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ON THE DEFICIENCY OF HOLOMORPHIC CURVES WITH MAXIMAL DEFICIENCY SUM

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1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\},$$

where n is a positive integer.

We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$

$$\|\mathbf{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2},$$

$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$

$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z).$$

The characteristic function $T(r, f)$ of f is defined as follows (see [11]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| \, d\theta - \log\|f(0)\|.$$

On the other hand, put

$$U(z) = \max_{1 \le j \le n+1} |f_j(z)|,$$

then it is known ([1]) that

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log U(re^{i\theta}) \, d\theta + O(1).$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty$$

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and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

It is well-known that f is linearly non-degenerate if and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions ([6], [7]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r, \boldsymbol{a}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\boldsymbol{a}\| \, \|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|} \, d\theta,$$

$$N(r, \boldsymbol{a}, f) = N\left(r, \frac{1}{(\boldsymbol{a}, f)}\right).$$

We then have the first fundamental theorem

$$T(r, f) = N(r, \boldsymbol{a}, f) + m(r, \boldsymbol{a}, f) + O(1)$$

([11], p. 76). We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \boldsymbol{a}, f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the deficiency of $\boldsymbol{a}$ with respect to f. We have

$$0 \le \delta(\boldsymbol{a}, f) \le 1 \tag{1}$$

from the first fundamental theorem since $N(r, \boldsymbol{a}, f) \ge 0$ for $r \ge 1$ and $m(r, \boldsymbol{a}, f) \ge 0$ for $r > 0$.

Let X be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position; that is to say, $\sharp X \ge N + 1$ and any $N + 1$ elements of X generate $\boldsymbol{C}^{n+1}$, where N is an integer satisfying $N \ge n$.

Cartan ([1], $N = n$) and Nochka ([8], $N > n$) gave the following

Theorem A. *For any q elements $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ of X $(2N - n + 1 < q < \infty)$,*

(I) (*Second fundamental inequality*)

$$(q - 2N + n - 1)T(r, f) < \sum_{j=1}^{q} N(r, \boldsymbol{a}_j, f) - \frac{N+1}{n+1} N\left(r, \frac{1}{W}\right) + S(r, f),$$

where $S(r, f)$ is any quantity satisfying

$$S(r, f) = o(T(r, f))$$

when r tends to ∞ outside a subset of r of at most finite linear measure;

(II) (*Defect relation*)

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le 2N - n + 1.$$

(see also [2] or [5].)

As in the case of meromorphic functions ([3]), we are interested in holomorphic curves with maximal deficiency sum; that is to say, we would like to study the extremal holomorphic curves for the defect relation given in Theorem A-(II). Some results are given in [4] *and* [10].

The main purpose of this paper is to show that if the equality holds in the defect relation for f and if $(n+1, 2N-n+1) = 1$ then there are at least $(2N-n+1)/(n+1)$ vectors $\boldsymbol{a}$ in $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ such that $\delta(\boldsymbol{a}, f) = 1$ (Theorem 3).

2. Preliminaries

Let $f = [f_1, \dots, f_{n+1}]$ and X etc. be as in Section 1. Let q be an integer satisfying $2N - n + 1 < q < \infty$ and put

$$Q = \{1, 2, \dots, q\}.$$

Let $\{\boldsymbol{a}_j \mid j \in Q\}$ be a family of vectors in X. For a non-empty subset P of Q, we denote

$$V(P) = \text{the vector space spanned by } \{\boldsymbol{a}_j \mid j \in P\}, \quad d(P) = \dim V(P)$$

and we put

$$\mathcal{O} = \{P \subset Q \mid 0 < \sharp P \leq N+1\}.$$

For $\{\boldsymbol{a}_j \mid j \in Q\}$, let $\omega : Q \to (0, 1]$ be the Nochka weight function given in [5, p. 72] and θ the reciprocal number of the Nochka constant "θ" given in [5, p. 72]. Then, they possess the following properties:

Lemma 1 (see [5], Theorem 2.4.11). (a) $0 < \omega(j)\theta \leq 1$ *for all* $j \in Q$;
(b) $q - 2N + n - 1 = \theta(\sum_{j=1}^{q} \omega(j) - n - 1)$;
(c) $(N+1)/(n+1) \leq \theta \leq (2N-n+1)/(n+1)$;
(d) *For any* $P \in \mathcal{O}$, $\sum_{j \in P} \omega(j) \leq d(P)$.

Lemma 2 (see pp. 109–110 in [5]). *Let* f, $\{\boldsymbol{a}_j \mid j \in Q\}$ *and* ω *be as in Lemma* 1. *Then, we have the inequality*

$$\sum_{j=1}^{q} \omega(j)\delta(\boldsymbol{a}_j, f) \leq n+1.$$

Lemma 3. *Suppose that* $N > n$. *For* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(2N - n + 1 < q < \infty)$ *the maximal deficiency sum*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1$$

holds if and only if the following two relations hold:
1) $(1 - \theta\omega(j))(1 - \delta(\boldsymbol{a}_j, f)) = 0$ $(j = 1, \dots, q)$;
2) $\sum_{j=1}^{q} \omega(j)\delta(\boldsymbol{a}_j, f) = n + 1$.

Proof. In general, we have the following equality, which can be proved easily,

$$\theta\sum_{j=1}^{q}\omega(j)\delta(\boldsymbol{a}_j,f)+q-\theta\sum_{j=1}^{q}\omega(j)=\sum_{j=1}^{q}\{\delta(\boldsymbol{a}_j,f)+(1-\theta\omega(j))(1-\delta(\boldsymbol{a}_j,f))\},$$

which reduces to

$$\text{(2)}\quad \theta\left(\sum_{j=1}^{q}\omega(j)\delta(\boldsymbol{a}_j,f)-n-1\right)=\sum_{j=1}^{q}\{\delta(\boldsymbol{a}_j,f)+(1-\theta\omega(j))(1-\delta(\boldsymbol{a}_j,f))\}-(2N-n+1)$$

by Lemma 1(b). By Lemma 2 and Theorem A-(II), we easily obtain this lemma since

$$(1-\theta\omega(j))(1-\delta(\boldsymbol{a}_j,f))\geq 0\quad (j=1,\dots,q)$$

from (1) and Lemma 1(a).

Corollary 1. *Suppose that $N>n$ and that for $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q\in X$ $(2N-n+1<q<\infty)$ the maximal deficiency sum*

$$\text{(3)}\quad \sum_{j=1}^{q}\delta(\boldsymbol{a}_j,f)=2N-n+1$$

holds.

(I) *If there exists some $j\in Q$ satisfying $\theta\omega(j)<1$, then $\delta(\boldsymbol{a}_j,f)=1$ for this j.*

(II) *If $\delta(\boldsymbol{a}_j,f)<1$ for all $j\in Q$, then*

$$\omega(j)=\frac{1}{\theta}=\frac{n+1}{2N-n+1}\quad (j=1,2,\dots,q).$$

Proof. (I) This is obvious from Lemma 3-1).

(II) From Lemma 3-1), we have

$$\omega(j)=\frac{1}{\theta}\quad (j=1,2,\dots,q)$$

and from Lemma 3-2) and (3) we obtain

$$\frac{1}{\theta}=\frac{n+1}{2N-n+1}.$$

Proposition 1. *Suppose that there exists a function*

$$\sigma : Q\to(0,1]$$

which satisfies the following condition (*):

(*) *For any $P\in\mathcal{O}$,*

$$\sum_{j \in P} \sigma(j) \leq d(P).$$

Then, for any element $A \in \mathcal{O}$ satisfying $\sharp A = N+1$ and for real numbers $E_1, \dots, E_q$ satisfying $E_j \geq 1$ $(j \in Q)$, there exists a subset B of A which satisfies the followings.

(a) $\sharp B = n+1$;

(b) $\{\boldsymbol{a}_j \mid j \in B\}$ is a basis of $\boldsymbol{C}^{n+1}$;

(c) $\prod_{j \in A} E_j^{\sigma(j)} \leq \prod_{j \in B} E_j$.

Proof. Due to the assumption (*), we can prove this proposition as in the case of Proposition 2.4.15 in [5], p. 75. To make sure of it we shall give a proof of this proposition. We suppose without loss of generality that

$$E_1 \geq E_2 \geq \cdots \geq E_q.$$

We choose $j_1, \dots, j_{n+1}$ by induction as follows.

1) Let j_1 be the minimum number of A. we put

$$A_1 = \{j_1\} \quad \text{and} \quad S_1 = \{j \in A \mid \boldsymbol{a}_j \in V(A_1)\}.$$

2) Suppose that $j_1, \dots, j_k$ are chosen. We put for $k \geq 1$

$$A_k = \{j_1, \dots, j_k\} \quad \text{and} \quad S_k = \left\{ j \in A - \bigcup_{\ell=0}^{k-1} S_\ell \,\middle|\, \boldsymbol{a}_j \in V(A_k) \right\},$$

where $S_0 = \phi$. We choose j_{k+1} $(1 \leq k \leq n)$ as follows.

$$j_{k+1} = \min\{j \in A \mid \boldsymbol{a}_j \notin V(A_k)\}$$

and put

$$A_{k+1} = \{j_1, \dots, j_{k+1}\} \quad \text{and} \quad S_{k+1} = \left\{ j \in A - \bigcup_{\ell=0}^{k} S_\ell \,\middle|\, \boldsymbol{a}_j \in V(A_{k+1}) \right\}.$$

Then, it is easy to see that

(i) $S_1, \dots, S_{n+1}$ are mutually disjoint and $\boldsymbol{a}_{j_1}, \dots, \boldsymbol{a}_{j_{n+1}}$ are linearly independent;

(ii) $A = \bigcup_{k=1}^{n+1} S_k$;

(iii) $E_j \leq E_{j_k}$ for $j \in S_k$;

(iv) $E_{j_1} \geq E_{j_2} \geq \cdots \geq E_{j_{n+1}} \geq 1$.

We put for $k = 1, \dots, n+1$

$$T_k = S_1 \cup \cdots \cup S_k \quad \text{and} \quad d_k = \sum_{j \in S_k} \sigma(j).$$

Then,

(4) $$\sum_{k=1}^{m} d_k \leq m \quad (m = 1, \dots, n+1)$$

since

$$\sum_{k=1}^{m} d_k = \sum_{k=1}^{m} \sum_{j \in S_k} \sigma(j) \leq d(T_m) = m$$

by (*).

Put $B = \{j_1, \dots, j_{n+1}\}$. Then B satisfies (a), (b) and (c). It is easy to see that (a) and (b) hold. We have only to prove (c).

Now by (4), (iii) and (iv) we have the inequality (c):

$$\begin{aligned}
\prod_{j \in A} E_j^{\sigma(j)} &= \prod_{k=1}^{n+1} \prod_{j \in S_k} E_j^{\sigma(j)} \\
&\leq \prod_{k=1}^{n+1} \prod_{j \in S_k} E_{j_k}^{\sigma(j)} = \prod_{k=1}^{n+1} E_{j_k}^{d_k} \\
&= E_{j_1} E_{j_1}^{-1+d_1} \prod_{k=2}^{n+1} E_{j_k}^{d_k} \leq E_{j_1} E_{j_2}^{-1+d_1} \prod_{k=2}^{n+1} E_{j_k}^{d_k} \\
&= E_{j_1} E_{j_2}^{-1+d_1+d_2} \prod_{k=3}^{n+1} E_{j_k}^{d_k} \\
&= E_{j_1} E_{j_2} E_{j_2}^{-2+d_1+d_2} \prod_{k=3}^{n+1} E_{j_k}^{d_k} \\
&\leq E_{j_1} E_{j_2} E_{j_3}^{-2+d_1+d_2} \prod_{k=3}^{n+1} E_{j_k}^{d_k} \\
&= E_{j_1} E_{j_2} E_{j_3} E_{j_3}^{-3+d_1+d_2+d_3} \prod_{k=4}^{n+1} E_{j_k}^{d_k} \\
&\cdots\cdots\cdots \\
&\leq E_{j_1} E_{j_2} \cdots E_{j_{n+1}} E_{j_{n+1}}^{-n-1+d_1+\cdots+d_{n+1}} \\
&\leq E_{j_1} E_{j_2} \cdots E_{j_{n+1}} = \prod_{j \in B} E_j.
\end{aligned}$$

Definition 1. We put

$$\lambda = \min_{P \in \mathcal{O}} \frac{d(P)}{\sharp P}$$

and for $j \in Q$

$$\sigma(j) = \lambda.$$

Proposition 2. (a) $1/(N-n+1) \leq \lambda \leq (n+1)/(N+1)$.
(b) *For any* $P \in \mathcal{O}$,

$$\sum_{j \in P} \sigma(j) \leq d(P).$$

Proof. (a) As X is in N-subgeneral position, when $\sharp P = N+1$, $d(P) = n+1$. This means that $\lambda \leq (n+1)/(N+1)$.

As

$$\sharp P - d(P) \leq N - n$$

for any $P \in \mathcal{O}$ (see (2.4.3) in [5], p. 68), we have

$$\frac{d(P)}{\sharp P} \geq \frac{d(P)}{N-n+d(P)} \geq \frac{1}{N-n+1},$$

so that

$$\frac{1}{N-n+1} \leq \lambda.$$

(b) By Definition 1

$$\sum_{j \in P} \sigma(j) = \lambda \sharp P \leq \frac{d(P)}{\sharp P} \sharp P = d(P).$$

Remark 1. By the definition of ω (see [5], p. 72), if

$$\lambda < \frac{n+1}{2N-n+1},$$

then

$$\lambda = \min_{1 \leq j \leq q} \omega(j) \quad \text{and} \quad \omega(j) = \lambda \quad \text{for } j \in P_o \in \mathcal{O} \quad \text{such that} \quad \frac{d(P_o)}{\sharp P_o} = \lambda.$$

Further, $\theta < (2N-n+1)/(n+1)$.

3. Theorem

Let $f = [f_1, \dots, f_{n+1}]$ and X etc. be as in Sections 1 and 2.

Theorem 1. *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(2N-n+1 < q < \infty)$ *we have the inequality*

$$\sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f) \leq \frac{n+1}{\lambda} T(r,f) - \frac{1}{\lambda} N\left(r, \frac{1}{W}\right) + S(r,f).$$

Proof. We put

$$F_j = (\boldsymbol{a}_j, f) \quad (j = 1, \dots, q).$$

For any $z(\neq 0)$ arbitrarily fixed in $|z| < \infty$ for which $F_j(z) \neq 0$ $(j = 1, \dots, q)$, let

$$|F_{j_1}(z)| \leq |F_{j_2}(z)| \leq \cdots \leq |F_{j_q}(z)|,$$

where $j_1, \dots, j_q$ are distinct and $1 \leq j_1, \dots, j_q \leq q$. Then, there is a positive constant K such that

$$\text{(5)} \qquad \begin{aligned} &\|f(z)\| \leq K|F_{j_\nu}(z)| \quad (\nu = N+1, \dots, q), \\ &|F_{j_\nu}(z)| \leq K\|f(z)\| \quad (\nu = 1, \dots, q). \end{aligned}$$

(From now on we denote by K a positive constant, which may be different from each other when it appears.)

For $\sigma(j) = \lambda$ $(j = 1, \dots, q)$ we have by (5), Proposition 1 and Proposition 2-(b)

$$\text{(6)} \qquad \begin{aligned} \prod_{j=1}^{q} \left(\frac{\|\boldsymbol{a}_j\| \, \|f(z)\|}{|(\boldsymbol{a}_j, f(z))|} \right)^{\sigma(j)} &\leq K \prod_{\nu=1}^{N+1} \left(\frac{\|\boldsymbol{a}_{j_\nu}\| \, \|f(z)\|}{|F_{j_\nu}(z)|} \right)^{\sigma(j_\nu)} \\ &= K \prod_{j_\nu \in B} \frac{\|\boldsymbol{a}_{j_\nu}\| \, \|f(z)\|}{|F_{j_\nu}(z)|} \\ &= K \frac{\|f(z)\|^{n+1}}{|W(z)|} \cdot \frac{|W_B(z)|}{\Pi_{j_\nu \in B} |F_{j_\nu}(z)|}, \end{aligned}$$

where B is the subset of $A = \{j_1, \dots, j_{N+1}\}$ given in Proposition 1 and $W_B(z)$ is the Wronskian of F_{j_ν} $(j_\nu \in B)$. Note that $W_B(z) = cW(z)$ ($c \neq 0$, constant). As $\sigma(j) = \lambda$ for all $j \in Q$, we obtain from (6) that

$$\begin{aligned} \lambda \sum_{j=1}^{q} \log \frac{\|\boldsymbol{a}_j\| \, \|f(z)\|}{|(\boldsymbol{a}_j, f(z))|} &\leq (n+1) \log\|f(z)\| - \log|W(z)| \\ &\quad + \sum_{B \subset Q} \log^+ \frac{|W_B(z)|}{\Pi_{j_\nu \in B} |F_{j_\nu}(z)|} + \log K, \end{aligned}$$

where summation $\sum_{B \subset Q}$ is taken over all $B \subset Q$ satisfying that $\{\boldsymbol{a}_j | j \in B\}$ is a basis of $\boldsymbol{C}^{n+1}$. From this inequality we obtain the inequality

$$\lambda \sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f) \leq (n+1) T(r, f) - N\left(r, \frac{1}{W}\right) + S(r, f).$$

as usual.

Corollary 2 (Defect relation). *For* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(2N - n + 1 < q < \infty)$,

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le \min\left(2N - n + 1, \frac{n+1}{\lambda}\right).$$

Proof. From Theorem 1 we obtain

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le \frac{n+1}{\lambda}.$$

as usual. Combining this inequality with the defect relation of Theorem A-(II), we obtain our corollary.

4. Defects of holomorphic curves with maximal deficiency sum

Let $f = [f_1, \ldots, f_{n+1}]$, X etc. be as in Sections 1, 2 and 3.

THEOREM 2. *Suppose that* $N > n \ge 2$ *and that there are vectors* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q$ *in* X *such that*

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1, \tag{7}$$

where $2N - n + 1 < q < \infty$. *If*

$$\lambda < \frac{n+1}{2N - n + 1}, \tag{8}$$

then there are at least

$$\left[\frac{2N - n + 1}{n+1}\right] + 1$$

vectors $\boldsymbol{a} \in \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q\}$ *satisfying* $\delta(\boldsymbol{a}, f) = 1$.

Proof. By the definition of λ, there is a set P_o in $\mathcal{O}$ such that

$$\lambda = d(P_o)/\sharp P_o.$$

Then, by (8) and Remark 1,

$$\omega(j) = \lambda < \frac{1}{\theta} \quad (j \in P_o),$$

so that

$$\theta\omega(j) < 1 \quad (j \in P_o).$$

By Corollary 1

$$\delta(\boldsymbol{a}_j, f) = 1 \quad (j \in P_o)$$

since (7) is assumed. As

$$\sharp P_o = \frac{1}{\lambda} d(P_o) > \frac{2N - n + 1}{n + 1} d(P_o) \geq \frac{2N - n + 1}{n + 1}$$

we have our theorem.

THEOREM 3. *Suppose that*

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) = 2N - n + 1.$$

If $(n + 1, 2N - n + 1) = 1$, *then there are at least*

$$\left[\frac{2N - n + 1}{n + 1}\right] + 1$$

vectors $\boldsymbol{a} \in X$ *satisfying* $\delta(\boldsymbol{a}, f) = 1$.

Proof. We first note that $N > n \geq 2$ under the condition $(n + 1, 2N - n + 1) = 1$. By Theorem A-(II), it is easy to see that the set

$$Y = \{\boldsymbol{a} \in X \mid \delta(\boldsymbol{a}, f) > 0\}$$

is at most countable and

$$\sum_{\boldsymbol{a} \in Y} \delta(\boldsymbol{a}, f) \leq 2N - n + 1.$$

(A) The case when Y is a finite set. Let

$$Y = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_q\}.$$

Then, from the assumption of this theorem we have

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) = 2N - n + 1, \tag{9}$$

where $2N - n + 1 \leq q < \infty$.

There is nothing to prove when $q = 2N - n + 1$. Suppose $2N - n + 1 < q < \infty$. From (9) and Corollary 2, we have the inequality

$$\lambda \leq \frac{n + 1}{2N - n + 1}. \tag{10}$$

On the other hand, by the definition of λ, there is an element P_o in $\mathcal{O}$ such that

$$\lambda = d(P_o)/\sharp P_o.$$

As $\sharp P_o \leq N + 1 < 2N - n + 1$ and $(n + 1, 2N - n + 1) = 1$ by our assumption, the equality in (10) cannot hold. That is to say, it must hold that

$$\lambda < \frac{n+1}{2N-n+1}.$$

Then, we have our theorem by Theorem 2 in this case.

(B) The case when Y is not finite. Let

$$Y = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots\}.$$

Then,

$$\sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f) = 2N - n + 1. \tag{11}$$

We put

$$\mathcal{O}_\infty = \{P \subset \boldsymbol{N} \mid 0 < \sharp P \le N+1\},$$

where $\boldsymbol{N}$ is the set of positive integers, and for any finite subset $P \ne \phi$ of $\boldsymbol{N}$, we use

$$V(P) \quad \text{and} \quad d(P)$$

as in Section 2.

Further, we put

$$\lambda_\infty = \min_{P \in \mathcal{O}_\infty} \frac{d(P)}{\sharp P} \quad \text{and} \quad \sigma(j) = \lambda_\infty \quad (j \in \boldsymbol{N}).$$

Note that the set $\{d(P)/\sharp P | P \in \mathcal{O}_\infty\}$ is a finite set.

As in the case of Proposition 2, we have the followings.

(a_∞) $1/(N-n+1) \le \lambda_\infty \le (n+1)/(N+1)$.
(b_∞) For any $P \in \mathcal{O}_\infty$, $\sum_{j \in P} \sigma(j) \le d(P)$.

Further, we have the inequality

$$\sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f) \le (n+1)/\lambda_\infty. \tag{12}$$

In fact, for any $q(> 2N - n + 1)$ in $\boldsymbol{N}$, we have the inequality

$$\lambda_\infty \sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f) \le (n+1)T(r,f) - N(r, 1/W) + S(r,f)$$

by (b_∞) and Proposition 1 as in the case of Theorem 1, from which we have

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le (n+1)/\lambda_\infty$$

as usual and letting q tend to ∞ we have the inequality (12).

Now, there is an element P_o of $\mathcal{O}_\infty$ satisfying

$$\lambda_\infty = d(P_o)/\sharp P_o.$$

Then we shall prove that

$$\delta(\boldsymbol{a}_j, f) = 1 \quad (j \in P_o).$$

Suppose to the contrary that

$$\min_{j \in P_o} \delta(\boldsymbol{a}_j, f) = \delta < 1.$$

By the assumption (11) of this theorem we obtain from (12) that

$$\lambda_\infty \le (n+1)/(2N-n+1).$$

Further as $(n+1, 2N-n+1) = 1$ and $\sharp P_o \le N+1 < 2N-n+1$, the inequality

$$\lambda_\infty < (n+1)/(2N-n+1). \tag{13}$$

must hold. As (13) holds, for any positive number ε satisfying

$$0 < \varepsilon < \left(1 - \frac{2N-n+1}{n+1}\lambda_\infty\right)(1-\delta), \tag{14}$$

we choose $q \in \boldsymbol{N}$ satisfying $Q = \{1, 2, \dots, q\} \supset P_o$ and

$$2N - n + 1 - \varepsilon < \sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f). \tag{15}$$

For this Q, we use θ_q, ω_q and λ_q instead of θ, ω and λ in Section 2 respectively. By the choice of q, $\lambda_\infty = \lambda_q$.

By Lemma 2 and (2) we obtain

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) + \sum_{j=1}^{q} (1 - \theta_q \omega_q(j))(1 - \delta(\boldsymbol{a}_j, f)) \le 2N - n + 1. \tag{16}$$

From (15) and (16) we have

$$\sum_{j=1}^{q} (1 - \theta_q \omega_q(j))(1 - \delta(\boldsymbol{a}_j, f)) < \varepsilon. \tag{17}$$

By the definition of ω_q (see p. 72 in [5]), for $j \in P_o$

$$\omega_q(j) = \lambda_q = \lambda_\infty \tag{18}$$

and by (13) and Lemma 1(c)

$$\lambda_\infty < \frac{n+1}{2N-n+1} \le \frac{1}{\theta_q}. \tag{19}$$

From (17), (18) and (19) for some $j \in P_o$ with $\delta(\boldsymbol{a}_j, f) = \delta$

$$\left(1 - \frac{2N-n+1}{n+1}\lambda_\infty\right)(1-\delta) \le (1 - \theta_q\lambda_\infty)(1-\delta) = (1 - \theta_q\omega_q(j))(1 - \delta(\boldsymbol{a}_j, f)) < \varepsilon,$$

which contradicts (14). This means that δ must be equal to 1. As

$$\frac{2N-n+1}{n+1} \leq \frac{2N-n+1}{n+1} d(P_o) < \sharp P_o,$$

we have our theorem.

Remark 2. This is a generalization of Corollary 1 in [9].

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