}_H^{*0}$ *(or* $\mathcal{C}_H^0$*)* $\Leftrightarrow |a_2| \le 1/4$;
(c) $\Phi[f] \in \mathcal{K}_H^0 \Leftrightarrow |a_2| \le 1/8$.
The constants $1/4$ *and* $1/8$ *are best possible.*

Proof. (a) If $a_2 = 0$, then we have nothing to prove. Therefore, assume that $a_2 \neq 0$. If $\Phi[f] \in \mathcal{S}_H^0$, then $\operatorname{Re} f' > 1/2$ in $\mathbf{D}$ by Theorem 2.1(i). It is easy to deduce that $\operatorname{Re}(1 + 2a_2 z) \ge 1/2$ on $|z| = 1$. In particular, for $z = -e^{-i\arg(a_2)}$, we have $1 - 2|a_2| \ge 1/2$ which simplifies to $|a_2| \le 1/4$. Conversely, if $|a_2| \le 1/4$, then $|f'(z) - 1| = 2|a_2|\,|z| < 2|a_2| \le 1/2$ so that $\operatorname{Re} f'(z) > 1/2$ for all $z \in \mathbf{D}$. By Theorem 2.1(ii), $\Phi[f] \in \mathcal{S}_H^0$.

(b) If $\Phi[f] \in \mathcal{S}_H^{*0}$ or $\mathcal{C}_H^0$, then by part (a), $|a_2| \le 1/4$. Conversely, let $|a_2| \le 1/4$. Then $\Phi[f] \in \mathcal{C}_H^0$ by Theorem 2.1(ii), since a domain convex in the direction of real axis is close-to-convex. Also, $\Phi[f] \in \mathcal{S}_H^{*0}$ since $2|a_2| \le 1/2$ (by the discussion preceding Theorem 2.3).

(c) Let $\Phi[f] \in \mathcal{K}_H^0$. Without loss of generality, we may assume that $a_2 \ge 0$. Since $\Phi[f](\mathbf{D})$ is a convex set, we have

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}\Phi[f](e^{i\theta})\right\}\right) \ge 0, \quad 0 \le \theta < 2\pi.$$

By a straightforward calculation, the last expression reduces to

$$\operatorname{Re}\left(\frac{z + 8a_2 \operatorname{Re}(z^2)}{z + 4ia_2 \operatorname{Im}(z^2)}\right) \ge 0 \quad \text{for } |z| = 1.$$

In particular, at $z = -1$, we have $1 - 8a_2 \ge 0$ which gives the desired result. As $4|a_2| \le 1/2$, the converse part is obvious.

For sharpness of the results, consider the analytic functions $g(z) = z + z^2/4$ and $h(z) = z + z^2/8$. Figure 2 depicts that the harmonic functions

$$\Phi[g](z) = z + \frac{z^2}{4} + \frac{\bar{z}^2}{4} \quad \text{and} \quad \Phi[h](z) = z + \frac{z^2}{8} + \frac{\bar{z}^2}{8}$$

map $\mathbf{D}$ onto starlike and convex domain respectively. □

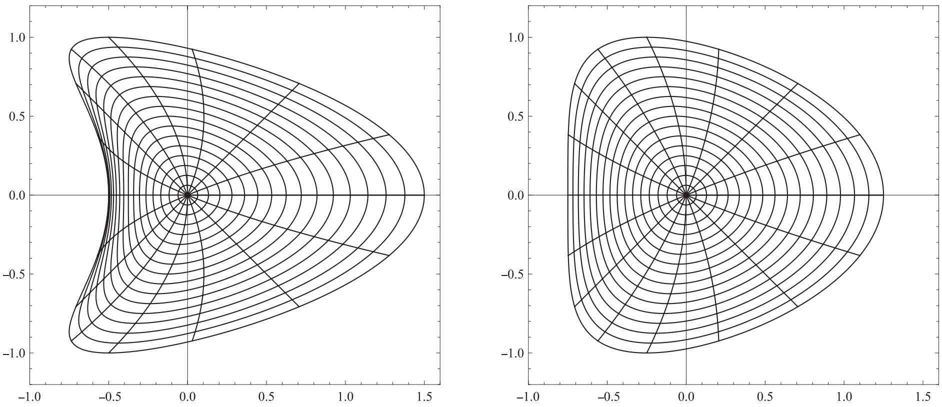

FIGURE 2. Images of the unit disk under $\Phi[z+z^2/4]$ and $\Phi[z+z^2/8]$.

The study of convolution properties of harmonic mappings is a fairly active area of research (see [6–8, 15, 17]). Given two analytic functions $f(z)=\sum_{n=1}^{\infty} a_n z^n$ and $F(z)=\sum_{n=1}^{\infty} A_n z^n$, their analytic convolution is defined as $(f*F)(z)=\sum_{n=1}^{\infty} a_n A_n z^n$. In the harmonic case, with $f=h+\overline{g}$ and $F=H+\overline{G}$, their harmonic convolution is defined as $f*F=h*H+\overline{g*G}$. The following theorem investigates the convolution properties of the function $\Phi[f]$.

THEOREM 2.4. (a) *If $f_1, f_2 \in \mathscr{A}$ with $\operatorname{Re}(f_1*f_2)'>1/2$ in $\mathbf{D}$, then $\Phi[f_1]*\Phi[f_2]\in \mathscr{S}_H^0$ and is convex in the direction of real axis.* (b) *If $f\in\mathscr{A}$ and L is the harmonic half-plane mapping defined as*

$$L(z)=M(z)+\overline{N(z)}, \quad M(z):=\frac{z-\frac{1}{2}z^2}{(1-z)^2}, \quad N(z):=\frac{-\frac{1}{2}z^2}{(1-z)^2}, \quad z\in\mathbf{D}$$

then $L*\Phi[f]$ is univalent and convex in the direction of imaginary axis if and only if $f\in\mathscr{K}$.

Proof. (a) It is easy to see that $(\Phi[f_1]*\Phi[f_2])(z)=(f_1(z)+\overline{f_1(z)-z})*(f_2(z)+\overline{f_2(z)-z})=(f_1*f_2)(z)+\overline{(f_1*f_2)(z)-z}=\Phi[f_1*f_2](z)$ so that the result follows by invoking Theorem 2.1(ii).

(b) Observe that

$$(L*\Phi[f])(z)=\frac{1}{2}(f(z)+zf'(z))+\overline{\frac{1}{2}(f(z)-zf'(z))}=T_1[f](z), \quad z\in\mathbf{D}$$

where $T_c[f]$ $(c>0)$ is the operator defined by Muir [11]. By [11, Theorem 3.2, p. 225], it follows that $L*\Phi[f]$ is univalent and convex in the direction of imaginary axis if and only if $f\in\mathscr{K}$. $\square$

Note that Theorem 2.4(a) was independently proved by the last two authors [17, Corollary 2.2, p. 1330]. If $f = h + \bar{g} \in \mathscr{H}$, then the δ-neighborhood of f denoted by $N_\delta(f)$ (see [2]) is the set consisting of all harmonic functions

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n}, \quad z \in \mathbf{D} \tag{2.2}$$

satisfying $\sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - b_n|) \le \delta$. The last result of this section deals with the neighborhood of $\Phi[f]$.

THEOREM 2.5. *If* $f \in \mathscr{A}$ *is given by* (1.2) *with* $\sum_{n=2}^{\infty} n^2|a_n| \le 1/2$, *then* $N_\delta(\Phi[f]) \subset \mathscr{S}_H^{*0}$ *for* $0 < \delta \le 1/2$.

Proof. Let $F \in N_\delta(\Phi[f])$ be given by (2.2). Then

$$\sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - b_n|) \le \delta,$$

so that

$$\begin{aligned}\sum_{n=2}^{\infty} n(|A_n| + |B_n|) &\le \sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - a_n|) + 2\sum_{n=2}^{\infty} n|a_n| \\ &\le \delta + \sum_{n=2}^{\infty} n^2|a_n| \le \delta + \frac{1}{2} \le 1.\end{aligned}$$

By Lemma 1.1, $F \in \mathscr{S}_H^{*0}$. □

3. Radii constants

By Figure 1, it is evident that if a function $f \in \mathscr{A}$ satisfies $\operatorname{Re} f'(z) > 1/2$ for all $z \in \mathbf{D}$, then $\Phi[f] \in \mathscr{S}_H^0$ need not map $\mathbf{D}$ onto a convex domain. Therefore it is interesting to determine the largest radius $\rho < 1$ for which the functions $\Phi[f]$ with the condition $\operatorname{Re} f'(z) > 1/2$ map the subdisk $|z| < \rho$ onto a convex domain. This is achieved in the next theorem which makes use of the result that for every $r > 0$ and every harmonic mapping $f = h + \bar{g}$ in a disk $\{z \in \mathbf{C} : |z| < R\}$ with $R > r$, the curve $[0, 2\pi] \ni \theta \mapsto f(re^{i\theta})$ is convex if and only if for every $\theta \in [0, 2\pi]$,

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right) = \operatorname{Re}\left(\frac{zh'(z) + z^2h''(z) + \overline{zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right) \ge 0$$

where $z = re^{i\theta}$.

THEOREM 3.1. *Let* $f \in \mathscr{A}$ *with* $\operatorname{Re} f'(z) > 1/2$ *for all* $z \in \mathbf{D}$. *Then* $\Phi[f] \in \mathscr{S}_H^0$ *and maps the disk* $|z| < \sqrt{2} - 1$ *onto a convex domain. The bound* $\sqrt{2} - 1$ *is best possible.*

Proof. By Theorem 2.1(ii), $\Phi[f]$ is univalent in $\mathbf{D}$. Consequently, it suffices to show that $\operatorname{Re}((zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)})(\overline{zh'(z)}-zg'(z)))>0$ for $|z|<\sqrt{2}-1$, where $\Phi[f]=h+\bar{g}$. Observe that

$$\begin{aligned}
&\operatorname{Re}((zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)})(\overline{zh'(z)}-zg'(z)))\\
&\quad=|z|^2|h'(z)|^2+|z|^2\operatorname{Re} zh''(z)\overline{h'(z)}-\operatorname{Re} z^3h''(z)g'(z)\\
&\qquad-|z|^2|g'(z)|^2+\operatorname{Re} z^3h'(z)g''(z)-|z|^2\operatorname{Re} zg''(z)\overline{g'(z)}.
\end{aligned}$$

On substituting $h(z)=f(z)$ and $g(z)=f(z)-z$, the last expression simplifies to

$$\begin{aligned}
&\operatorname{Re}((zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)})(\overline{zh'(z)}-zg'(z)))\\
&\quad=|z|^2|f'(z)|^2+\operatorname{Re} z^3f''(z)-|z|^2|f'(z)-1|^2+|z|^2\operatorname{Re} zf''(z)\\
&\quad=2|z|^2\operatorname{Re} f'(z)-|z|^2+|z|^2\operatorname{Re} zf''(z)+\operatorname{Re} z^3f''(z)\\
&\quad\geq 2|z|^2\operatorname{Re} f'(z)-|z|^2-2|z|^3|f''(z)|\\
&\quad=|z|^2(2\operatorname{Re} f'(z)-1-2|z|\,|f''(z)|).
\end{aligned}$$

Making use of the fact that [13, Corollary 3, p. 213] an analytic function p in $\mathbf{D}$ with $p(0)=1$ and $\operatorname{Re} p(z)>\alpha$ for all $z\in\mathbf{D}$ and $\alpha\in[0,1)$ satisfies

$$|p'(z)|\leq\frac{2(\operatorname{Re} p(z)-\alpha)}{1-|z|^2},$$

it is easy to deduce that

$$|f''(z)|\leq\frac{2\operatorname{Re} f'(z)-1}{1-|z|^2}$$

so that

$$\begin{aligned}
&\operatorname{Re}((zh'(z)+z^2h''(z)+\overline{zg'(z)+z^2g''(z)})(\overline{zh'(z)}-zg'(z)))\\
&\quad\geq|z|^2\left(2\operatorname{Re} f'(z)-1-\frac{2|z|(2\operatorname{Re} f'(z)-1)}{1-|z|^2}\right)\\
&\quad=|z|^2(2\operatorname{Re} f'(z)-1)\left(\frac{1-2|z|-|z|^2}{1-|z|^2}\right)
\end{aligned}$$

for all $z\in\mathbf{D}$. The right hand side of the above expression is positive provided $|z|<\sqrt{2}-1$. For the function $\Phi[f_0]$ given by (2.2), we have

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}\Phi[f_0](re^{i\theta})\right\}\right)\bigg|_{\theta=\pi,\,r=\sqrt{2}-1}=0$$

which verifies the sharpness of the result. □

For $f \in \mathscr{A}$, it is worth to note that all the following three conditions imply that $\operatorname{Re} f'(z) > 1/2$ for all $z \in \mathbf{D}$ (see [18, Theorem 1, p. 64] and [18, Corollary 2, p. 67]):

(i) $\operatorname{Re}(1 + zf''(z)/f'(z)) > 1/2$ for all $z \in \mathbf{D}$;

(ii) $|f'(z) - 1| < 1/2$ for all $z \in \mathbf{D}$;

(iii) $|f''(z)| \le 1/4$ for all $z \in \mathbf{D}$.

Thus $\Phi[f] \in \mathscr{S}_H^0$ by Theorem 2.1(ii) and the next corollary determines the largest disk $|z| < \rho$ mapped by $\Phi[f]$ onto a convex domain in each case. In particular, Corollary 3.2(iii) determines a subclass of $\mathscr{K}$ which is mapped by the operator Φ into $\mathscr{K}_H^0$.

Corollary 3.2. *Let $f \in \mathscr{A}$.*

(i) *If $\operatorname{Re}(1 + zf''(z)/f'(z)) > 1/2$ for all $z \in \mathbf{D}$, then $\Phi[f] \in \mathscr{S}_H^0$ and maps the disk $|z| < \sqrt{2} - 1$ onto a convex domain. The bound $\sqrt{2} - 1$ is best possible.*

(ii) *If $|f'(z) - 1| < 1/2$ for all $z \in \mathbf{D}$, then $\Phi[f] \in \mathscr{S}_H^0$ and maps the disk $|z| < 1/2$ onto a convex domain. The bound $1/2$ is best possible.*

(iii) *If $|f''(z)| \le 1/4$ for all $z \in \mathbf{D}$, then $\Phi[f] \in \mathscr{K}_H^0$.*

Proof. (i) Since the function $f_0(z) = -\log(1 - z)$ satisfies $\operatorname{Re}(1 + zf''(z)/f'(z)) > 1/2$ for all $z \in \mathbf{D}$, therefore the result follows by invoking Theorem 3.1.

For the next two parts, write $\Phi[f] = h + \bar{g}$, where $h(z) = f(z)$ and $g(z) = f(z) - z$. Let $F_\varepsilon = h + \varepsilon g$ for $|\varepsilon| = 1$.

(ii) Note that $|F_\varepsilon'(z) - 1| = |h'(z) + \varepsilon g'(z) - 1| = |(1 + \varepsilon)(f'(z) - 1)| \le 2|f'(z) - 1| < 1$ for all $z \in \mathbf{D}$ and $|\varepsilon| = 1$. By [12, Theorem 5, p. 314], F_ε is convex in $|z| < 1/2$ for each $|\varepsilon| = 1$. Thus $\Phi[f]$ is convex in $|z| < 1/2$ by [16, Theorem 2.3, p. 89].

For sharpness, consider the function $h_0(z) = z + z^2/4$. Clearly, $|h_0'(z) - 1| = |z|/2 < 1/2$ for all $z \in \mathbf{D}$ and

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}\Phi[h_0](re^{i\theta})\right\}\right)\Bigg|_{\theta=\pi,\, r=1/2} = 0$$

(iii) Since $|F_\varepsilon''(z)| = |(1 + \varepsilon)f''(z)| \le 2|f''(z)| \le 1/2$ for all $z \in \mathbf{D}$, therefore F_ε is convex in $\mathbf{D}$ for each $|\varepsilon| = 1$ by [14, Theorem 2, p. 33] and hence $\Phi[f] \in \mathscr{K}_H^0$. □

If $f \in \mathscr{A}$ with $\Phi[f] \in \mathscr{S}_H^0$, then $|a_n| \le 1/n$ for $n = 1, 2, \ldots$ by Corollary 2.2. However, if $f \in \mathscr{A}$ is given by (1.2) with $|a_n| \le 1/n$ for $n \ge 1$, then $\Phi[f]$ need not be univalent in $\mathbf{D}$. If we consider the function $f(z) = z + z^2/2$, then it is easy to see that the harmonic function $\Phi[f](z) = 1 + z^2/2 + \bar{z}^2/2$ is not univalent in $\mathbf{D}$, since its Jacobian vanishes at the point $z = -1/2$. The next result determines the radius of univalence of functions $\Phi[f]$ with the prescribed coefficient bounds.

Theorem 3.3. *If $f \in \mathscr{A}$ is given by* (1.2) *with $|a_n| \le 1/n$ for $n \ge 1$, then $\Phi[f]$ is univalent in $|z| < 1/3$ and the radius $1/3$ is best possible.*

Proof. For $r \in (0,1)$, let $\Phi_r[f] : \mathbf{D} \to \mathbf{C}$ be defined by

$$\Phi_r[f](z) = \frac{\Phi[f](rz)}{r} = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} a_n r^{n-1} z^n}$$

for all $z \in \mathbf{D}$. We shall show that $\Phi_r[f] \in \mathcal{S}_H^0$ for $r \leq 1/3$. Since $|a_n| \leq 1/n$ for $n = 2, 3, \ldots$, note that

$$S := 2\sum_{n=2}^{\infty} n|a_n| r^{n-1} \leq 2\sum_{n=2}^{\infty} r^{n-1} = \frac{2r}{1-r}.$$

Thus $S \leq 1$ if r satisfies the inequality $r \leq 1/3$. By Lemma 1.1, $\Phi_r[f] \in \mathcal{S}_H^{*0}$ for $r \leq 1/3$. In particular $\Phi[f]$ is univalent in $|z| < 1/3$.

For sharpness of the bound 1/3, consider the function

$$f(z) = 2z + \log(1-z) = z - \sum_{n=2}^{\infty} \frac{1}{n} z^n, \quad z \in \mathbf{D}.$$

The Jacobian of the harmonic function $\Phi[f]$ is given by

$$J_{\Phi[f]}(z) = |f'(z)|^2 - |f'(z) - 1|^2 = 3 - 2\,\mathrm{Re}\left(\frac{1}{1-z}\right)$$

which vanishes at $z = 1/3$. Therefore $\Phi[f]$ is not univalent in $|z| < r$ if $r > 1/3$. □

As observed earlier, if $f \in \mathcal{K}$, then $\Phi[f]$ need not be univalent in $\mathbf{D}$. The last theorem of this section determines the radius of univalence of the class $\{\Phi[f] : f \in \mathcal{K}\}$.

THEOREM 3.4. *If $f \in \mathcal{K}$, then $\Phi[f]$ is univalent in $|z| < \sqrt{2} - 1$ and the result is sharp for the function $l(z) = z/(1-z)$.*

Proof. Since $f \in \mathcal{K}$, $f'(z) \prec 1/(1-z)^2$ in $\mathbf{D}$ by Marx Strohhäcker theorem [10, Theorem 2.6(b), p. 60]. Using subordination, it follows that for every $r \in (0,1)$, $f'(\{z \in \mathbf{C} : |z| \leq r\}) \subseteq g(\{z \in \mathbf{C} : |z| \leq r\})$, where $g(z) = 1/(1-z)^2$. Consequently, for $|z| \leq r_0 := \sqrt{2} - 1$, we have

$$\mathrm{Re}\, f'(z) \geq \min_{|z| \leq r_0} \mathrm{Re}\, f'(z) \geq \min_{|z| \leq r_0} \mathrm{Re}\, g(z) = \min_{|z| = r_0} \mathrm{Re}\, g(z).$$

In view of these inequalities and Theorem 2.1, it suffices to show that

$$\min_{|z| = r_0} \mathrm{Re}\, g(z) = \frac{1}{2}.$$

For $z = r_0 e^{i\theta}$, note that

$$\operatorname{Re} g(z) = \frac{1 - 2\operatorname{Re} z + \operatorname{Re} z^2}{(1 - 2\operatorname{Re} z + |z|^2)^2} = \frac{1 - 2r_0 \cos\theta + r_0^2 \cos 2\theta}{(1 - 2r_0 \cos\theta + r_0^2)^2}$$

which attains its minimum at $\theta = \pm\pi$. Therefore

$$\min_{|z|=r_0} \operatorname{Re} g(z) = \frac{1}{(1+r_0)^2} = \frac{1}{2}.$$

Thus $\operatorname{Re} f'(z) > 1/2$ in $|z| < r_0$ and hence $\Phi[f] \in \mathcal{S}_H^0$ in $|z| < \sqrt{2} - 1$ by Theorem 2.1(ii). □

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