On Thompson's p-complement theorems for saturated fusion systems

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Abstract In this short note we prove that a saturated fusion system admitting some special type of automorphism is nilpotent. This generalizes classical results by J. G. Thompson.

1. Introduction

In his Ph.D. thesis, John G. Thompson [13] proved the long-standing conjecture that the Frobenius kernel of a Frobenius group is nilpotent. The Frobenius kernel always admits a fixed-point-free automorphism of prime order, which turned out to be a sufficient condition for the nilpotency. In fact, both results, the nilpotency of the Frobenius kernel and the nilpotency of a finite group admitting a fixed-point-free automorphism of prime order, are equivalent. To prove this result, Thompson introduced his famous p-nilpotency criterion for an odd prime p: a group G is p-nilpotent if and only if the normalizer of the J subgroup of a Sylow p-subgroup and the centralizer of the center of a Sylow p-subgroup are p-nilpotent (see [15]). In fact, Thompson proved that a group G admitting automorphisms that leave invariant some special subgroups is p-nilpotent. The following translates [14, Theorem A] for saturated fusion systems, and includes an extra hypothesis to cover the $p=2$ case (see Definition 3.3).

THEOREM 1.1

Let $\mathcal{F}$ be a saturated fusion system over a p-group S such that either p is odd, or $p=2$ and $\mathcal{F}$ is Σ_4-free. Let $\mathcal{U}$ be a group of automorphisms of $(S,\mathcal{F})$. Suppose that, for every $\mathcal{U}$-invariant normal subgroup $Q\trianglelefteq S$, $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group. Then $\mathcal{F}=\mathcal{F}_S(S)$.

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As a consequence of [14, Theorem A], Thompson proved that a group G admitting a fixed-point-free automorphism of prime order is p-nilpotent for all primes, and therefore the group is nilpotent (see [13, Theorem 1]).

Given a finite group G and ϕ a prime-order automorphism without fixed points, ϕ fixes a Sylow p-subgroup S of G. For a fixed subgroup H one has that $N_G(H)$ and $C_G(H)$ are also fixed. Therefore, ϕ acts on the quotient group $N_G(H)/C_G(H)$ and the action on this quotient is without fixed points (see Theorem 2.9(b)). So one can translate the concept of a fixed-point-free automorphism of prime order from the category of finite groups to the category of saturated fusion systems (see Definition 2.10, Proposition 2.11). It turned out that the concept of fixed-point-free automorphism in the category of saturated fusion systems is more general since there exist finite groups admitting automorphisms of prime order with fixed points such that the induced automorphism on the fusion category is fixed-point-free (see Example 3.5). Nevertheless, p-nilpotency holds under this mild assumption, as we prove in the following generalization of [13, Theorem 1].

THEOREM 1.2

Let $\mathcal{F}$ be a saturated fusion system over a p-group S such that either p is odd, or $p=2$ and $\mathcal{F}$ is Σ_4-free. If $(S,\mathcal{F})$ admits a prime-order fixed-point-free automorphism, then $\mathcal{F}=\mathcal{F}_S(S)$.

These results contribute to the list of nilpotency criteria for saturated fusion systems that generalize classical criteria for finite groups and fit within the framework of previous work by Kessar and Linckelmann [9], Díaz, Glesser, Mazza, and Park [6], Díaz, Glesser, Park, and Stancu [7], Cantarero, Scherer, and Viruel [3], and Craven [5]. Indeed, Theorem 1.1 can also be deduced from [6, Corollary 4.6], although the proof of Thompson's p-nilpotency criterion in [6] resorts to the group case, while the proof presented here is purely fusion-theoretical. Another independent fusion-theoretical proof of the odd prime case in Theorem 1.2 can be found in [5].

This note is organized as follows. In Section 2 we recall the main properties of saturated fusion systems and we introduce the concept of a fixed-point-free automorphism of a saturated fusion system. In Section 3 we provide a unified proof of Theorems 1.1 and 1.2.

2. Background on saturated fusion systems and finite groups

In this section we review the concept of a saturated fusion system over a p-group S as defined in [2], and define fixed-point-free automorphism of a saturated fusion system.

DEFINITION 2.1

A fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all subgroups P and Q of S;
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

We say that two subgroups $P,Q \leq S$ are *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$. As all the morphisms are injective by Definition 2.1(b), we denote by $\mathrm{Aut}_{\mathcal{F}}(P)$ the group $\mathrm{Hom}_{\mathcal{F}}(P,P)$. We denote by $\mathrm{Out}_{\mathcal{F}}(P)$ the quotient group $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$.

The fusion systems that we consider are saturated, so we need the following definitions.

DEFINITION 2.2
Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized in $\mathcal{F}$* if $|C_S(P)| \geq |C_S(P')|$ for all P' which are $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully normalized in $\mathcal{F}$* if $|N_S(P)| \geq |N_S(P')|$ for all P' which are $\mathcal{F}$-conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold.

(a) Every fully normalized in $\mathcal{F}$ subgroup $P \leq S$ is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
(b) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that φP is fully centralized and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

As expected, every finite group G gives rise to a saturated fusion system over S, a Sylow p-subgroup of G, denoted by $(S, \mathcal{F}_S(G))$ (see [2, Proposition 1.3]). But there exist saturated fusion systems that are not the fusion system of any finite group (see, e.g., [2, Section 9] or [12]).

Let $\mathcal{F}$ be a fusion system over a p-group S, and let Q be a subgroup of S. We can take the *normalizer of Q in $\mathcal{F}$* as the fusion system over the normalizer of Q in S, $N_S(Q)$, with morphisms:

$$\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P,P') = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P') \mid \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, P'Q), \psi|_P = \varphi\}.$$

Although $(N_S(Q), N_{\mathcal{F}}(Q))$ is not always a saturated fusion system, it is so when Q is fully normalized (see [2, Proposition A.6]).

PROPOSITION 2.3
Let $\mathcal{F}$ be a saturated fusion system over a p-group S. If $Q \leq S$ is fully normalized in $\mathcal{F}$, then $(N_S(Q), N_{\mathcal{F}}(Q))$ is a saturated fusion system.

We also need results concerning the quotients of saturated fusion systems. Recall that if $(S,\mathcal{F})$ is a saturated fusion system, then we say that $Q \leq S$ is a *weakly $\mathcal{F}$-closed* subgroup if Q is not $\mathcal{F}$-conjugate to any other subgroup of S.

In [10, Lemma 2.6] we can find the following result.

LEMMA 2.4

Let $(S,\mathcal{F})$ be a saturated fusion system, and let $Q \lhd S$ be a weakly $\mathcal{F}$-closed subgroup. Let $\mathcal{F}/Q$ be the fusion system over S/Q defined by setting

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q,P'/Q) = \{\varphi/Q \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P')\}$$

for all P, $P' \leq S$ which contain Q. Then $\mathcal{F}/Q$ is saturated.

Some classical results for finite groups can be generalized to saturated fusion systems, for example, Alperin's fusion theorem for saturated fusion systems (see [2, Theorem A.10]).

DEFINITION 2.5

Let $\mathcal{F}$ be any fusion system over a p-group S. A subgroup $P \leq S$ is

- *$\mathcal{F}$-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers,
- *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, that is, if $\mathrm{Out}_{\mathcal{F}}(P)$ has no nontrivial normal p-subgroups.

THEOREM 2.6

Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(P,P')$, there exist sequences of subgroups of S,

$$P = P_0, P_1, \ldots, P_k = P' \qquad \textit{and} \qquad Q_1, Q_2, \ldots, Q_k,$$

and morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$ such that

- *Q_i is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical, and $\mathcal{F}$-centric for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$.*

Here we will recall some definitions and results concerning groups (see [8, Chapter 2]) and group automorphisms (see [8, Chapter 10]).

A finite group G is *nilpotent* if the lower central series of G, defined as

$$\gamma_1(G) = G, \qquad \gamma_i(G) = [\gamma_{i-1}(G), G] \quad \text{for } i \geq 2,$$

satisfies that there exists m such that $\gamma_m = \{1\}$.

As examples we have that every finite p-group is nilpotent (see [8, Theorem 2.3.3(iii)]), and more generally, we have the following result.

THEOREM 2.7

A finite group G is nilpotent if and only if it is the direct product of its Sylow p-subgroups.

So the fusion on S, a Sylow p-subgroup of a nilpotent group G, satisfies $\mathcal{F}_S(G) = \mathcal{F}_S(S)$.

DEFINITION 2.8

An automorphism φ of a group G is said to be *fixed-point-free* if it leaves only the identity element of G fixed.

The following result shows that fixed-point-free morphisms are compatible with the p-local structure (see [8, Theorem 10.1.2, Lemma 10.1.3]).

THEOREM 2.9

Let G be a finite group, and let p be a prime dividing the order of G. If $\varphi\colon G \to G$ is a fixed-point-free morphism, then the following hold.

(a) *There exists a unique φ-invariant Sylow p-subgroup S of G, and it contains every φ-invariant p-subgroup of G.*

(b) *If H is a φ-invariant normal subgroup of G, then φ induces a fixed-point-free automorphism of G/H.*

Let $\mathrm{Aut}(\mathcal{F})$ denote the group of automorphisms of $\mathcal{F}$:

$$\begin{aligned}\mathrm{Aut}(\mathcal{F}) = \big\{\varphi \in \mathrm{Aut}(S) \,\big|\, &\text{ if } \alpha \in \mathrm{Hom}_{\mathcal{F}}(P, Q),\\ &\text{then } \varphi|_Q \circ \alpha \circ (\varphi|_P)^{-1} \in \mathrm{Hom}_{\mathcal{F}}\big(\varphi(P), \varphi(Q)\big)\big\}.\end{aligned}$$

We are now ready to give a definition for a fixed-point-free automorphism of a saturated fusion system.

DEFINITION 2.10

Let $(S, \mathcal{F})$ be a saturated fusion system. Then $\varphi \in \mathrm{Aut}(\mathcal{F})$ is a *fixed-point-free* automorphism if the following hold:

- $\varphi\colon S \to S$ is fixed-point-free, and
- $\varphi_\sharp\colon \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$, defined as $\varphi_\sharp(\alpha) = \varphi \circ \alpha \circ (\varphi|_P)^{-1}$, is fixed-point-free for all φ-invariant subgroups $P \leq S$.

The next result shows that this definition generalizes the concept of a fixed-point-free automorphism of a finite group.

PROPOSITION 2.11

Let G be a finite group, and let p be a prime dividing the order of G. If $\varphi\colon G \to G$ is a fixed-point-free automorphism, then φ induces a fixed-point-free automorphism of $\mathcal{F}_S(G)$, where S is the only φ-invariant Sylow p-subgroup of G.

Proof

According to Theorem 2.9(a) there exists S a unique φ-invariant Sylow p-subgroup of G. As $\varphi|_S$ is the restriction of a fixed-point-free group morphism, it is also fixed-point-free.

Consider now $P \leq S$ such that $\varphi(P) = P$. Then, $C_G(P)$ and $N_G(P)$ are φ-invariant and $\varphi|_{C_G(P)}$ and $\varphi|_{N_G(P)}$ are fixed-point-free. Using Theorem 2.9(b), we have that φ induces a fixed-point-free group morphism

$$\varphi_\sharp\colon \operatorname{Aut}_{\mathcal{F}_S(G)}(P) \cong N_G(P)/C_G(P) \to N_G(P)/C_G(P) \cong \operatorname{Aut}_{\mathcal{F}_S(G)}(P). \qquad \square$$

3. Thompson theorems for saturated fusion systems

In this section we give a unified proof of Theorems 1.1 and 1.2. In order to do it, we introduce the concept of $\mathcal{T}$-automorphism.

DEFINITION 3.1

Let $(S, \mathcal{F})$ be a saturated fusion system. We say that $\mathcal{F}$ *admits $\mathcal{T}$-automorphisms* if there exists $\mathcal{U} \leq \operatorname{Aut}(\mathcal{F})$ such that one of the following holds.

- $\mathcal{U} = \langle \varphi \rangle$ where φ is a fixed-point-free automorphism of prime order.
- For every $\mathcal{U}$-invariant normal subgroup $Q \trianglelefteq S$, $\operatorname{Aut}_{\mathcal{F}}(Q)$ is a p-group.

The following lemma is a particular case for finite groups which can be proved directly.

LEMMA 3.2

Let G be the semidirect product $V \rtimes H$ where V is an elementary abelian p-group V and H is a group with an element of order prime to p which does not centralize V. Then, given any p-Sylow subgroup S of G, $\mathcal{F}_S(G)$ does not admit $\mathcal{T}$-automorphisms that leave V invariant.

Proof

Assume that $G = V \rtimes H$ is a minimal counterexample to the statement, let S be a p-Sylow subgroup of G, and let $\mathcal{U}$ be a group of $\mathcal{T}$-automorphisms of $\mathcal{F}_S(G)$ that leave V invariant. Since V is $\mathcal{U}$-invariant normal in G, it is so in S. According to the hypothesis, $\operatorname{Aut}_{\mathcal{F}}(V) = N_G(V)/C_G(V)$ contains an element of order prime to p (coming from H); hence we may assume that $\mathcal{U} = \langle \varphi \rangle$ where φ is a fixed-point-free automorphism of prime order r. Without loss of generality, we may assume that φ is an honest fixed-point-free automorphism of G. The automorphism φ restricts to a fixed-point-free automorphism of V, and according to Theorem 2.9(b), φ induces a fixed-point-free automorphism on $G/V \cong H$, namely, $\widetilde{\varphi}$.

Let q be any prime dividing the order of H. By Theorem 2.9(a), there exists Q, a $\widetilde{\varphi}$-invariant Sylow q-subgroup in H. Now, if $Q \lneq H$, then φ induces a fixed-point-free automorphism on $V \rtimes Q \lneq G$ in contradiction to the minimality of G. Therefore, H must be a q-group. Moreover, if H is not abelian, then the center $Z(Q) \lneq H$ is $\widetilde{\varphi}$-invariant and φ induces a fixed-point-free automorphism on $V \rtimes Z(H) \lneq G$ in contradiction to the minimality of G. Hence, H is abelian. Finally, if H is not elementary abelian, then the characteristic subgroup $\Omega(H) \lneq H$ of elements of order q is $\widetilde{\varphi}$-invariant and φ induces a fixed-point-free automorphism on $V \rtimes \Omega(H) \lneq G$ in contradiction to the minimality of G.

Now r, the order of φ, is different from p: the restriction of φ to V gives a fixed-point-free action over V, a p-group, so $r \neq p$.

We can assume that H acts over V without fixed points. Consider a set $A = \{h_1, \ldots, h_n\}$ of generators of H. Also $\varphi(A)$ generates H. If $x \in V$ and $h \in H$, then we have the identity $\varphi(h(x)) = \varphi(h)(\varphi(x))$, so if $h(x) = x$ for all $h \in H$, then $\varphi(x) = \varphi(h)(\varphi(x))$. But $H = \{\varphi(h)\}_{h \in H}$ and the fixed points by H form a φ-invariant subgroup. Let N be the subgroup of V of fixed points by H, and assume that N is not trivial. Then we can construct the group $(V/N) \rtimes H$ and φ induces a fixed-point-free action by Theorem 2.9(b), and we get a contradiction with the minimality of G.

Consider L the semidirect product $H \rtimes \mathbb{Z}/r$. The centralizer of H in L is itself, and we are assuming that V is a faithful H-module and that $p \neq r$. Then we can apply [8, Theorem 3.4.4] to deduce that φ cannot restrict to a fixed-point-free automorphism of V, getting a contradiction. □

We now recall the definition of an H-free fusion system, for a finite group H (see [9, Definition 1.1]). If $\mathcal{F}$ is a saturated fusion system over a p-group S and P is an $\mathcal{F}$-centric fully normalized in $\mathcal{F}$ subgroup of S, then $N_{\mathcal{F}}(P)$ is a constrained fusion system (see [1, Definition 4.1]) and there is, up to isomorphism, a unique finite group $L = L_P^{\mathcal{F}}$ having $N_S(P)$ as a Sylow p-subgroup such that $C_L(P) = Z(P)$ and $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(L)$ (see [1, Proposition 4.3]).

DEFINITION 3.3

Let H be a finite group, and let $\mathcal{F}$ be a saturated fusion system over a p-group S. We say that $\mathcal{F}$ *is H-free* if H is not involved in any of the groups $L_P^{\mathcal{F}}$, with P running over the set of $\mathcal{F}$-centric fully normalized in $\mathcal{F}$ subgroups of S.

Proof of Theorems 1.1 and 1.2

We will proceed by considering a minimal counterexample and getting a contradiction. So, let S be the smallest p-group, and let $\mathcal{F}$ be the saturated fusion system with a minimal number of morphisms such that $(S, \mathcal{F})$ admits $\mathcal{U}$ a group of $\mathcal{T}$-automorphisms and $\mathcal{F}_S(S) \lneq \mathcal{F}$.

Step 1. There exists a nontrivial elementary abelian proper subgroup $W(S) \lneq S$ such that $(S, \mathcal{F}) = (S, N_{\mathcal{F}}(W(S)))$.

Given a group G, let $Z(G)$, $J(G)$, and $\Omega(G)$ denote the center, the Thompson subgroup, and the group generated by the elements of order p of G, respectively. Let $W(S)$ be the characteristic subgroup of S defined in [11, Section 4]. Then $\Omega(Z(S)) \leq W(S) \leq \Omega(Z(J(S)))$, and $W(S)$ is nontrivial and elementary abelian. Moreover, as $W(S)$ is characteristic, it is $\mathcal{U}$-invariant and normal in S. This implies that $W(S)$ is fully normalized in $\mathcal{F}$ so, by Proposition 2.3, $N_{\mathcal{F}}(W(S))$ is a saturated fusion system over S.

Let us note that $(S,\mathcal{F}) = (S, N_{\mathcal{F}}(W(S)))$. Assume that $N_{\mathcal{F}}(W(S)) \lneq \mathcal{F}$. Then $\mathcal{U}$ induces a group of $\mathcal{T}$-automorphisms in $(S, N_{\mathcal{F}}(W(S)))$, so if there are morphisms in $\mathcal{F}$ which are not in $N_{\mathcal{F}}(W(S))$, then by the minimality of $\mathcal{F}$, $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$. But, as p is an odd prime (respectively, $p=2$ and $\mathcal{F}$ is Σ_4-free), by [11, Theorem 1.3] (respectively, [11, Theorem 1.1]) this implies that $\mathcal{F} = \mathcal{F}_S(S)$, so $N_{\mathcal{F}}(W(S)) = \mathcal{F}$, a contradiction.

If $W(S) = S$, then by applying Lemma 3.2, $(S,\mathcal{F}) = (S, \mathcal{F}_S(S))$, so it is not a counterexample.

Step 2. According to Lemma 2.4, $(S,\mathcal{F})$ projects onto a saturated fusion system $(S/W(S), \mathcal{F}/W(S))$ and $\mathcal{U}$ induces a group of $\mathcal{T}$-automorphisms of $\mathcal{F}/W(S)$. So, by the minimality hypothesis, $\mathcal{F}/W(S) = \mathcal{F}_{S/W(S)}(S/W(S))$.

Step 3. There is an element α of prime order q in $\mathrm{Aut}_{\mathcal{F}}(W(S))$, with $q \neq p$.

As we are assuming that $\mathcal{F}_S(S) \lneq \mathcal{F}$, by using Alperin's theorem for saturated fusion systems (Theorem 2.6), there exists P, a fully normalized in $\mathcal{F}$, $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of S with an $\mathcal{F}$-automorphism $\tilde{\alpha}$ of prime order q, $q \neq p$. By [1, Proposition 1.6], as $W(S)$ is normal in $\mathcal{F}$ and P is $\mathcal{F}$-radical, $W(S)$ is contained in P and we have induced maps in the normal series $1 \trianglelefteq W(S) \trianglelefteq P$. By Step 2, $\tilde{\alpha}$ projects to the identity in $P/W(S)$. If the restriction of $\tilde{\alpha}$ to $W(S)$ is also the identity, by [8, Theorem 5.3.2] the order of $\tilde{\alpha}$ is a power of p, a contradiction. So $\tilde{\alpha}$ restricts to an automorphism α on $W(S)$ of order q.

Last step. We finish the proof by considering $G = W(S) \rtimes H$ where $H = \mathrm{Aut}_{\mathcal{F}}(W(S))$; then the automorphism group induced by $\mathcal{U}$ on G is a group of $\mathcal{T}$-automorphisms that leave $W(S)$ invariant, which contradicts Lemma 3.2. Therefore there is no minimal counterexample $(S,\mathcal{F})$. □

We finish this section with a couple of examples that illustrate the scope of these results. The first example, which seems to be well known to the experts (see comments below [13, Theorem A]), shows that the Σ_4-free hypothesis in the $p=2$ case is a necessary one.

EXAMPLE 3.4

Let G be the simple group $L(2,17)$, and let $S \leq G$ be a 2-Sylow subgroup. According to [4, p. 9], the 2-fusion is completely determined by its self-normalizing Sylow $S \cong D_{16}$ and two conjugacy classes of rank-two elementary abelian subgroups of type $2A^2$ whose normalizer is isomorphic to Σ_4. So $\mathcal{F} = \mathcal{F}_S(G)$ is not Σ_4-free, $\mathcal{F} \neq \mathcal{F}_S(S)$, and for every normal $Q \trianglelefteq S$, $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a 2-group (since the only maximal subgroup of G of index coprime to 2 is S). Therefore, for every $\mathcal{U} \leq \mathrm{Aut}(\mathcal{F})$ and $\mathcal{U}$-invariant normal subgroup $Q \trianglelefteq S$, $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a 2-group, but $\mathcal{F} \neq \mathcal{F}_S(S)$.

The last example shows that group automorphisms with fixed points can induce a fixed-point-free automorphism of a saturated fusion system.

EXAMPLE 3.5

Let S be a finite p-group such that there exists a fixed-point-free automorphism ϕ of S. Consider $H = S \rtimes \langle\phi\rangle$, and let $f : H \to \mathrm{Aut}(N)$ be a homomorphism, where N is a group whose order is coprime to p. The homomorphism f defines a semidirect product $G = N \rtimes H$, where the subgroup $K = SN$ is a normal subgroup of G and ϕ acts on K not necessarily without fixed points, but it acts on the fusion category $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ without fixed points.

An example of this situation is the group A_4 and $\phi \in \mathrm{Aut}(A_4)$ given by conjugation in S_4 by the transposition $(1,2)$. Then ϕ fixes the Sylow 3-subgroup $S = \langle(1,2,3)\rangle$, and furthermore it acts without fixed points on the fusion category $\mathcal{F}_S(A_4)$.

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