Koecher–Maass series of the Ikeda lift for $U(m,m)$

Hidenori Katsurada

In memory of Professor Hiroshi Saito

Abstract Let $K=\mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$, and let χ be the Dirichlet character corresponding to the extension $K/\mathbf{Q}$. Let $m=2n$ or $2n+1$ with n a positive integer. Let f be a primitive form of weight $2k+1$ and character χ for $\Gamma_0(D)$ or a primitive form of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$ according to whether $m=2n$ or $m=2n+1$. For such an f let $I_m(f)$ be the lift of f to the space of Hermitian modular forms constructed by Ikeda. We then give an explicit formula of the Koecher–Maass series $L(s,I_m(f))$ of $I_m(f)$. This is a generalization of Mizuno.

1. Introduction

Mizuno [M] gave explicit formulas of the Koecher–Maass series of the Hermitian Eisenstein series of degree 2 and of the Hermitian Maass lift. In this paper, we give an explicit formula of the Koecher–Maass series of the Hermitian Ikeda lift. Let $K=\mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$. Let $\mathcal{O}$ be the ring of integers in K, and let χ be the Kronecker character corresponding to the extension $K/\mathbf{Q}$. For a nondegenerate Hermitian matrix or alternating matrix T with entries in K, let $\mathcal{U}_T$ be the unitary group defined over $\mathbf{Q}$ whose group $\mathcal{U}_T(R)$ of R-valued points is given by

$$\mathcal{U}_T(R)=\left\{g\in \mathrm{GL}_m(R\otimes K)\,\middle|\,{}^t\overline{g}Tg=T\right\}$$

for any $\mathbf{Q}$-algebra R, where $\overline{g}$ denotes the automorphism of $M_n(R\otimes K)$ induced by the nontrivial automorphism of K over $\mathbf{Q}$. We also define the special unitary group $\mathcal{SU}_T$ over $\mathbf{Q}_p$ by $\mathcal{SU}_T=\mathcal{U}_T\cap R_{K/\mathbf{Q}}(\mathrm{SL}_m)$, where $R_{K/\mathbf{Q}}$ is the Weil restriction. In particular, we write $\mathcal{U}_T$ as $\mathcal{U}^{(m)}$ or $U(m,m)$ if $T=\left(\begin{smallmatrix} O & -1_m \\ 1_m & O\end{smallmatrix}\right)$. For a more precise description of $\mathcal{U}^{(m)}$ see Section 2. Put $\Gamma_K^{(m)}=U(m,m)(\mathbf{Q})\cap \mathrm{GL}_{2m}(\mathcal{O})$. For a modular form F of weight $2l$ and character ψ for $\Gamma_K^{(m)}$ we define the Koecher–Maass series $L(s,F)$ of F by

Kyoto Journal of Mathematics, Vol. 55, No. 2 (2015), 321–364
DOI 10.1215/21562261-2871776,
Received August 19, 2013. Revised March 6, 2014. Accepted March 6, 2014.
2010 Mathematics Subject Classification: Primary 11F55, 11F67.
Author's work partly supported by Japan Society for the Promotion of Science KAKENHI Grant Number 24540005.

$$L(s,F)=\sum_T \frac{c_F(T)}{e^*(T)(\det T)^s},$$

where T runs over all $\mathrm{SL}_m(\mathcal{O})$-equivalence classes of positive definite semi-integral Hermitian matrices of degree m, $c_F(T)$ denotes the Tth Fourier coefficient of F, and $e^*(T)=\#(\mathcal{SU}_T(\mathbf{Q})\cap \mathrm{SL}_m(\mathcal{O}))$.

Let k be a nonnegative integer. Then for a primitive form $f\in\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ Ikeda [I2] constructed a lift $I_{2n}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_K^{(2n)}$. This is a generalization of the Maass lift considered by Kojima [Ko], Gritsenko [G], Krieg [Kr], and Sugano [Su]. Similarly for a primitive form $f\in\mathfrak{S}_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ he constructed a lift $I_{2n+1}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_K^{(2n+1)}$. For the rest of this section, let $m=2n$ or $m=2n+1$. We then call $I_m(f)$ the Ikeda lift of f for $U(m,m)$ or the Hermitian Ikeda lift of degree m. Ikeda also showed that the automorphic form $\mathit{Lift}^{(m)}(f)$ on the adèle group $\mathcal{U}^{(m)}(\mathbf{A})$ associated with $I_m(f)$ is a cuspidal Hecke eigenform whose standard L-function coincides with

$$\prod_{i=1}^m L(s+k+n-i+1/2,f)L(s+k+n-i+1/2,f,\chi),$$

where $L(s+k+n-i+1/2,f)$ is the Hecke L-function of f and $L(s+k+n-i+1/2,f,\chi)$ is its “modified twist” by χ. For the precise definition of $L(s+k+n-i+1/2,f,\chi)$ see Section 2. We also call $\mathit{Lift}^{(m)}(f)$ the adèlic Ikeda lift of f for $U(m,m)$. Then we express the Koecher–Maass series of $I_m(f)$ in terms of the L-functions related to f. This result was already obtained in the case $m=2$ by Mizuno [M].

The method we use is similar to that in the proof of the main result of [IK1] or [IK2]. We explain it more precisely. In Section 3, we reduce our computation to a computation of a certain formal power series $\hat{P}_{m,p}(d;X,t)$ in t associated with local Siegel series similarly to [IK1] (see Theorem 3.4 and Section 5).

Section 4 is devoted to the computation of them. This computation is similar to that in [IK1], but we should be careful in dealing with the case where p is ramified in K. After such an elaborate computation, we can get explicit formulas of $\hat{P}_{m,p}(d;X,t)$ for all prime numbers p (see Theorems 4.3.1, 4.3.2, and 4.3.6). In Section 5, by using explicit formulas for $\hat{P}_{m,p}(d;X,t)$, we immediately get an explicit formula for $L(s,I_m(f))$.

Using the same argument as in the proof of our main result, we can give an explicit formula of the Koecher–Maass series of the Hermitian Eisenstein series of any degree, which can be regarded as a zeta function of a certain prehomogeneous vector space. We also note that the method used in this paper is useful for giving an explicit formula for the Rankin–Selberg series of the Hermitian Ikeda lift, and as a result we can prove the period relation of the Hermitian Ikeda lift, which was conjectured by Ikeda [I2]. We will discuss these topics in subsequent papers [Ka1] and [Ka2].

NOTATION

Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of nonzero elements of R and the unit group of R, respectively. For a subset S of R we denote by $M_{mn}(S)$ the set of (m,n)-matrices with entries in S. In particular, put $M_n(S)=M_{nn}(S)$. Put $\mathrm{GL}_m(R)=\{A\in M_m(R) \mid \det A\in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. Let K_0 be a field, and let K be a quadratic extension of K_0 or $K=K_0\oplus K_0$. In the latter case, we regard K_0 as a subring of K via the diagonal embedding. We also identify $M_{mn}(K)$ with $M_{mn}(K_0)\oplus M_{mn}(K_0)$ in this case. If K is a quadratic extension of K_0, then let ρ be the nontrivial automorphism of K over K_0, and if $K=K_0\oplus K_0$, then let ρ be the automorphism of K defined by $\rho(a,b)=(b,a)$ for $(a,b)\in K$. We sometimes write $\overline{x}$ instead of $\rho(x)$ for $x\in K$ in both cases. Let R be a subring of K. For an (m,n)-matrix $X=(x_{ij})_{m\times n}$ write $X^*=(\overline{x_{ji}})_{n\times m}$, and for an (m,m)-matrix A, we write $A[X]=X^*AX$. Let $\mathrm{Her}_n(R)$ denote the set of Hermitian matrices of degree n with entries in R, that is, the subset of $M_n(R)$ consisting of matrices X such that $X^*=X$. Then a Hermitian matrix A of degree n with entries in K is said to be semi-integral over R if $\mathrm{tr}(AB)\in K_0\cap R$ for any $B\in\mathrm{Her}_n(R)$, where tr denotes the trace of a matrix. We denote by $\widehat{\mathrm{Her}}_n(R)$ the set of semi-integral matrices of degree n over R.

For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of nondegenerate matrices. If S is a subset of $\mathrm{Her}_n(\mathbf{C})$ with $\mathbf{C}$ the field of complex numbers, then we denote by S^+ the subset of S consisting of positive definite matrices. The group $\mathrm{GL}_n(R)$ acts on the set $\mathrm{Her}_n(R)$ in the following way:

$$\mathrm{GL}_n(R)\times\mathrm{Her}_n(R)\ni(g,A)\longrightarrow g^*Ag\in\mathrm{Her}_n(R).$$

Let G be a subgroup of $\mathrm{GL}_n(R)$. For a G-stable subset $\mathcal{B}$ of $\mathrm{Her}_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/\mathrm{GL}_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Two Hermitian matrices A and A' with entries in R are said to be G-equivalent and we write $A\sim_G A'$ if there is an element X of G such that $A'=A[X]$. For square matrices X and Y we write $X\perp Y=\left(\begin{smallmatrix}X&O\\O&Y\end{smallmatrix}\right)$.

We put $\mathbf{e}(x)=\exp(2\pi\sqrt{-1}x)$ for $x\in\mathbf{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x)=\mathbf{e}(x)$ for $x\in\mathbf{Z}[p^{-1}]$.

For a prime number p we denote by $\mathrm{ord}_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\mathrm{ord}_p(p)=1$, and put $|x|_p=p^{-\mathrm{ord}_p(x)}$. Moreover, we denote by $|x|_\infty$ the absolute value of $x\in\mathbf{C}$. Let K be an imaginary quadratic field, and let $\mathcal{O}$ be the ring of integers in K. For a prime number p put $K_p=K\otimes\mathbf{Q}_p$, and put $\mathcal{O}_p=\mathcal{O}\otimes\mathbf{Z}_p$. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p\cong\mathbf{Q}_p\oplus\mathbf{Q}_p$. In the former case, for $x\in K_p$, we denote by $\overline{x}$ the conjugate of x over $\mathbf{Q}_p$. In the latter case, we identify K_p with $\mathbf{Q}_p\oplus\mathbf{Q}_p$, and for $x=(x_1,x_2)$ with $x_i\in\mathbf{Q}_p$, we put $\overline{x}=(x_2,x_1)$. For $x\in K_p$ we define the norm $N_{K_p/\mathbf{Q}_p}(x)$ by $N_{K_p/\mathbf{Q}_p}(x)=x\overline{x}$, put $\nu_{K_p}(x) = \mathrm{ord}_p(N_{K_p/\mathbf{Q}_p}(x))$, and put $|x|_{K_p} = |N_{K_p/\mathbf{Q}_p}(x)|_p$. Moreover, put $|x|_{K_\infty} = |x\overline{x}|_\infty$ for $x \in \mathbf{C}$.

2. Main results

For a positive integer N let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$

and for a Dirichlet character ψ mod N, we denote by $\mathfrak{M}_l(\Gamma_0(N), \psi)$ the space of modular forms of weight l for $\Gamma_0(N)$ and nebentype ψ, and by $\mathfrak{S}_l(\Gamma_0(N), \psi)$ its subspace consisting of cusp forms. We simply write $\mathfrak{M}_l(\Gamma_0(N), \psi)$ (resp., $\mathfrak{S}_l(\Gamma_0(N), \psi)$) as $\mathfrak{M}_l(\Gamma_0(N))$ (resp., as $\mathfrak{S}_l(\Gamma_0(N))$) if ψ is the trivial character.

Throughout the paper, we fix an imaginary quadratic extension K of $\mathbf{Q}$ with discriminant $-D$, and denote by $\mathcal{O}$ the ring of integers in K. For such a K let $\mathcal{U}^{(m)} = U(m,m)$ be the unitary group defined in Section 1. Put $J_m = \left(\begin{smallmatrix} O_m & -1_m \\ 1_m & O_m \end{smallmatrix}\right)$, where 1_m denotes the unit matrix of degree m. Then

$$\mathcal{U}^{(m)}(\mathbf{Q}) = \left\{ M \in \mathrm{GL}_{2m}(K) \;\middle|\; J_m[M] = J_m \right\}.$$

Put

$$\Gamma^{(m)} = \Gamma_K^{(m)} = \mathcal{U}^{(m)}(\mathbf{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O}).$$

Let $\mathfrak{H}_m$ be the Hermitian upper half-space defined by

$$\mathfrak{H}_m = \left\{ Z \in M_m(\mathbf{C}) \;\middle|\; \frac{1}{2\sqrt{-1}}(Z - Z^*) \text{ is positive definite} \right\}.$$

The group $\mathcal{U}^{(m)}(\mathbf{R})$ acts on $\mathfrak{H}_m$ by

$$g\langle Z \rangle = (AZ+B)(CZ+D)^{-1} \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{U}^{(m)}(\mathbf{R}), Z \in \mathfrak{H}_m.$$

We also put $j(g,Z) = \det(CZ+D)$ for such Z and g. Let l be an integer. For a subgroup Γ of $\mathcal{U}^{(m)}(\mathbf{Q})$ commensurable with $\Gamma^{(m)}$ and a character ψ of Γ, we denote by $\mathfrak{M}_l(\Gamma, \psi)$ the space of holomorphic modular forms of weight l with character ψ for Γ. We denote by $\mathfrak{S}_l(\Gamma, \psi)$ the subspace of $\mathfrak{M}_l(\Gamma, \psi)$ consisting of cusp forms. In particular, if ψ is the character of Γ defined by $\psi(\gamma) = (\det \gamma)^{-l}$ for $\gamma \in \Gamma$, then we write $\mathfrak{M}_{2l}(\Gamma, \psi)$ as $\mathfrak{M}_{2l}(\Gamma, \det^{-l})$, and so on. Let $F(z)$ be an element of $\mathfrak{M}_{2l}(\Gamma^{(m)}, \det^{-l})$. We then define the Koecher–Maass series $L(s,F)$ for F by

$$L(s,F) = \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+ / \mathrm{SL}_n(\mathcal{O})} \frac{c_F(T)}{(\det T)^s e^*(T)},$$

where $c_F(T)$ denotes the Tth Fourier coefficient of F, and $e^*(T) = \#(\mathcal{SU}_T(\mathbf{Q}) \cap \mathrm{SL}_m(\mathcal{O}))$.

Now we consider the adèlic modular form. Let $\mathbf{A}$ be the adèle ring of $\mathbf{Q}$, and let $\mathbf{A}_f$ be the non-archimedean factor of $\mathbf{A}$. Let $h = h_K$ be a class number of K. Let $G^{(m)} = \mathrm{Res}_{K/\mathbf{Q}}(\mathrm{GL}_m)$, and let $G^{(m)}(\mathbf{A})$ be the adèlization of $G^{(m)}$.

Moreover, put $\mathcal{C}^{(m)} = \prod_p \mathrm{GL}_m(\mathcal{O}_p)$. Let $\mathcal{U}^{(m)}(\mathbf{A})$ be the adèlization of $\mathcal{U}^{(m)}$. We define the compact subgroup $\mathcal{K}_0^{(m)}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$ by $\mathcal{U}^{(m)}(\mathbf{A}) \cap \prod_p \mathrm{GL}_{2m}(\mathcal{O}_p)$, where p runs over all rational primes. Then we have that

$$\mathcal{U}^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} \mathcal{U}^{(m)}(\mathbf{Q})\gamma_i \mathcal{K}_0^{(m)} \mathcal{U}^{(m)}(\mathbf{R})$$

with some subset $\{\gamma_1, \dots, \gamma_h\}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$. We can take γ_i as

$$\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{*-1} \end{pmatrix},$$

where $\{t_i\}_{i=1}^h = \{(t_{i,p})\}_{i=1}^h$ is a certain subset of $G^{(m)}(\mathbf{A}_f)$ such that $t_1 = 1$ and

$$G^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} G^{(m)}(\mathbf{Q}) t_i G^{(m)}(\mathbf{R}) \mathcal{C}^{(m)}.$$

Put $\Gamma_i = \mathcal{U}^{(m)}(\mathbf{Q}) \cap \gamma_i \mathcal{K}_0 \gamma_i^{-1} \mathcal{U}^{(m)}(\mathbf{R})$. Then for an element $(F_1, \dots, F_h) \in \bigoplus_{i=1}^h \mathfrak{M}_{2l}(\Gamma_i, \det^{-l})$, we define $(F_1, \dots, F_h)^\sharp$ by

$$(F_1, \dots, F_h)^\sharp(g) = F_i\bigl(x\langle \mathbf{i}\rangle\bigr) j(x, \mathbf{i})^{-2l} (\det x)^l$$

for $g = u\gamma_i x\kappa$ with $u \in \mathcal{U}^{(m)}(\mathbf{Q})$, $x \in \mathcal{U}^{(m)}(\mathbf{R})$, and $\kappa \in \mathcal{K}_0$. We denote by $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q}) \backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ the space of automorphic forms obtained in this way. We also put

$$\mathcal{S}_{2l}\bigl(\mathcal{U}^{(m)}(\mathbf{Q}) \backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l}\bigr) = \bigl\{(F_1, \dots, F_h)^\sharp \bigm| F_i \in \mathfrak{S}_{2l}(\Gamma_i, \det^{-l})\bigr\}.$$

We can define the Hecke operators which act on the space $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q}) \backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$. For the precise definition of them, see [I2].

Let $\widehat{\mathrm{Her}}_m(\mathcal{O})$ be the set of semi-integral Hermitian matrices over $\mathcal{O}$ of degree m as in the Notation. We note that A belongs to $\widehat{\mathrm{Her}}_m(\mathcal{O})$ if and only if its diagonal components are rational integers and $\sqrt{-D}A \in \mathrm{Her}_m(\mathcal{O})$. For a non-degenerate Hermitian matrix B with entries in K_p of degree m, put $\gamma(B) = (-D)^{[m/2]} \det B$.

Let $\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$ be the set of semi-integral matrices over $\mathcal{O}_p$ of degree m as in the Notation. We put $\xi_p = 1, -1$, or 0 according to whether $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, K_p is an unramified quadratic extension of $\mathbf{Q}_p$, or K_p is a ramified quadratic extension of $\mathbf{Q}_p$. For $T \in \widehat{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ we define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in \mathrm{Her}_n(K_p)/\,\mathrm{Her}_n(\mathcal{O}_p)} \mathbf{e}_p\bigl(\mathrm{tr}(TR)\bigr) p^{-\mathrm{ord}_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathcal{O}_p^m + \mathcal{O}_p^m : \mathcal{O}_p^m]$. We remark that there exists a unique polynomial $F_p(T, X)$ in X such that (see [Sh1])

$$b_p(T, s) = F_p(T, p^{-s}) \prod_{i=0}^{[(m-1)/2]} (1 - p^{2i-s}) \prod_{i=1}^{[m/2]} (1 - \xi_p p^{2i-1-s}).$$

We then define a Laurent polynomial $\widetilde{F}_p(T, X)$ as

$$\widetilde{F}_p(T,X) = X^{-\operatorname{ord}_p(\gamma(T))} F_p(T, p^{-m}X^2).$$

We remark that we have (see [I2])

$$\widetilde{F}_p(T,X^{-1}) = \bigl(-D,\gamma(T)\bigr)_p \widetilde{F}_p(T,X) \quad \text{if } m \text{ is even},$$

$$\widetilde{F}_p(T,\xi_p X^{-1}) = \widetilde{F}_p(T,X) \quad \text{if } m \text{ is even and } p \nmid D,$$

and

$$\widetilde{F}_p(T,X^{-1}) = \widetilde{F}_p(T,X) \quad \text{if } m \text{ is odd}.$$

Here $(a,b)_p$ is the Hilbert symbol of $a,b \in \mathbf{Q}_p^\times$. Hence we have that

$$\widetilde{F}_p(T,X) = \bigl(-D,\gamma(B)\bigr)_p^{m-1} X^{\operatorname{ord}_p(\gamma(T))} F_p(T,p^{-m}X^{-2}).$$

Now we put

$$\widehat{\operatorname{Her}}_m(\mathcal{O})_i^+ = \bigl\{T \in \operatorname{Her}_m(K)^+ \bigm| t_{i,p}^* T t_{i,p} \in \widehat{\operatorname{Her}}_m(\mathcal{O}_p) \text{ for any } p\bigr\}.$$

First let k be a nonnegative integer, and let $m = 2n$ be a positive even integer. Let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$. For a prime number p not dividing D let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \chi(p)\alpha_p^{-1} = p^{-k}a(p)$, and for $p \mid D$ put $\alpha_p = p^{-k}a(p)$. We note that $\alpha_p \neq 0$ even if $p \mid D$. Then for the Kronecker character χ we define Hecke's L-function $L(s,f,\chi^i)$ twisted by χ^i as

$$L(s,f,\chi^i) = \prod_{p \nmid D} \bigl\{\bigl(1-\alpha_p p^{-s+k}\chi(p)^i\bigr)\bigl(1-\alpha_p^{-1}p^{-s+k}\chi(p)^{i+1}\bigr)\bigr\}^{-1}$$

$$\times \begin{cases} \prod_{p \mid D}(1-\alpha_p p^{-s+k})^{-1} & \text{if } i \text{ is even},\\ \prod_{p \mid D}(1-\alpha_p^{-1}p^{-s+k})^{-1} & \text{if } i \text{ is odd}.\end{cases}$$

In particular, if i is even, then we sometimes write $L(s,f,\chi^i)$ as $L(s,f)$ as usual. Moreover, for $i = 1,\ldots,h$ we define a Fourier series

$$I_m(f)_i(Z) = \sum_{T \in \widehat{\operatorname{Her}}_m(\mathcal{O})_i^+} a_{I_m(f)_i}(T)\mathbf{e}\bigl(\operatorname{tr}(TZ)\bigr),$$

where

$$a_{I_{2n}(f)_i}(T) = \bigl|\gamma(T)\bigr|^k \prod_p \bigl|\det(t_{i,p})\det(\overline{t_{i,p}})\bigr|_p^n \widetilde{F}_p(t_{i,p}^* T t_{i,p}, \alpha_p^{-1}).$$

Next let k be a positive integer, and let $m = 2n+1$ be a positive odd integer. Let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k}(\operatorname{SL}_2(\mathbf{Z}))$. For a prime number p let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+1/2}a(p)$. Then we define Hecke's L-function $L(s,f,\chi^i)$ twisted by χ^i as

$$L(s,f,\chi^i)=\prod_p\bigl\{\bigl(1-\alpha_p p^{-s+k-1/2}\chi(p)^i\bigr)\bigl(1-\alpha_p^{-1}p^{-s+k-1/2}\chi(p)^i\bigr)\bigr\}^{-1}.$$

In particular, if i is even, then we write $L(s,f,\chi^i)$ as $L(s,f)$ as usual. Moreover, for $i=1,\dots,h$ we define a Fourier series

$$I_{2n+1}(f)_i(Z)=\sum_{T\in\widehat{\mathrm{Her}}_{2n+1}(\mathcal{O})_i^+}a_{I_{2n+1}(f)_i}(T)\mathbf{e}\bigl(\mathrm{tr}(TZ)\bigr),$$

where

$$a_{I_{2n+1}(f)_i}(T)=\bigl|\gamma(T)\bigr|^{k-1/2}\prod_p\bigl|\det(t_{i,p})\det(\overline{t_{i,p}})\bigr|_p^{n+1/2}\widetilde{F}_p(t_{i,p}^*Tt_{i,p},\alpha_p^{-1}).$$

REMARK

Ikeda [I2] defined $\widetilde{F}_p(T,X)$ as

$$\widetilde{F}_p(T,X)=X^{\mathrm{ord}_p(\gamma(T))}F_p(T,p^{-m}X^{-2}),$$

and we define it by replacing X with X^{-1} in this paper. This change does not affect the results.

Then Ikeda [I2] showed the following.

THEOREM 2.1

Let $m=2n$ or $2n+1$. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ according to whether $m=2n$ or $m=2n+1$. Moreover, let Γ_i be the subgroup of $\mathcal{U}^{(m)}$ defined as above. Then $I_m(f)_i(Z)$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma_i,\det^{-k-n})$ for any i. In particular, $I_m(f):=I_m(f)_1$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma^{(m)},\det^{-k-n})$.

This is a Hermitian analogue of the lifting constructed in [I1]. We call $I_m(f)$ the Ikeda lift of f for $\mathcal{U}^{(m)}$.

It follows from Theorem 2.1 that we can define an element $(I_m(f)_1,\dots,I_m(f)_h)^\sharp$ of $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-k-n})$, which we call $\mathit{Lift}^{(m)}(f)$.

THEOREM 2.2

Let $m=2n$ or $2n+1$. Suppose that $\mathit{Lift}^{(m)}(f)$ is not identically zero. Then $\mathit{Lift}^{(m)}(f)$ is a Hecke eigenform in $\mathcal{S}_{2k+2n}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-k-n})$ and its standard L-function $L(s,\mathit{Lift}^{(m)}(f),\mathrm{st})$ coincides with

$$\prod_{i=1}^m L(s+k+n-i+1/2,f)L(s+k+n-i+1/2,f,\chi)$$

up to bad Euler factors.

We call $\mathit{Lift}^{(m)}(f)$ the adèlic Ikeda lift of f for $\mathcal{U}^{(m)}$.

Let Q_D be the set of prime divisors of D. For each prime $q \in Q_D$, put $D_q = q^{\mathrm{ord}_q(D)}$. We define a Dirichlet character χ_q by

$$\chi_q(a) = \begin{cases} \chi(a') & \text{if } (a,q)=1, \\ 0 & \text{if } q \mid a, \end{cases}$$

where a' is an integer such that

$$a' \equiv a \bmod D_q \qquad \text{and} \qquad a' \equiv 1 \bmod DD_q^{-1}.$$

For a subset Q of Q_D put $\chi_Q = \prod_{q \in Q} \chi_q$ and $\chi'_Q = \prod_{q \in Q_D, q \notin Q} \chi_q$. Here we make the convention that $\chi_Q = 1$ and $\chi'_Q = \chi$ if Q is the empty set. Let

$$f(z) = \sum_{N=1}^{\infty} c_f(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then there exists a primitive form

$$f_Q(z) = \sum_{N=1}^{\infty} c_{f_Q}(N)\mathbf{e}(Nz)$$

such that

$$c_{f_Q}(p) = \chi_Q(p)c_f(p) \quad \text{for } p \notin Q$$

and

$$c_{f_Q}(p) = \chi'_Q(p)\overline{c_f(p)} \quad \text{for } p \in Q.$$

Let $L(s,\chi^i) = \zeta(s)$ or $L(s,\chi)$ according to whether i is even or odd, where $\zeta(s)$ and $L(s,\chi)$ are Riemann's zeta function and the Dirichlet L-function for χ, respectively. Moreover, we define $\widetilde{\Lambda}(s,\chi^i)$ by

$$\widetilde{\Lambda}(s,\chi^i) = 2(2\pi)^{-s}\Gamma(s)L(s,\chi^i)$$

with $\Gamma(s)$ the Gamma function.

Then our main results in this paper are as follows.

THEOREM 2.3

Let k be a nonnegative integer, and let n be a positive integer. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then, we have

$$L\big(s, I_{2n}(f)\big) = D^{ns+n^2-n/2-1/2}2^{-2n+1} \times \prod_{i=2}^{2n} \widetilde{\Lambda}(i,\chi^i) \sum_{Q \subset Q_D} \chi_Q\big((-1)^n\big) \prod_{j=1}^{2n} L(s-2n+j, f_Q, \chi^{j-1}).$$

THEOREM 2.4

Let k be a positive integer, and let n be a nonnegative integer. Let f be a primitive form in $\mathfrak{S}_{2k}(\mathrm{SL}_2(\mathbf{Z}))$. Then, we have that

$$L\big(s, I_{2n+1}(f)\big) = D^{ns+n^2+3n/2}2^{-2n} \prod_{i=2}^{2n+1} \widetilde{\Lambda}(i,\chi^i) \prod_{j=1}^{2n+1} L(s-2n-1+j, f, \chi^{j-1}).$$

REMARK
We note that $L(s, I_{2n+1}(f))$ has an Euler product.

3. Reduction to local computations

To prove our main result, we reduce the problem to local computations. Let $K_p = K \otimes \mathbf{Q}_p$ and $\mathcal{O}_p = \mathcal{O} \otimes \mathbf{Z}_p$ as in the Notation. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case let f_p be the exponent of the conductor of $K_p/\mathbf{Q}_p$. If K_p is ramified over $\mathbf{Q}_p$, then put $e_p = f_p - \delta_{2,p}$, where $\delta_{2,p}$ is Kronecker's delta. If K_p is unramified over $\mathbf{Q}_p$, then put $e_p = f_p = 0$. In the latter case, put $e_p = f_p = 0$. Let K_p be a quadratic extension of $\mathbf{Q}_p$, and let $\varpi = \varpi_p$ and $\pi = \pi_p$ be prime elements of K_p and $\mathbf{Q}_p$, respectively. If K_p is unramified over $\mathbf{Q}_p$, then we take $\varpi = \pi = p$. If K_p is ramified over $\mathbf{Q}_p$, then we take π so that $\pi = N_{K_p/\mathbf{Q}_p}(\varpi)$. Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then put $\varpi = \pi = p$. Let χ_{K_p} be the quadratic character of $\mathbf{Q}_p^\times$ corresponding to the quadratic extension $K_p/\mathbf{Q}_p$. We note that we have $\chi_{K_p}(a) = (-D_0, a)_p$ for $a \in \mathbf{Q}_p^\times$ if $K_p = \mathbf{Q}_p(\sqrt{-D_0})$ with $D_0 \in \mathbf{Z}_p$. Moreover, put $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = p^{e_p}\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$. We note that $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = \mathrm{Her}_m(\mathcal{O}_p)$ if K_p is not ramified over $\mathbf{Q}_p$. Let K be an imaginary quadratic extension of $\mathbf{Q}$ with discriminant $-D$. We then put $\widetilde{D} = \prod_{p \mid D} p^{e_p}$ and $\widetilde{\mathrm{Her}}_m(\mathcal{O}) = \widetilde{D}\,\mathrm{Her}_m(\mathcal{O})$. An element $X \in M_{ml}(\mathcal{O}_p)$ with $m \geq l$ is said to be primitive if there is an element Y of $M_{m,m-l}(\mathcal{O}_p)$ such that $(XY) \in \mathrm{GL}_m(\mathcal{O}_p)$. If K_p is a field, then this is equivalent to saying that $\mathrm{rank}_{\mathcal{O}_p/\varpi\mathcal{O}_p} X = l$. If $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$ and $X = (X_1, X_2) \in M_{ml}(\mathbf{Z}_p) \oplus M_{ml}(\mathbf{Z}_p)$, then this is equivalent to saying that $\mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X_1 = \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X_2 = l$. Now let m and l be positive integers such that $m \geq l$. Then for an integer a and $A \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p), B \in \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)$ put

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X] - B \in p^a \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)\},$$

and

$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid X \text{ is primitive}\}.$$

Suppose that A and B are nondegenerate. Then the number $p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B)$ is independent of a if a is sufficiently large. Hence we define the local density $\alpha_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B).$$

Similarly we can define the primitive local density $\beta_p(A,B)$ as

$$\beta_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{B}_a(A,B)$$

if A is nondegenerate. We remark that the primitive local density $\beta_p(A,B)$ can be defined even if B is not nondegenerate. In particular, we write $\alpha_p(A) = \alpha_p(A,A)$. We also define $\upsilon_p(A)$ for $A \in \mathrm{Her}_m(\mathcal{O}_p)^\times$ as

$$\upsilon_p(A) = \lim_{a\to\infty} p^{-am^2}\#(\Upsilon_a(A)),$$

where

$$\Upsilon_a(A) = \{X \in M_m(\mathcal{O}_p)/p^a M_m(\mathcal{O}_p) \mid A[X] - A \in p^a \operatorname{Her}_m(\mathcal{O}_p)\}.$$

The relation between $\alpha_p(A)$ and $\upsilon_p(A)$ is as follows.

LEMMA 3.1
Let $T \in \widetilde{\operatorname{Her}}_m(\mathcal{O}_p)^\times$. Suppose that K_p is ramified over $\mathbf{Q}_p$. Then we have that

$$\alpha_p(T) = p^{-m(m+1)f_p/2+m^2\delta_{2,p}}\upsilon_p(T).$$

Otherwise, $\alpha_p(T) = \upsilon_p(T)$.

Proof
The proof is similar to that for [Ki3, Lemma 5.6.5], and we here give an outline of the proof. The last assertion is trivial. Suppose that K_p is ramified over $\mathbf{Q}_p$. Let $\{T_i\}_{i=1}^l$ be a complete set of representatives of $\operatorname{Her}_m(\mathcal{O}_p)/p^{r+e_p}\operatorname{Her}_m(\mathcal{O}_p)$ such that $T_i \equiv T \bmod p^r \widetilde{\operatorname{Her}}_m(\mathcal{O}_p)$. Then it is easily seen that

$$l = [p^r \widetilde{\operatorname{Her}}_m(\mathcal{O}_p) : p^{r+e_p}\operatorname{Her}_m(\mathcal{O}_p)] = p^{m(m-1)f_p/2}.$$

Define a mapping

$$\phi : \bigsqcup_{i=1}^{l} \Upsilon_{r+e_p}(T_i) \longrightarrow \mathcal{A}_r(T,T)$$

by $\phi(X) = X \bmod p^r$. For $X \in \mathcal{A}_r(T,T)$ and $Y \in M_m(\mathcal{O}_p)$ we have that

$$T[X + p^r Y] \equiv T[X] \bmod p^r \widetilde{\operatorname{Her}}_m(\mathcal{O}_p).$$

Namely, $X + p^r Y$ belongs to $\Upsilon_{r+e_p}(T_i)$ for some i and therefore ϕ is surjective. Moreover, for $X \in \mathcal{A}_r(T,T)$ we have that $\#(\phi^{-1}(X)) = p^{2m^2 e_p}$. For a sufficiently large integer r we have that $\#\Upsilon_{r+e_p}(T_i) = \#\Upsilon_{r+e_p}(T)$ for any i. Hence

$$\begin{aligned} p^{m(m-1)f_p/2}\#\Upsilon_{r+e_p}(T) &= \sum_{i=1}^{l} \#\Upsilon_{r+e_p}(T_i) \\ &= p^{2m^2 e_p}\#\mathcal{A}_r(T,T) = p^{m^2 e_p}\#\mathcal{A}_{r+e_p}(T,T). \end{aligned}$$

Recall that $e_p = f_p - \delta_{2,p}$. Hence

$$\#\Upsilon_{r+e_p}(T) = p^{m(m+1)f_p/2 - m^2\delta_{2p}}\#\mathcal{A}_{r+e_p}(T,T).$$

This proves the assertion. □

For $T \in \operatorname{Her}_m(K)^+$, let $\mathcal{G}(T)$ denote the set of $\mathrm{SL}_m(\mathcal{O})$-equivalence classes of positive definite Hermitian matrices T' such that T' is $\mathrm{SL}_m(\mathcal{O}_p)$-equivalent to T for any prime number p. Moreover, put

$$M^*(T) = \sum_{T' \in \mathcal{G}(T)} \frac{1}{e^*(T')}$$

for a positive definite Hermitian matrix T of degree m with entries in $\mathcal{O}$.

Let $\mathcal{U}_1$ be the unitary group defined in Section 1. Namely, let

$$\mathcal{U}_1 = \{u \in R_{K/\mathbf{Q}}(\mathrm{GL}_1) \mid \overline{u}u = 1\}.$$

For an element $T \in \mathrm{Her}_m(\mathcal{O}_p)$, let

$$\widetilde{U_{p,T}} = \{\det X \mid X \in \mathcal{U}_T(K_p) \cap \mathrm{GL}_m(\mathcal{O}_p)\},$$

and put $U_{1,p} = \mathcal{U}_1(K_p) \cap \mathcal{O}_p^*$. Then $\widetilde{U_{p,T}}$ is a subgroup of $U_{1,p}$ of finite index. We then put $l_{p,T} = [U_{1,p} : \widetilde{U_{p,T}}]$. We also put

$$u_p = \begin{cases} (1+p^{-1})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified,} \\ (1-p^{-1})^{-1} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p, \\ 2^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

To state the mass formula for $\mathcal{SU}_T$, put $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

PROPOSITION 3.2

Let $T \in \mathrm{Her}_m(\mathcal{O})^+$. *Then*

$$M^*(T) = \frac{(\det T)^m \prod_{i=2}^m D^{i/2}\Gamma_{\mathbf{C}}(i)}{2^{m-1}\prod_p l_{p,T} u_p \upsilon_p(T)}.$$

Proof

The assertion is more or less well known (see [R]). But for the sake of completeness we here give an outline of the proof. Let $\mathcal{SU}_T(\mathbf{A})$ be the adèlization of $\mathcal{SU}_T$, and let $\{x_i\}_{i=1}^H$ be a subset of $\mathcal{SU}_T(\mathbf{A})$ such that

$$\mathcal{SU}_T(\mathbf{A}) = \bigsqcup_{i=1}^H \mathcal{Q} x_i \mathcal{SU}_T(\mathbf{Q}),$$

where $\mathcal{Q} = \mathcal{SU}_T(\mathbf{R})\prod_{p<\infty}(\mathcal{SU}_T(K_p) \cap \mathrm{SL}_m(\mathcal{O}_p))$. We note that the strong approximation theorem holds for SL_m. Hence, by using the standard method we can prove that

$$M^*(T) = \sum_{i=1}^H \frac{1}{\#(x_i^{-1}\mathcal{Q}x_i \cap \mathcal{SU}_T(\mathbf{Q}))}.$$

We recall that the Tamagawa number of $\mathcal{SU}_T$ is 1 (see [W]). Hence, by [R, (1.1) and (4.5)], we have that

$$M^*(T) = \frac{(\det T)^m \prod_{i=2}^m D^{i/2}\Gamma_{\mathbf{C}}(i)}{2^{m-1}\prod_p l_{p,T}} \frac{\upsilon_p(1)}{\upsilon_p(T)}.$$

We can easily show that $\upsilon_p(1) = u_p^{-1}$. This completes the assertion. □

COROLLARY

Let $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+$. *Then*

$$M^*(T) = \frac{2^{c_D m^2} (\det T)^m \prod_{i=2}^m \Gamma_{\mathbf{C}}(i)}{2^{m-1} D^{m(m+1)/4+1/2} \prod_p u_p l_{p,T} \alpha_p(T)},$$

where $c_D = 1$ or 0 according to whether 2 divides D or not.

For a subset $\mathcal{T}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{T}) = \mathrm{Her}_m(\mathcal{O}_p) \cap M_m(\mathcal{T}),$$

and for a subset $\mathcal{S}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{S}, \mathcal{T}) = \{A \in \mathrm{Her}_m(\mathcal{T}) \mid \det A \in \mathcal{S}\}$$

and $\widetilde{\mathrm{Her}}_m(\mathcal{S}, \mathcal{T}) = \mathrm{Her}_m(\mathcal{S}, \mathcal{T}) \cap \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. In particular, if $\mathcal{S}$ consists of a single element d, then we write $\mathrm{Her}_m(\mathcal{S}, \mathcal{T})$ as $\mathrm{Her}_m(d, \mathcal{T})$, and so on. For $d \in \mathbf{Z}_{>0}$ we also define the set $\widetilde{\mathrm{Her}}_m(d, \mathcal{O})^+$ in a similar way. For each $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ put

$$F_p^{(0)}(T, X) = F_p(p^{-e_p} T, X)$$

and

$$\widetilde{F}_p^{(0)}(T, X) = \widetilde{F}_p(p^{-e_p} T, X).$$

We remark that

$$\widetilde{F}_p^{(0)}(T, X) = X^{-\mathrm{ord}_p(\det T)} X^{e_p m - f_p [m/2]} F_p^{(0)}(T, p^{-m} X^2).$$

For $d \in \mathbf{Z}_p^\times$ put

$$\lambda_{m,p}(d, X) = \sum_{A \in \widetilde{\mathrm{Her}}_m(d, \mathcal{O}_p)/\mathrm{SL}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X)}{u_p l_{p,A} \alpha_p(A)}.$$

An explicit formula for $\lambda_{m,p}(p^i d_0, X)$ will be given in the next section for $d_0 \in \mathbf{Z}_p^*$ and $i \geq 0$.

Now let $\widetilde{\mathsf{Her}}_m = \prod_p (\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/\mathrm{SL}_m(\mathcal{O}_p))$. Then the diagonal embedding induces a mapping

$$\phi : \widetilde{\mathrm{Her}}_m(O)^+ / \prod_p \mathrm{SL}_m(\mathcal{O}_p) \longrightarrow \widetilde{\mathsf{Her}}_m.$$

PROPOSITION 3.3

In addition to the above notation and the assumption, for a positive integer d let

$$\widetilde{\mathsf{Her}}_m(d) = \prod_p \big(\widetilde{\mathrm{Her}}_m(d, \mathcal{O}_p)/\mathrm{SL}_m(\mathcal{O}_p)\big).$$

Then the mapping ϕ induces a bijection from $\widetilde{\mathrm{Her}}_m(d, O)^+ / \prod_p \mathrm{SL}_m(\mathcal{O}_p)$ to $\widetilde{\mathsf{Her}}_m(d)$, which will be denoted also by ϕ.

Proof

The proof is similar to that of [IS, Proposition 2.1], but it is a little bit more complex because the class number of K is not necessarily 1. It is easily seen that

ϕ is injective. Let $(x_p) \in \widetilde{\mathsf{Her}}_m(d)$. Then by [Sc, Theorem 6.9], there exists an element y in $\mathrm{Her}_m(K)^+$ such that $\det y \in dN_{K/\mathbf{Q}}(K^\times)$. Then we have that $\det y \in \det x_p N_{K_p/\mathbf{Q}_p}(K_p^\times)$ for any p. Thus by [J, Theorem 3.1] we have $x_p = g_p^* y g_p$ with some $g_p \in \mathrm{GL}_m(K_p)$ for any prime number p. For p not dividing Dd we may suppose that $g_p \in \mathrm{GL}_m(O_p)$. Hence, (g_p) defines an element of $R_{K/\mathbf{Q}}(\mathrm{GL}_m)(\mathbf{A}_f)$. Since we have $d^{-1}\det y \in \mathbf{Q}^\times \cap \prod_p N_{K_p/\mathbf{Q}_p}(K_p)$, we see that $d^{-1}\det y = N_{K/\mathbf{Q}}(u)$ with some $u \in K^\times$. Thus, by replacing y with $\left(\begin{smallmatrix} 1_{m-1} & O \\ O & \overline{u}^{-1}\end{smallmatrix}\right) y \left(\begin{smallmatrix} 1_{m-1} & O \\ O & u^{-1}\end{smallmatrix}\right)$, we may suppose that $\det y = d$. Then we have $N_{K_p/\mathbf{Q}_p}(\det g_p) = 1$. It is easily seen that there exists an element $\delta_p \in \mathrm{GL}_m(K_p)$ such that $\det \delta_p = \det g_p^{-1}$ and $\delta_p^* x_p \delta_p = x_p$. Thus we have $g_p\delta_p \in \mathrm{SL}_m(K_p)$ and

$$x_p = (g_p\delta_p)^* y g_p \delta_p.$$

By the strong approximation theorem for SL_m there exists an element $\gamma \in \mathrm{SL}_m(K), \gamma_\infty \in \mathrm{SL}_m(\mathbf{C})$, and $(\gamma_p) \in \prod_p \mathrm{SL}_m(O_p)$ such that

$$(g_p\delta_p) = \gamma\gamma_\infty(\gamma_p).$$

Put $x = \gamma^* y \gamma$. Then x belongs to $\widetilde{\mathrm{Her}}_m(d,\mathcal{O})^+$, and $\phi(x) = (x_p)$. This proves the surjectivity of ϕ. $\square$

THEOREM 3.4

Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ according to whether $m = 2n$ or $2n+1$. For such an f and a positive integer d_0 put

$$b_m(f;d_0) = \prod_p \lambda_{m,p}(d_0, \alpha_p^{-1}),$$

where α_p is the Satake p-parameter of f. Moreover, put

$$\mu_{m,k,D} = D^{m(s-k+l_0/2)+(k-l_0/2)[m/2]-m(m+1)/4-1/2} \times 2^{-c_D m(s-k-2n-l_0/2)-m+1} \prod_{i=2}^{m} \Gamma_{\mathbf{C}}(i),$$

where $l_0 = 0$ or 1 according to whether m is even or odd. Then for $\mathrm{Re}(s) \gg 0$, we have that

$$L\bigl(s, I_m(f)\bigr) = \mu_{m,k,D} \sum_{d_0=1}^{\infty} b_m(f;d_0) d_0^{-s+k+2n+l_0/2}.$$

Proof

We note that $L(s, I_m(f))$ can be rewritten as

$$L\bigl(s, I_m(f)\bigr) = \widetilde{D}^{ms} \sum_{T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+/\,\mathrm{SL}_m(\mathcal{O})} \frac{a_{I_m(f)}(\widetilde{D}^{-1}T)}{e^*(T)(\det T)^s}.$$

For $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+$ the Fourier coefficient $a_{I_m(f)}(\widetilde{D}^{-1}T)$ of $I_m(f)$ is uniquely determined by the genus to which T belongs, and can be expressed as

$$a_{I_m(f)}(\widetilde{D}^{-1}T) = (D^{[m/2]}\widetilde{D}^{-m}\det T)^{k-l_0/2}\prod_p \widetilde{F}_p^{(0)}(T,\alpha_p^{-1}).$$

Thus the assertion follows from the Corollary to Proposition 3.2 and Proposition 3.3 similarly as in [IS]. □

4. Formal power series associated with local Siegel series

For $d_0 \in \mathbf{Z}_p^\times$ put

$$\hat{P}_{m,p}(d_0,X,t) = \sum_{i=0}^{\infty} \lambda_{m,p}^*(p^i d_0,X)t^i,$$

where for $d \in \mathbf{Z}_p^\times$ we define $\lambda_{m,p}^*(d,X)$ as

$$\lambda_{m,p}^*(d,X) = \sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A,X)}{\alpha_p(A)}.$$

We note that

$$\sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A,X^{-1})}{\alpha_p(A)}$$

is $\chi_{K_p}((-1)^{m/2}d)\lambda_{m,p}^*(d,X)$ or $\lambda_{m,p}^*(d,X)$ according to whether m is even and K_p is a field, or not. In Proposition 4.3.7 we will show that we have

$$\lambda_{m,p}^*(d,X) = u_p\lambda_{m,p}(d,X)$$

for $d \in \mathbf{Z}_p^\times$ and therefore

$$\hat{P}_{m,p}(d_0,X,t) = u_p\sum_{i=0}^{\infty} \lambda_{m,p}(p^i d_0,X)t^i.$$

We also define $P_{m,p}(d_0,X,t)$ as

$$P_{m,p}(d_0,X,t) = \sum_{i=0}^{\infty} \lambda_{m,p}^*(\pi_p^i d_0,X)t^i.$$

We note that $P_{m,p}(d_0,X,t) = \hat{P}_{m,p}(d_0,X,t)$ if K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, but it is not necessarily the case if K_p is ramified over $\mathbf{Q}_p$. In this section, we give explicit formulas of $P_{m,p}(d_0,X,t)$ for all prime numbers p (see Theorems 4.3.1 and 4.3.2) and therefore explicit formulas for $\hat{P}_{m,p}(d_0,X,t)$ (see Theorem 4.3.6).

From now on we fix a prime number p. Throughout this section we simply write ord_p as ord and so on if the prime number p is clear from the context. We also write ν_{K_p} as ν. We also simply write $\widetilde{\mathrm{Her}}_{m,p}$ instead of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, and so on.

4.1. Preliminaries

Let m be a positive integer. For a nonnegative integer $i \le m$ let

$$\mathcal{D}_{m,i} = \mathrm{GL}_m(\mathcal{O}_p)\begin{pmatrix} 1_{m-i} & 0 \\ 0 & \varpi 1_i \end{pmatrix}\mathrm{GL}_m(\mathcal{O}_p),$$

and for $W \in \mathcal{D}_{m,i}$, put $\mathit{\Pi}_p(W) = (-1)^i p^{i(i-1)a/2}$, where $a = 2$ or 1 according to whether K_p is unramified over $\mathbf{Q}_p$ or not. Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for a pair $i = (i_1, i_2)$ of nonnegative integers such that $i_1, i_2 \le m$, let

$$\mathcal{D}_{m,i} = \mathrm{GL}_m(\mathcal{O}_p)\left(\begin{pmatrix} 1_{m-i_1} & 0 \\ 0 & p1_{i_1} \end{pmatrix}, \begin{pmatrix} 1_{m-i_2} & 0 \\ 0 & p1_{i_2} \end{pmatrix}\right)\mathrm{GL}_m(\mathcal{O}_p),$$

and for $W \in \mathcal{D}_{m,i}$ put $\mathit{\Pi}_p(W) = (-1)^{i_1+i_2} p^{i_1(i_1-1)/2 + i_2(i_2-1)/2}$. In either the case where K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, we put $\mathit{\Pi}_p(W) = 0$ for $W \in M_n(\mathcal{O}_p^\times) \setminus \bigcup_{i=0}^m \mathcal{D}_{m,i}$.

First we give the following lemma, which can easily be proved by the usual Newton approximation method in $\mathcal{O}_p$.

LEMMA 4.1.1

Let $A, B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$. Let e be an integer such that $p^e A^{-1} \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Suppose that $A \equiv B \bmod p^{e+1}\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then there exists a matrix $U \in \mathrm{GL}_m(\mathcal{O}_p)$ such that $B = A[U]$.

LEMMA 4.1.2

Let $S \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ and $T \in \widetilde{\mathrm{Her}}_n(\mathcal{O}_p)^\times$ with $m \ge n$. Then

$$\alpha_p(S,T) = \sum_{W \in \mathrm{GL}_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{(n-m)\nu(\det W)}\beta_p\bigl(S, T[W^{-1}]\bigr).$$

Proof

The assertion can be proved by using the same argument as in the proof of [Ki3, Theorem 5.6.1]. We here give an outline of the proof. For each $W \in M_n(\mathcal{O}_p)$, put

$$\mathcal{B}_e(S,T;W) = \bigl\{X \in \mathcal{A}_e(S,T) \bigm| XW^{-1} \text{ is primitive}\bigr\}.$$

Then we have that

$$\mathcal{A}_e(S,T) = \bigsqcup_{W \in \mathrm{GL}_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} \mathcal{B}_e(S,T;W).$$

Take a sufficiently large integer e, and for an element W of $M_n(\mathcal{O}_p)$, let $\{R_i\}_{i=1}^r$ be a complete set of representatives of $p^e\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)[W^{-1}]/p^e\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have $r = p^{\nu(\det W)n}$. Put

$$\begin{aligned}\widetilde{\mathcal{B}}_e(S,T;W) = \bigl\{X \in M_{mn}(\mathcal{O}_p)/p^e M_{mn}(\mathcal{O}_p)W \bigm| \\ S[X] \equiv T \bmod p^e\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) \text{ and } XW^{-1} \text{ is primitive}\bigr\}.\end{aligned}$$

Then

$$\#\bigl(\widetilde{\mathcal{B}}_e(S,T;W)\bigr) = p^{\nu(\det W)m}\#\bigl(\mathcal{B}_e(S,T;W)\bigr).$$

It is easily seen that

$$S[XW^{-1}] \equiv T[W^{-1}] + R_i \bmod p^e\, \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$$

for some i. Hence the mapping $X \mapsto XW^{-1}$ induces a bijection from $\widetilde{\mathcal{B}}_e(S,T;W)$ to $\bigsqcup_{i=1}^r \mathcal{B}_e(S,T[W^{-1}]+R_i)$. Recall that $\nu(W) \le \mathrm{ord}(\det T)$. Hence

$$R_i \equiv O \bmod p^{[e/2]}\, \widetilde{\mathrm{Her}}_m(\mathcal{O}_p),$$

and therefore by Lemma 4.1.1,

$$T[W^{-1}] + R_i = T[W^{-1}][G]$$

for some $G \in \mathrm{GL}_n(\mathcal{O}_p)$. Hence

$$\#\bigl(\widetilde{\mathcal{B}}_e(S,T;W)\bigr) = p^{\nu(\det W)n} \#\bigl(\mathcal{B}_e\bigl(S,T[W^{-1}]\bigr)\bigr).$$

Hence

$$\begin{aligned}\alpha_p(S,T) &= p^{-2mne+n^2e} \#\bigl(\mathcal{A}_e(S,T)\bigr)\\ &= p^{-2mne+n^2e} \sum_{W \in \mathrm{GL}_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{\nu(\det W)(-m+n)} \#\bigl(\mathcal{B}_e\bigl(S,T[W^{-1}]\bigr)\bigr).\end{aligned}$$

This proves the assertion. $\square$

Now by using the same argument as in the proof of [Ki1, Theorem 1], we obtain the following result.

COROLLARY

Under the same notation as above, we have that

$$\beta_p(S,T) = \sum_{W \in \mathrm{GL}_n(\mathcal{O}_p)\backslash M_n(\mathcal{O}_p)^\times} p^{(n-m)\nu(\det W)} \varPi_p(W) \alpha_p\bigl(S,T[W^{-1}]\bigr).$$

For two elements $A, A' \in \mathrm{Her}_m(\mathcal{O}_p)$ we simply write $A \sim_{\mathrm{GL}_m(\mathcal{O}_p)} A'$ as $A \sim A'$ if there is no fear of confusion. For variables U and q put

$$(U,q)_m = \prod_{i-1}^{m} (1 - q^{i-1}U), \qquad \phi_m(q) = (q,q)_m.$$

We note that $\phi_m(q) = \prod_{i=1}^m (1-q^i)$. Moreover, for a prime number p put

$$\phi_{m,p}(q) = \begin{cases} \phi_m(q^2) & \text{if } K_p/\mathbf{Q}_p \text{ is unramified},\\ \phi_m(q)^2 & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p,\\ \phi_m(q) & \text{if } K_p/\mathbf{Q}_p \text{ is ramified}.\end{cases}$$

LEMMA 4.1.3

(a) *Let $\Omega(S,T) = \{w \in M_m(\mathcal{O}_p) \mid S[w] \sim T\}$. Then we have that*

$$\frac{\alpha_p(S,T)}{\alpha_p(T)} = \#\bigl(\Omega(S,T)/\mathrm{GL}_m(\mathcal{O}_p)\bigr) p^{-m(\mathrm{ord}(\det T) - \mathrm{ord}(\det S))}.$$

(b) *Let* $\widetilde{\Omega}(S,T)=\{w\in M_m(\mathbf{Z}) \mid S\sim T[w^{-1}]\}$. *Then we have that*

$$\frac{\alpha_p(S,T)}{\alpha_p(S)}=\#\bigl(\mathrm{GL}_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(S,T)\bigr).$$

Proof

(a) The proof is similar to that of [BS, Lemma 2.2]. First we prove that

$$\int_{\Omega(S,T)}|dx|=\phi_{m,p}(p^{-1})\frac{\alpha_p(S,T)}{\alpha_p(T)},$$

where $|dx|$ is the Haar measure on $M_m(K_p)$ normalized so that

$$\int_{M_m(\mathcal{O}_p)}|dx|=1.$$

To prove this, for a positive integer e let $T_1,\ldots,T_l$ be a complete set of representatives of $\{T[\gamma] \bmod p^e \mid \gamma\in\mathrm{GL}_m(\mathcal{O}_p)\}$. Then it is easy to see that

$$\int_{\Omega(S,T)}|dx|=p^{-2m^2e}\sum_{i=1}^{l}\#\bigl(\mathcal{A}_e(S,T_i)\bigr),$$

and by Lemma 4.1.1, T_i is $\mathrm{GL}_m(\mathcal{O}_p)$-equivalent to T if e is sufficiently large. Hence, we have that

$$\#\bigl(\mathcal{A}_e(S,T_i)\bigr)=\#\bigl(\mathcal{A}_e(S,T)\bigr)$$

for any i. Moreover, we have that

$$l=\#\bigl(\mathrm{GL}_m(\mathcal{O}_p/p^e\mathcal{O}_p)\bigr)/\#\bigl(\mathcal{A}_e(T,T)\bigr)=p^{m^2e}\phi_{m,p}(p^{-1})/\alpha_p(T).$$

Hence

$$\int_{\Omega(S,T)}|dx|=lp^{-2m^2e}\#\bigl(\mathcal{A}_e(S,T)\bigr)=\phi_{m,p}(p^{-1})\frac{\alpha_p(S,T)}{\alpha_p(T)},$$

which proves the above equality. Now we have that

$$\int_{\Omega(S,T)}|dx|=\sum_{W\in\Omega(S,T)/\mathrm{GL}_m(\mathcal{O}_p)}|\det W|_{K_p}^m=\sum_{W\in\Omega(S,T)/\mathrm{GL}_m(\mathcal{O}_p)}|\det W\,\overline{\det \mathrm{W}}\,|_p^m.$$

We remark that $|\det W\,\overline{\det \mathrm{W}}\,|_p=p^{-m(\mathrm{ord}(\det T)-\mathrm{ord}(\det S))}$ for any $W\in\Omega(S,T)/\mathrm{GL}_m(\mathcal{O}_p)$. Thus the assertion has been proved.

(b) By Lemma 4.1.2 we have that

$$\alpha_p(S,T)=\sum_{W\in\mathrm{GL}_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^\times}\beta_p\bigl(S,T[W^{-1}]\bigr).$$

Then we have that $\beta_p(S,T[W^{-1}])=\alpha_p(S)$ or 0 according to whether $S\sim T[W^{-1}]$ or not. Thus the assertion (b) holds. $\square$

For a subset $\mathcal{T}$ of $\mathcal{O}_p$, we put

$$\mathrm{Her}_m(\mathcal{T})_k=\bigl\{A=(a_{ij})\in\mathrm{Her}_m(\mathcal{T})\bigm| a_{ii}\in\pi^k\mathbf{Z}_p\bigr\}.$$

From now on put

$$\mathrm{Her}_{m,*}(\mathcal{O}_p) = \begin{cases} \mathrm{Her}_m(\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p=3, \\ \mathrm{Her}_m(\varpi\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p=2, \\ \mathrm{Her}_m(\mathcal{O}_p) & \text{otherwise,} \end{cases}$$

where ϖ is a prime element of K_p. Moreover, put $i_p = 0$ or 1 according to whether $p=2$ and $f_2 = 2$, or not. Suppose that $K_p/\mathbf{Q}_p$ is unramified or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ can be expressed as $B \sim_{\mathrm{GL}_m(\mathcal{O}_p)} 1_r \bot pB_2$ with some integer r and $B_2 \in \mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. Suppose that $K_p/\mathbf{Q}_p$ is ramified. For an even positive integer r, define Θ_r by

$$\Theta_r = \overbrace{\begin{pmatrix} 0 & \varpi^{i_p} \\ \bar{\varpi}^{i_p} & 0 \end{pmatrix} \bot \cdots \bot \begin{pmatrix} 0 & \varpi^{i_p} \\ \bar{\varpi}^{i_p} & 0 \end{pmatrix}}^{r/2},$$

where $\bar{\varpi}$ is the conjugate of ϖ over $\mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ is expressed as $B \sim_{\mathrm{GL}_m(\mathcal{O}_p)} \Theta_r \bot \pi^{i_p} B_2$ with some even integer r and $B_2 \in \mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. For these results, see [J].

A nondegenerate square matrix $W = (d_{ij})_{m\times m}$ with entries in $\mathcal{O}_p$ is called reduced if W satisfies the following conditions: $d_{ii} = p^{e_i}$ with e_i a nonnegative integer, and d_{ij} is a nonnegative integer less than or equal to $p^{e_j} - 1$ for $i < j$, and $d_{ij} = 0$ for $i > j$. It is well known that we can take the set of all reduced matrices as a complete set of representatives of $\mathrm{GL}_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^{\times}$. Let m be an integer. For $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ put

$$\widetilde{\Omega}(B) = \bigl\{W \in \mathrm{GL}_m(K_p) \cap M_m(\mathcal{O}_p) \bigm| B[W^{-1}] \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)\bigr\}.$$

Let $r \le m$, and let $\psi_{r,m}$ be the mapping from $\mathrm{GL}_r(K_p)$ into $\mathrm{GL}_m(K_p)$ defined by $\psi_{r,m}(W) = 1_{m-r} \bot W$.

LEMMA 4.1.4

(a) *Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $\mathrm{GL}_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(B_1)$ to $\mathrm{GL}_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\bot B_1)$, which will also be denoted by $\psi_{m-n_0,m}$.*

(b) *Assume that K_p is ramified over $\mathbf{Q}_p$ and that n_0 is even. Let $B_1 \in \widetilde{\mathrm{Her}}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $\mathrm{GL}_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(B_1)$ to $\mathrm{GL}_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(\Theta_{n_0}\bot B_1)$, which will also be denoted by $\psi_{m-n_0,m}$. Here i_p is the integer defined above.*

Proof

(a) Clearly $\psi_{m-n_0,m}$ is injective. To prove the surjectivity, take a representative W of an element of $\mathrm{GL}_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\bot B_1)$. Without loss of generality we may assume that W is a reduced matrix. Since we have that $(1_{n_0}\bot B_1)[W^{-1}] \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, we have that $W = \left(\begin{smallmatrix} 1_{n_0} & 0 \\ 0 & W_1 \end{smallmatrix}\right)$ with $W_1 \in \widetilde{\Omega}(B_1)$. This proves the assertion.

(b) The assertion can be proved in the same manner as (a). □

LEMMA 4.1.5
Let $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$. Then we have that

$$\alpha_p(\pi^r dB) = p^{rm^2}\alpha_p(B)$$

for any nonnegative integer r and $d \in \mathbf{Z}_p^*$.

Proof
The assertion can be proved by using the same argument as in the proof of [Ki3, Theorem 5.6.4(a)]. □

Now we prove induction formulas for local densities different from Lemma 4.1.2 (see Lemmas 4.1.6, 4.1.7, and 4.1.8). For technical reasons, we formulate and prove them in terms of Hermitian modules. Let M be $\mathcal{O}_p$ free module, and let b be a mapping from $M \times M$ to K_p such that

$$b(\lambda_1 u + \lambda_2 u_2, v) = \lambda_1 b(u_1, v) + \lambda_2 b(u_2, v)$$

for $u, v \in M$ and $\lambda_1, \lambda_2 \in \mathcal{O}_p$, and

$$b(u,v) = \overline{b(v,u)} \quad \text{for } u, v \in M.$$

We call such an M a Hermitian module with a Hermitian inner product b. We set $q(u) = b(u,u)$ for $u \in M$. Take an $\mathcal{O}_p$-basis $\{u_i\}_{i=1}^m$ of M, and put $T_M = (b(u_i,u_j))_{1\le i,j\le m}$. Then T_M is a Hermitian matrix, and its determinant is uniquely determined, up to $N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$, by M. We say M is nondegenerate if $\det T_M \neq 0$. Conversely for a Hermitian matrix T of degree m, we can define a Hermitian module M_T so that

$$M_T = \mathcal{O}_p u_1 + \mathcal{O}_p u_2 + \cdots + \mathcal{O}_p u_m$$

with $(b(u_i,u_j))_{1\le i,j\le m} = T$. Let M_1 and M_2 be submodules of M. We then write $M = M_1 \perp M_2$ if $M = M_1 + M_2$, and $b(u,v) = 0$ for any $u \in M_1, v \in M_2$. Let M and N be Hermitian modules. Then a homomorphism $\sigma : N \longrightarrow M$ is said to be an isometry if σ is injective and $b(\sigma(u), \sigma(v)) = b(u,v)$ for any $u, v \in N$. In particular, M is said to be isometric to N if σ is an isomorphism. We denote by U'_M the group of isometries of M to M itself. From now on we assume that $T_M \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ for a Hermitian module M of rank m. For Hermitian modules M and N over $\mathcal{O}_p$ of rank m and n, respectively, put

$$\mathcal{A}'_a(N,M) = \bigl\{\sigma : N \longrightarrow M/p^a M \bigm| q\bigl(\sigma(u)\bigr) \equiv q(u) \bmod p^{e_p + a}\bigr\},$$

and

$$\mathcal{B}'_a(N,M) := \bigl\{\sigma \in \mathcal{A}'_a(N,M) \bigm| \sigma \text{ is primitive}\bigr\}.$$

Here a homomorphism $\sigma : N \longrightarrow M$ is said to be primitive if ϕ induces an injective mapping from $N/\varpi N$ to $M/\varpi M$. Then we can define the local density $\alpha'_p(N,M)$ as

$$\alpha'_p(N,M) = \lim_{a\to\infty} p^{-a(2mn-n^2)} \#\bigl(\mathcal{A}'_a(N,M)\bigr)$$

if M and N are nondegenerate, and we can define the primitive local density $\beta'_p(N,M)$ as

$$\beta'_p(N,M) = \lim_{a\to\infty} p^{-a(2mn-n^2)} \#\big(\mathcal{B}'_a(N,M)\big)$$

if M is nondegenerate as in the matrix case. It is easily seen that

$$\alpha_p(S,T) = \alpha'_p(M_T, M_S)$$

and

$$\beta_p(S,T) = \beta'_p(M_T, M_S).$$

Let N_1 be a submodule of N. For each $\phi_1 \in \mathcal{B}'_a(N_1,M)$, put

$$\mathcal{B}'_a(N,M;\phi_1) = \big\{\phi \in \mathcal{B}'_a(N,M) \bigm| \phi|_{N_1} = \phi_1\big\}.$$

We note that we have

$$\mathcal{B}'_a(N,M) = \bigsqcup_{\phi_1 \in \mathcal{B}'_a(N_1,M)} \mathcal{B}'_a(N,M;\phi_1).$$

Suppose that K_p is unramified over $\mathbf{Q}_p$. Then put $\Xi_m = 1_m$. Suppose that K_p is ramified over $\mathbf{Q}_p$ and that m is even. Then put $\Xi_m = \Theta_m$.

LEMMA 4.1.6

Let m_1, m_2, n_1, and n_2 be nonnegative integers such that $m_1 \geq n_1$ and $m_1+m_2 \geq n_1+n_2$. Moreover, suppose that m_1 and n_1 are even if K_p is ramified over $\mathbf{Q}_p$. Let $A_2 \in \widetilde{\mathrm{Her}}_{m_2}(\mathcal{O}_p)$, and let $B_2 \in \widetilde{\mathrm{Her}}_{n_2}(\mathcal{O}_p)$. Then we have that

$$\beta_p(\Xi_{m_1}\perp A_2, \Xi_{n_1}\perp B_2) = \beta_p(\Xi_{m_1}\perp A_2, \Xi_{n_1})\beta_p(\Xi_{m_1-n_1}\perp A_2, B_2),$$

and in particular, we have that

$$\beta_p(\Xi_{n_1}\perp A_2, \Xi_{n_1}\perp B_2) = \beta_p(\Xi_{n_1}\perp A_2, \Xi_{n_1})\beta_p(A_2, B_2).$$

Proof

Let $M = M_{\Xi_{m_1}\perp A_2}$, $N_1 = M_{\Xi_{n_1}}$, $N_2 = M_{B_2}$, and $N = N_1 \perp N_2$. Let a be a sufficiently large positive integer. Let $N_1 = \mathcal{O}_p v_1 \oplus \cdots \oplus \mathcal{O}_p v_{n_1}$ and $N_2 = \mathcal{O}_p v_{n_1+1} \oplus \cdots \oplus \mathcal{O}_p v_{n_1+n_2}$. For each $\phi_1 \in \mathcal{B}'_a(N_1,M)$, put $u_i = \phi_1(v_i)$ for $i = 1,\ldots,n_1$. Then we can take elements $u_{n_1+1},\ldots,u_{m_1+m_2} \in M$ such that

$$(u_i,u_j) = 0 \quad (i = 1,\ldots,n_1, j = n_1+1,\ldots,m_1+m_2),$$

and

$$\big((u_i,u_j)\big)_{n_1+1\leq i,j\leq m_1+m_2} = \Xi_{m_1-n_1}\perp A_2.$$

Put $N'_1 = \mathcal{O}_p u_1 \oplus \cdots \oplus \mathcal{O}_p u_{n_1}$. Then we have $N'_1 = M_{\Xi_{n_1}}$. For $\phi \in \mathcal{B}'_a(N_1,M;\phi_1)$ and $i = 1,\ldots,n_2$ we have that

$$\phi(v_{n_1+i}) = \sum_{j=1}^{m_1+m_2} a_{n_1+i,j} u_j$$

with $a_{n_1+i,j} \in \mathcal{O}_p$. Put $\Xi_{n_1} = (b_{ij})_{1\leq i,j\leq n_1}$. Then we have that

$$\big(\phi(v_j),\phi(v_{n_1+i})\big)=\sum_{\gamma=1}^{n_1}\overline{a_{n_1+i,\gamma}}b_{j\gamma}=0$$

for $i=1,\ldots,n_2$ and $j=1,\ldots,n_1$. Hence we have $a_{n_1+i,\gamma}=0$ for $i=1,\ldots,n_2$ and $\gamma=1,\ldots,n_1$. This implies that $\phi|_{N_2}\in\mathcal{B}'_a(N_2,M_{A_2\perp\Xi_{m_1-n_1}})$. Then the mapping

$$\mathcal{B}'_a(N_1,M;\phi_1)\ni\phi\mapsto\phi|_{N_2}\in\mathcal{B}'_a(N_2,M_{A_2\perp\Xi_{m-n_1}})$$

is bijective. Thus we have that

$$\#\mathcal{B}'_a(N,M)=\#\mathcal{B}'_a(N_1,M)\#\mathcal{B}'_a(N_2,M_{\Xi_{m-n_1}\perp A_2}).$$

This implies that

$$\beta_p(\Xi_{m_1}\perp A_2,\Xi_{n_1}\perp B_2)=\beta_p(\Xi_{m_1}\perp A_2,\Xi_{n_1})\beta_p(\Xi_{m_1-n_1}\perp A,B_2).\qquad\square$$

LEMMA 4.1.7

In addition to the notation and the assumption in Lemma 4.1.6, suppose that A_1 and A_2 are nondegenerate. Then

$$\alpha_p(\Xi_{m_1}\perp A_2,\Xi_{n_1})=\beta_p(\Xi_{m_1}\perp A_2,\Xi_{n_1}),$$

and we have that

$$\alpha_p(\Xi_{m_1}\perp A_2,\Xi_{n_1}\perp B_2)=\alpha_p(\Xi_{m_1}\perp A_2,\Xi_{n_1})\alpha_p(\Xi_{m_1-n_1}\perp A_2,B_2),$$

and in particular, we have that

$$\alpha_p(\Xi_{n_1}\perp A_2,\Xi_{n_1}\perp B_2)=\alpha_p(\Xi_{n_1}\perp A_2,\Xi_{n_1})\alpha_p(A_2,B_2).$$

Proof

The first assertion can easily be proved. By Lemmas 4.1.2 and 4.1.4, we have

$$\begin{aligned}&\alpha_p(\Xi_{m_1}\perp A_2,\Xi_{n_1}\perp B_2)\\&\quad=\sum_{W\in\mathrm{GL}_{n_1+n_2}(\mathcal{O}_p)\backslash\widetilde{\Omega}(\Xi_{n_1}\perp B_2)}p^{(n_1+n_2-(m_1+m_2))\nu(\det W)}\\&\qquad\times\beta_p\big(\Xi_{m_1}\perp A_2,(\Xi_{n_1}\perp B_2)[W^{-1}]\big)\\&\quad=\sum_{X\in\mathrm{GL}_{n_2}(\mathcal{O}_p)\backslash\widetilde{\Omega}(B_2)}p^{(n_2-(m_1-n_1+m_2))\nu(\det X)}\beta_p\big(\Xi_{m_1}\perp A_2,\Xi_{n_1}\perp B_2[X^{-1}]\big).\end{aligned}$$

By Lemma 4.1.6 and the first assertion, we have that

$$\beta_p\big(\Xi_{m_1}\perp A_2,\Xi_{n_1}\perp B_2[X^{-1}]\big)=\alpha_p(\Xi_{m_1}\perp A_2,\Xi_{n_1})\beta_p\big(\Xi_{m_1-n_1}\perp A_2,B_2[X^{-1}]\big).$$

Hence again by Lemma 4.1.2, we prove the second assertion. $\square$

LEMMA 4.1.8

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Let $A\in\mathrm{Her}_l(\mathcal{O}_p)$, $B_1\in\mathrm{Her}_{n_1}(\mathcal{O}_p)$, and $B_2\in\mathrm{Her}_{n_2}(\mathcal{O}_p)$ with $m\geq 2n_1$. Then we have that*

$$\beta_p(1_m\perp A,B_1\perp B_2)=\beta_p(1_m\perp A,B_1)\beta_p\big((-B_1)\perp 1_{m-2n_1}\perp A,B_2\big).$$

(b) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let $A \in \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)$, $B_1 \in \widetilde{\mathrm{Her}}_{n_1}(\mathcal{O}_p)$, and $B_2 \in \widetilde{\mathrm{Her}}_{n_2}(\mathcal{O}_p)$ with $m \geq n_1$. Then we have that*

$$\beta_p(\Theta_{2m}\perp A, B_1\perp B_2) = \beta_p(\Theta_{2m}\perp A, B_1)\beta_p\big((-B_1)\perp\Theta_{2m-2n_1}\perp A, B_2\big).$$

Proof

First suppose that K_p is ramified over $\mathbf{Q}_p$. Let $M = M_{\Theta_{2m}\perp A}$, $N_1 = M_{B_1}$, $N_2 = M_{B_2}$, and $N = N_1\perp N_2$. Let a be a sufficiently large positive integer. Let $N_1 = \mathcal{O}_p v_1 \oplus \cdots \oplus \mathcal{O}_p v_{n_1}$ and $N_2 = \mathcal{O}_p v_{n_1+1} \oplus \cdots \oplus \mathcal{O}_p v_{n_1+n_2}$. For each $\phi_1 \in \mathcal{B}'_a(N_1, M)$, put $u_i = \phi_1(v_i)$ for $i = 1, \ldots, n_1$. Then we can take elements $u_{n_1+1}, \ldots, u_{2m+l} \in M$ such that

$$(u_i, u_{n_1+j}) = \delta_{ij}\varpi^{i_p}, \qquad (u_{n_1+i}, u_{n_1+j}) = 0 \quad (i, j = 1, \ldots, n_1),$$
$$(u_i, u_j) = 0 \quad (i = 1, \ldots, 2n_1, j = 2n_1 + 1, \ldots, 2m + l),$$

and

$$\big((u_i, u_j)\big)_{2n_1+1\leq i,j\leq 2m+l} = \Theta_{2m-2n_1}\perp A,$$

where δ_{ij} is Kronecker's delta. Let $B_1 = (b_{ij})_{1\leq i,j\leq n_1}$, put

$$u'_j = u_j - \bar{\varpi}^{-i_p}\sum_{\gamma=1}^{n_1}\bar{b}_{\gamma j}u_{n_1+\gamma}$$

for $j = 1, \ldots, n_1$, and put $M' = \mathcal{O}_p u'_1 \oplus \cdots \oplus \mathcal{O}_p u'_{n_1}$. Then we have $(u'_i, u'_j) = -b_{ij}$ and hence we have $M' = M_{(-B_1)}$. For $\phi \in \mathcal{B}'_a(N_1, M; \phi_1)$ and $i = 1, \ldots, n_2$ we have that

$$\phi(v_{n_1+i}) = \sum_{j=1}^{2m+l} a_{n_1+i,j}u_j$$

with $a_{n_1+i,j} \in \mathcal{O}_p$. Then we have that

$$\big(\phi(v_j), \phi(v_{n_1+i})\big) = \sum_{\gamma=1}^{n_1}\overline{a_{n_1+i,\gamma}}b_{j\gamma} + \overline{a_{n_1+i,n_1+j}}\varpi^{i_p} = 0$$

for $i = 1, \ldots, n_2$ and $j = 1, \ldots, n_1$. Hence we have that

$$\phi(v_{n_1+i}) = \sum_{j=1}^{n_1}a_{n_1+i,j}u'_j + \sum_{j=2n_1+1}^{2m+l}a_{n_1+i,j}u_j.$$

This implies that $\phi|_{N_2} \in \mathcal{B}'_a(N_2, M_{(-B_1)}\perp M_{A\perp\Theta_{2m-2n_1}})$. Then the mapping

$$\mathcal{B}'_a(N_1, M; \phi_1) \ni \phi \mapsto \phi|_{N_2} \in \mathcal{B}'_a(N_2, M_{(-B_1)}\perp M_{A\perp\Theta_{2m-2n_1}})$$

is bijective. Thus we have that

$$\#\mathcal{B}'_a(N, M) = \#\mathcal{B}'_a(N_1, M)\#\mathcal{B}'_a(N_2, M_{(-B_1)}\perp M_{\Theta_{2m-2n_1}\perp A}).$$

This implies that

$$\beta_p(\Theta_{2m}\perp A, B_1\perp B_2) = \beta_p(\Theta_{2m}\perp A, B_1)\beta_p\big((-B_1)\perp\Theta_{2m-2n_1}\perp A, B_2\big).$$

This proves (b). Next suppose that K_p is unramified over $\mathbf{Q}_p$. For an even positive integer r define Θ_r by

$$\Theta_r = \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{r/2}.$$

Then we have $\Theta_r \sim 1_r$. By using the same argument as above we can prove that

$$\beta_p(\Theta_m \perp A, B_1 \perp B_2) = \beta_p(\Theta_m \perp A, B_1)\beta_p\bigl((-B_1)\perp\Theta_{m-2n_1}\perp A, B_2\bigr)$$

or

$$\beta_p(\Theta_{m-1}\perp 1\perp A, B_1\perp B_2) = \beta_p(\Theta_{m-1}\perp 1\perp A, B_1)\beta_p\bigl((-B_1)\perp\Theta_{m-2n_1}\perp 1\perp A, B_2\bigr)$$

according to whether m is even or not. Thus we prove the assertion (a). $\square$

LEMMA 4.1.9

Let k be a positive integer.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$.*

(1) *Let $b \in \mathbf{Z}_p$. Then we have that*

$$\beta_p(1_{2k}, pb) = (1 - p^{-2k})(1 + p^{-2k+1}).$$

(2) *Let $b \in \mathbf{Z}_p^*$. Then we have that*

$$\alpha_p(1_{2k}, b) = \beta_p(1_{2k}, b) = 1 - p^{-2k}$$

and

$$\alpha_p(1_{2k-1}, b) = \beta_p(1_{2k-1}, b) = 1 + p^{-2k+1}.$$

(b) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*

(1) *Let $B \in \mathrm{Her}_{m,*}(\mathcal{O}_p)$ with $m \le 2$. Then we have that*

$$\beta_p(\Theta_{2k}, \pi^{i_p} B) = \prod_{i=0}^{m-1} (1 - p^{-2k+2i}).$$

(2) *Let $B = \left(\begin{smallmatrix} 0 & \varpi \\ \bar{\varpi} & 0 \end{smallmatrix}\right)$. Then we have that*

$$\alpha_p(\Theta_{2k}, B) = \beta_p(\Theta_{2k}, B) = 1 - p^{-2k}.$$

Proof

(a) Put $B = (b)$. Let $p \neq 2$. Then we have that $K_p = \mathbf{Q}_p(\sqrt{\varepsilon})$ with $\varepsilon \in \mathbf{Z}_p^*$ such that $(\varepsilon, p)_p = -1$. Then we have that

$$\#\mathcal{B}_a(1_{2k}, B) = \#\Bigl\{(x_i) \in M_{4k,1}(\mathbf{Z}_p)/p^a M_{4k,1}(\mathbf{Z}_p) \Bigm| (x_i) \not\equiv 0 \bmod p,$$
$$\sum_{i=1}^{2k} (x_{2i-1}^2 - \varepsilon x_{2i}^2) \equiv pb \bmod p^a\Bigr\}.$$

Let $p=2$. Then we have that $K_2=\mathbf{Q}_2(\sqrt{-3})$ and

$$\#\mathcal{B}_a(1_{2k},B)=\#\Big\{(x_i)\in M_{4k,1}(\mathbf{Z}_2)/2^aM_{4k,1}(\mathbf{Z}_2)\ \Big|\ (x_i)\not\equiv 0 \bmod 2,\ \sum_{i=1}^{2k}(x_{2i-1}^2+x_{2i-1}x_{2i}+x_{2i}^2)\equiv 2b \bmod 2^a\Big\}.$$

In any case, by [Ki2, Lemma 9], we have that

$$\#\mathcal{B}_a(1_{2k},B)=p^{(4k-1)a}(1-p^{-2k})(1+p^{-2k+1}).$$

This proves the assertion (a.1). Similarly the assertion (a.2) holds.

(b) First let $m=1$, and put $B=(b)$ with $b\in 2\mathbf{Z}_p$. Then $2^{-1}b\in\mathbf{Z}_p$. Let $p\neq 2$, or let $p=2$ and $f_2=3$. Then we have $K_p=\mathbf{Q}_p(\varpi)$ with ϖ a prime element of K_p such that $\bar{\varpi}=-\varpi$. Then an element $\mathbf{x}=(x_{2i-1}+\varpi x_{2i})_{1\le i\le 2k}$ of $M_{2k,1}(\mathcal{O}_p)/p^aM_{2k,1}(\mathcal{O}_p)$ is primitive if and only if $(x_{2i-1})_{1\le i\le 2k}\not\equiv 0 \bmod p$. Moreover, we have that

$$\Theta_{2k}[\mathbf{x}]=2\sum_{1\le i\le 2k}(x_{2i}x_{2i+1}-x_{2i-1}x_{2i+2})\pi.$$

Hence we have that

$$\#\mathcal{B}_a(1_{2k},B)=\#\Big\{(x_i)\in M_{4k,1}(\mathbf{Z}_p)/p^aM_{4k,1}(\mathbf{Z}_p)\ \Big|\ (x_{2i-1})_{1\le i\le 2k}\not\equiv 0 \bmod p\ \sum_{i=1}^{2k}(x_{2i}x_{2i+1}-x_{2i-1}x_{2i+2})\equiv 2^{-1}b \bmod p^a\Big\}.$$

Let $p=2$, and let $f_2=2$. Then we have that $K_2=\mathbf{Q}_2(\varpi)$ with ϖ a prime element of K_2 such that $\eta:=2^{-1}(\varpi+\bar{\varpi})\in\mathbf{Z}_2^*$. Then we have that

$$\begin{aligned}&\#\mathcal{B}_a(1_{2k},B)\\&=\#\Big\{(x_i)\in M_{4k,1}(\mathbf{Z}_2)/2^aM_{4k,1}(\mathbf{Z}_2)\ \Big|\ (x_{2i-1})_{1\le i\le 2k}\not\equiv 0 \bmod 2,\\&\sum_{i=1}^{2k}\big\{\eta(x_{2i}x_{2i+1}+x_{2i-1}x_{2i+2})+x_{2i-1}x_{2i+1}+\pi x_{2i}x_{2i+2}\big\}\equiv 2^{-1}b \bmod 2^a\Big\}.\end{aligned}$$

Thus, in any case, by a simple computation we have that

$$\#\mathcal{B}_a(1_{2k},B)=p^{(2k-1)a}(p^{2ka}-p^{2k(a-1)}).$$

Thus the assertion (b.1) has been proved for $m=1$. Next let $\pi^{i_p}B=(b_{ij})_{1\le i,j\le 2}\in \mathrm{Her}_{2,*}(\mathcal{O}_p)$. Let $M=M_{\Theta_{2k}}$, $N_1=M_{\pi^{i_p}b_{11}}$, and $N=M_B$. Let a be a sufficiently large positive integer. For each $\phi_1\in\mathcal{B}_a'(N_1,M)$, put

$$\mathcal{B}_a'(N,M;\phi_1)=\big\{\phi\in\mathcal{B}_a'(N,M)\ \big|\ \phi|_{N_1}=\phi_1\big\}.$$

Let $N=\mathcal{O}_pv_1\oplus\mathcal{O}_pv_2$, and put $u_1=\phi_1(v_1)$. Then we can take elements $u_2,\ldots,$ $u_{2k}\in M$ such that

$$M=\mathcal{O}_pu_1\oplus\mathcal{O}_pu_2\oplus\cdots\oplus\mathcal{O}_pu_{2k}$$

and

$$(u_1,u_2)=\varpi, \qquad (u_2,u_2)=0, \qquad (u_i,u_j)=0 \quad \text{for } i=1,2, j=3,\dots,2k,$$

and

$(u_i,u_j)_{3\le i,j\le 2k}=\Theta_{2k-2}$.

Then by the same argument as in the proof of Lemma 4.1.8, we can prove that

$$\begin{aligned}&\mathcal{B}'_a(N,M;\phi_1)\\ &\quad=\big\{(x_i)_{1\le i\le 2k-1}\in M_{2k-1,1}(\mathcal{O}_p)/p^aM_{2k-1,1}(\mathcal{O}_p)\,\big|\,(x_i)_{2\le i\le 2k-2}\not\equiv 0 \bmod \varpi,\\ &\qquad -x_1\bar{x}_1b_{11}-x_1b_{12}-\bar{x}_1\bar{b}_{12}+\Theta_{2k-2}\big[(x_i)_{2\le i\le 2k-2}\big]\equiv b_{22} \bmod p^a\big\}.\end{aligned}$$

Hence by the assertion for $m=1$, we have that

$$\begin{aligned}&\beta_p(\Theta_{2k},B)\\ &\quad=\beta_p(\Theta_{2k},b_{11})p^{-a}\sum_{x_1\in\mathcal{O}_p/\varpi^a\mathcal{O}_p}\beta_p(\Theta_{2k-2},b_{22}+b_{11}x_1\bar{x}_1+x_1b_{12}+\bar{x}_1\bar{b}_{12})\\ &\quad=(1-p^{-2k})(1-p^{-2k+2}).\end{aligned}$$

Thus the assertion (b.1) has been proved for $m=2$. The assertion (b.2) can be proved by using the same argument as above. □

LEMMA 4.1.10

Let k and m be integers with $k\ge m$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Let $A\in \mathrm{Her}_l(\mathcal{O}_p)$ and $B\in \mathrm{Her}_m(\mathcal{O}_p)$. Then we have that*

$$\beta_p(pA\perp 1_{2k},pB)=\beta_p(1_{2k},pB)=\prod_{i=0}^{2m-1}\big(1-(-1)^ip^{-2k+i}\big).$$

(b) *Let $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Let l be an integer. Let $B\in\mathrm{Her}_m(\mathcal{O}_p)$. Then we have that*

$$\beta_p(1_{2k},pB)=\prod_{i=0}^{2m-1}(1-p^{-2k+i}).$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let $A\in\mathrm{Her}_{l,*}(\mathcal{O}_p)$, and let $B\in \mathrm{Her}_{m,*}(\mathcal{O}_p)$. Then we have that*

$$\beta_p(\pi^{i_p}A\perp\Theta_{2k},\pi^{i_p}B)=\beta_p(\Theta_{2k},\pi^{i_p}B)=\prod_{i=0}^{m-1}(1-p^{-2k+2i}).$$

Proof

(a) Suppose that K_p is unramified over $\mathbf{Q}_p$. We prove the assertion by induction on m. Let $\deg B=1$, and let a be a sufficiently large integer. Then, by

Lemma 4.1.9, we have that

$$\begin{aligned}\beta_p(pA\perp 1_{2k}, pB) &= p^{-al}\sum_{\mathbf{x}\in M_{l1}(\mathcal{O}_p)/p^aM_{l1}(\mathcal{O}_p)}\beta_p\bigl(1_{2k}, pB-pA[\mathbf{x}]\bigr)\\ &=(1-p^{-2k})(1+p^{-2k+1}).\end{aligned}$$

This proves the assertion for $m=1$. Let $m>1$, and suppose that the assertion holds for $m-1$. Then B can be expressed as $B\sim_{\mathrm{GL}_m(\mathcal{O}_p)} B_1\perp B_2$ with $B_1\in \mathrm{Her}_1(\mathcal{O}_p)$ and $B_2\in \mathrm{Her}_{m-1}(\mathcal{O}_p)$. Then by Lemma 4.1.8, we have that

$$\beta_p(pA\perp 1_{2k}, pB_1\perp pB_2)=\beta_p(pA\perp 1_{2k}, pB_1)\beta_p\bigl(pA\perp(-pB_1)\perp 1_{2k-2}, pB_2\bigr).$$

Thus the assertion holds by the induction assumption.

(b) Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then we easily see that

$$\beta_p(1_{2k}, pB)=p^{(-4km+m^2)}\#\mathcal{B}_1(1_{2k}, O_m).$$

We have that

$$\begin{aligned}&\mathcal{B}_1(1_{2k}, O_m)\\ &\quad=\{(X,Y)\in M_{2k,l}(\mathbf{Z}_p)/pM_{2k,l}(\mathbf{Z}_p)\oplus M_{2k,l}(\mathbf{Z}_p)/pM_{2k,l}(\mathbf{Z}_p)\ \bigm|\\ &\qquad {}^tYX\equiv O_m \bmod pM_m(\mathbf{Z}_p) \text{ and } \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}X=\mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}Y=m\}.\end{aligned}$$

For each $X\in M_{2k,l}(\mathbf{Z}_p)/pM_{2k,l}(\mathbf{Z}_p)$ such that $\mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}X=m$, put

$$\begin{aligned}&\#\mathcal{B}_1(1_{2k}, O_m; X)\\ &\quad=\bigl\{Y\in M_{2k,l}(\mathbf{Z}_p)/pM_{2k,l}(\mathbf{Z}_p)\ \bigm|\\ &\qquad {}^tYX\equiv O_m \bmod pM_m(\mathbf{Z}_p) \text{ and } \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}Y=m\bigr\}.\end{aligned}$$

By a simple computation we have that

$$\#\bigl\{X\in M_{2k,l}(\mathbf{Z}_p)/pM_{2k,l}(\mathbf{Z}_p)\bigm| \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}X=m\bigr\}=\prod_{i=0}^{m-1}(p^{2k}-p^i),$$

and

$$\#\mathcal{B}_1(1_{2k}, O_m; X)=\prod_{i=0}^{m-1}(p^{2k-m}-p^i).$$

This proves the assertion.

(c) Suppose that K_p is ramified over $\mathbf{Q}_p$. We prove the assertion by induction on m. Let $\deg B=1$, and let a be a sufficiently large integer. Then, by Lemma 4.1.9, we have that

$$\begin{aligned}\beta_p(\pi^{i_p}A\perp\Theta_{2k}, \pi^{i_p}B) &= p^{-al}\sum_{\mathbf{x}\in M_{l1}(\mathcal{O}_p)/p^aM_{l1}(\mathcal{O}_p)}\beta_p\bigl(\Theta_{2k}, \pi^{i_p}B-\pi^{i_p}A[\mathbf{x}]\bigr)\\ &=1-p^{-2k}.\end{aligned}$$

Let $\deg B=2$. Then by Lemma 4.1.9, we have that

$$\begin{aligned}\beta_p(\pi^{i_p}A\bot\Theta_{2k},\pi^{i_p}B) &= p^{-2la}\sum_{\mathbf{x}\in M_{l2}(\mathcal{O}_p)/p^aM_{l2}(\mathcal{O}_p)}\beta_p\big(\Theta_{2k},\pi^{i_p}B-\pi^{i_p}A[\mathbf{x}]\big)\\ &= (1-p^{-2k})(1-p^{-2k+2}).\end{aligned}$$

Let $m\geq 3$. Then B can be expressed as $B\sim_{\mathrm{GL}_m(\mathcal{O}_p)}B_1\bot B_2$ with $\deg B_1\leq 2$. Then the assertion for m holds by Lemma 4.1.8, the induction hypothesis, and Lemma 4.1.9. □

LEMMA 4.1.11

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Let l and m be integers with $l\geq m$. Then we have that*

$$\alpha_p(1_l,1_m)=\beta_p(1_l,1_m)=\prod_{i=0}^{m-1}\big(1-(-p)^{-l+i}\big).$$

(b) *Let $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Let l and m be integers with $l\geq m$. Then we have*

$$\alpha_p(1_l,1_m)=\beta_p(1_l,1_m)=\prod_{i=0}^{m-1}(1-p^{-l+i}).$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let k and m be even integers with $k\geq m$. Then we have that*

$$\alpha_p(\Theta_{2k},\Theta_{2m})=\beta_p(\Theta_{2k},\Theta_{2m})=\prod_{i=0}^{m-1}(1-p^{-2k+2i}).$$

Proof

In any case, we easily see that the local density coincides with the primitive local density. Suppose that K_p is unramified over $\mathbf{Q}_p$. Then, by Lemma 4.1.7, we have

$$\alpha_p(1_l,1_m)=\alpha_p(1_l,1)\alpha_p(1_{l-1},1_{m-1}).$$

We easily see that

$$\alpha_p(1_l,1)=1-(-1)^lp^{-l}.$$

This proves the assertion (a). Suppose that K_p is ramified over $\mathbf{Q}_p$. Then by Lemma 4.1.7, we have that

$$\alpha_p(\Theta_{2k},\Theta_m)=\alpha_p(\Theta_{2k},\Theta_2)\alpha_p(\Theta_{2k-2},\Theta_{2m-2}).$$

Moreover, by Lemma 4.1.9, we have that

$$\alpha_p(\Theta_{2k},\Theta_2)=1-p^{-2k}.$$

This proves the assertion (c). Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then the assertion can be proved similarly to Lemma 4.1.10(b). □

4.2. Primitive densities

For an element $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, we define a polynomial $G_p(T,X)$ in X by

$$G_p(T,X) = \sum_{i=0}^{m} \sum_{W \in \mathrm{GL}_m(\mathcal{O}_p)\backslash \mathcal{D}_{m,i}} (Xp^m)^{\nu(\det W)} \Pi_p(W) F_p^{(0)}\big(T[W^{-1}],X\big).$$

LEMMA 4.2.1

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then we have that*

$$\alpha_p(1_{n_0} \perp pB_1) = \prod_{i=1}^{n_0}\big(1-(-p)^{-i}\big)\alpha_p(pB_1).$$

(b) *Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $B_1 \in \mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then we have that*

$$\alpha_p(1_{n_0} \perp pB_1) = \prod_{i=1}^{n_0}(1-p^{-i})\alpha_p(pB_1).$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let n_0 be an even integer. Let $B_1 \in \mathrm{Her}_{m-n_0,*}(\mathcal{O}_p)$. Then we have that*

$$\alpha_p(\Theta_{n_0} \perp \pi^{i_p} B_1) = \prod_{i=1}^{n_0/2}(1-p^{-2i})\alpha_p(\pi^{i_p} B_1).$$

Proof

Suppose that K_p is unramified over $\mathbf{Q}_p$. By Lemma 4.1.7, we have that

$$\alpha_p(1_{n_0} \perp pB_1) = \alpha_p(1_{n_0} \perp pB_1, 1_{n_0})\alpha_p(pB_1).$$

By using the same argument as in the proof of Lemma 4.1.10, we can prove that

$$\alpha_p(1_{n_0} \perp pB_1, 1_{n_0}) = \alpha_p(1_{n_0}),$$

and hence by Lemma 4.1.11, we have that

$$\alpha_p(1_{n_0} \perp pB_1) = \prod_{i=1}^{n_0}\big(1-(-p)^{-i}\big)\alpha_p(pB_1).$$

This proves the assertion (a). The assertions (b) and (c) can be proved similarly. $\square$

LEMMA 4.2.2

Let m be a positive integer, and let r be a nonnegative integer such that $r \le m$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Let $T = 1_{m-r} \perp pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then*

$$\beta_p(1_{2k},T) = \prod_{i=0}^{m+r-1}\big(1-p^{-2k+i}(-1)^i\big).$$

(b) *Suppose that* $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. *Let* $T = 1_{m-r} \perp pB_1$ *with* $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. *Then*

$$\beta_p(1_{2k}, T) = \prod_{i=0}^{m+r-1} (1 - p^{-2k+i}).$$

(c) *Suppose that* K_p *is ramified over* $\mathbf{Q}_p$, *and suppose that* $m-r$ *is even. Let* $T = \Theta_{m-r} \perp \pi^{i_p} B_1$ *with* $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. *Then*

$$\beta_p(\Theta_{2k}, T) = \prod_{i=0}^{(m+r-2)/2} (1 - p^{-2k+2i}).$$

Proof
Suppose that K_p is unramified over $\mathbf{Q}_p$. By Lemma 4.1.8, we have that

$$\beta_p(1_{2k}, T) = \beta_p(1_{2k}, pB_1)\beta_p\big((-pB_1) \perp 1_{2k-2r}, 1_{m-r}\big).$$

By using the same argument as in the proof of Lemma 4.1.11, we can prove that $\beta_p((-pB_1) \perp 1_{2k-2r}, 1_{m-r}) = \beta_p(1_{2k-2r}, 1_{m-r})$. Hence the assertion follows from Lemmas 4.1.10 and 4.1.11. The assertions (b) and (c) can be proved similarly. $\square$

COROLLARY

(a) *Suppose that* K_p *is unramified over* $\mathbf{Q}_p$ *or* $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. *Let* $T = 1_{m-r} \perp pB_1$ *with* $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. *Then we have that*

$$G_p(T, Y) = \prod_{i=0}^{r-1} \big(1 - (\xi_p p)^{m+i} Y\big).$$

(b) *Suppose that* K_p *is ramified over* $\mathbf{Q}_p$, *and suppose that* $m-r$ *is even. Let* $T = \Theta_{m-r} \perp \pi^{i_p} B_1$ *with* $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. *Then*

$$G_p(T, Y) = \prod_{i=0}^{[(r-2)/2]} (1 - p^{2i+2[(m+1)/2]} Y).$$

Proof
Let k be a positive integer such that $k \geq m$. Put $\Xi_{2k} = \Theta_{2k}$ or 1_{2k} according to whether K_p is ramified over $\mathbf{Q}_p$ or not. Then it follows from [Sh1, Lemma 14.8] that for $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^{\times}$ we have

$$b_p(p^{-e_p} B, 2k) = \alpha_p(\Xi_{2k}, B).$$

Hence, by the definition of $G_p(T, X)$ and the Corollary to Lemma 4.1.2, we have

$$\beta_p(\Xi_{2k}, T) = G_p(T, p^{-2k}) \prod_{i=0}^{[(m-1)/2]} (1 - p^{2i-2k}) \prod_{i=1}^{[m/2]} (1 - \xi_p p^{2i-1-2k}).$$

Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then by Lemma 4.2.2, we have that

$$G_p(T, p^{-2k}) = \prod_{i=0}^{r-1} \bigl(1 - (\xi_p p)^{m+i} p^{-2k}\bigr).$$

This equality holds for infinitely many positive integers k, and both sides of it are polynomials in p^{-2k}. Thus the assertion (a) holds. Similarly the assertion (b) holds. □

LEMMA 4.2.3

Let $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have that

$$F_p^{(0)}(B, X) = \sum_{W \in \mathrm{GL}_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B)} G_p\bigl(B[W^{-1}], X\bigr)(p^m X)^{\nu(\det W)}.$$

Proof

Let k be a positive integer such that $k \geq m$. By Lemma 4.1.2, we have that

$$\alpha_p(\Xi_{2k}, B) = \sum_{W \in \mathrm{GL}_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B)} \beta_p\bigl(\Xi_{2k}, B[W^{-1}]\bigr) p^{(-2k+m)\nu(\det W)}.$$

Then the assertion can be proved by using the same argument as in the proof of the Corollary to Lemma 4.2.2. □

COROLLARY

Let $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have that

$$\widetilde{F}^{(0)}(B, X) = X^{e_p m - f_p [m/2]} \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/\,\mathrm{GL}_m(\mathcal{O}_p)} X^{-\operatorname{ord}(\det B')} \frac{\alpha_p(B', B)}{\alpha_p(B')}$$
$$\times G_p(B', p^{-m} X^2) X^{\operatorname{ord}(\det B) - \operatorname{ord}(\det B')}.$$

Proof

We have that

$$\begin{aligned}
&\widetilde{F}^{(0)}(B, X) \\
&\quad = X^{e_p m - f_p [m/2]} X^{-\operatorname{ord}(\det B)} F^{(0)}(B, p^{-m} X^2) \\
&\quad = X^{e_p m - f_p [m/2]} \sum_{W \in \mathrm{GL}_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B)} X^{-\operatorname{ord}(\det B)} \\
&\qquad \times G_p\bigl(B[W^{-1}], p^{-m} X^2\bigr)(X^2)^{\nu(\det W)} \\
&\quad = X^{e_p m - f_p [m/2]} \\
&\qquad \times \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/\,\mathrm{GL}_m(\mathcal{O}_p)} \sum_{W \in \mathrm{GL}_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B', B)} X^{-\operatorname{ord}(\det B)} \\
&\qquad \times G_p(B', p^{-m} X^2)(X^2)^{\nu(\det W)}
\end{aligned}$$

$$= X^{e_p m - f_p[m/2]} \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)} X^{-\operatorname{ord}(\det B')} \#\big(\mathrm{GL}_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B',B)\big)$$

$$\times G_p(B', p^{-m}X^2) X^{\operatorname{ord}(\det B) - \operatorname{ord}(\det B')}.$$

Thus the assertion follows from Lemma 4.1.3(b). □

Let

$$\widetilde{\mathcal{F}}_{m,p}(d_0) = \bigcup_{i=0}^{\infty} \widetilde{\mathrm{Her}}_m\big(\pi^i d_0 N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p\big),$$

and let

$$\mathcal{F}_{m,p,*}(d_0) = \widetilde{\mathcal{F}}_{m,p}(d_0) \cap \mathrm{Her}_{m,*}(\mathcal{O}_p).$$

First suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let H_m be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ satisfying the following condition: $H_m(1_{m-r} \perp pB) = H_r(pB)$ for any $B \in \mathrm{Her}_r(\mathcal{O}_p)$.

Let $d_0 \in \mathbf{Z}_p^*$. Then we put

$$Q(d_0, H_m, r, t) = \sum_{B \in p^{-1}\widetilde{\mathcal{F}}_{r,p}(d_0) \cap \mathrm{Her}_r(\mathcal{O}_p)} \frac{H_m(1_{m-r} \perp pB)}{\alpha_p(1_{m-r} \perp pB)} t^{\operatorname{ord}(\det(pB))}.$$

Next suppose that K_p is ramified over $\mathbf{Q}_p$. Let H_m be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ satisfying the following condition:

$$H_m(\Theta_{m-r} \perp \pi^{i_p} B) = H_r(\pi^{i_p} B) \quad \text{for any } B \in \mathrm{Her}_{r,*}(\mathcal{O}_p) \text{ if } m-r \text{ is even.}$$

Let $d_0 \in \mathbf{Z}_p^*$, and let $m-r$ be even. Then we put

$$Q(d_0, H_m, r, t) = \sum_{B \in \pi^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0) \cap \mathrm{Her}_{r,*}(\mathcal{O}_p)} \frac{H_m(\Theta_{m-r} \perp \pi^{i_p} B)}{\alpha_p(\Theta_{m-r} \perp \pi^{i_p} B)} t^{\operatorname{ord}(\det(\pi^{i_p} B))}.$$

Then by Lemma 4.2.1 we easily obtain the following.

PROPOSITION 4.2.4

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a nonnegative integer r we have that*

$$Q(d_0, H_m, r, t) = \frac{Q(d_0, H_r, r, t)}{\phi_{m-r}(\xi_p p^{-1})}.$$

(b) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a nonnegative integer r such that $m-r$ is even, we have that*

$$Q(d_0, H_m, r, t) = \frac{Q(d_0, H_r, r, t)}{\phi_{(m-r)/2}(p^{-2})}.$$

4.3. Explicit formulas of formal power series of Koecher–Maass type

In this section we give an explicit formula for $P_m(d_0, X, t)$.

THEOREM 4.3.1

Let m be even, and let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(-p^{-1})\prod_{i=1}^m(1-t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}.$$

(b) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(p^{-1})\prod_{i=1}^m(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{t^{mi_p/2}}{2\phi_{m/2}(p^{-2})}\Big\{\frac{1}{\prod_{i=1}^{m/2}(1-tp^{-2i+1}X^{-1})(1-tp^{-2i}X)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)}{\prod_{i=1}^{m/2}(1-tp^{-2i}X^{-1})(1-tp^{-2i+1}X)}\Big\}.$$

THEOREM 4.3.2

Let m be odd, and let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(-p^{-1})\prod_{i=1}^m(1+t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}.$$

(b) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{1}{\phi_m(p^{-1})\prod_{i=1}^m(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t) = \frac{t^{(m+1)i_p/2+\delta_{2p}}}{2\phi_{(m-1)/2}(p^{-2})\prod_{i=1}^{(m+1)/2}(1-tp^{-2i+1}X)(1-tp^{-2i+1}X^{-1})}.$$

To prove Theorems 4.3.1 and 4.3.2, put

$$K_m(d_0,X,t) = \sum_{B'\in\widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{G_p(B',p^{-m}X^2)}{\alpha_p(B')}(tX^{-1})^{\mathrm{ord}(\det B')}.$$

PROPOSITION 4.3.3

Let m and d_0 be as above. Then we have that

$$P_m(d_0,X,t) = X^{me_p-[m/2]f_p}K_m(d_0,X,t) \times \begin{cases} \prod_{i=1}^m(1-t^2X^2p^{2i-2-2m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified,} \\ \prod_{i=1}^m(1-tXp^{i-1-m})^{-2} & \text{if } K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p, \\ \prod_{i=1}^m(1-tXp^{i-1-m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

Proof
We note that B' belongs to $\widetilde{\mathrm{Her}}_{m,p}(d_0)$ if B belongs to $\widetilde{\mathrm{Her}}_{m-l,p}(d_0)$ and $\alpha_p(B', B) \neq 0$. Hence by the Corollary to Lemma 4.2.3 we have that

$$\begin{aligned}
&P_m(d_0, X, t)\\
&= X^{me_p-[m/2]f_p} \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B'} \frac{G_p(B', p^{-m}X^2) X^{-\operatorname{ord}(\det B')} \alpha_p(B', B)}{\alpha_p(B')}\\
&\quad \times X^{\operatorname{ord}(\det B)-\operatorname{ord}(\det B')} t^{\operatorname{ord}(\det B)}\\
&= X^{me_p-[m/2]f_p} \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B', p^{-m}X^2)}{\alpha_p(B')} (tX^{-1})^{\operatorname{ord}(\det B')}\\
&\quad \times \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\alpha_p(B', B)}{\alpha_p(B)} (tX)^{\operatorname{ord}(\det B)-\operatorname{ord}(\det B')}.
\end{aligned}$$

Hence by using the same argument as in the proof of [BS, Theorem 5] and by Lemma 4.1.3(a), we have that

$$\begin{aligned}
&\sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\alpha_p(B', B)}{\alpha_p(B)} (tX)^{\operatorname{ord}(\det B)-\operatorname{ord}(\det B')}\\
&= \sum_{W \in M_m(\mathcal{O}_p)^\times / \mathrm{GL}_m(\mathcal{O}_p)} (tXp^{-m})^{\nu(\det W)}\\
&= \begin{cases} \prod_{i=1}^m (1 - t^2X^2p^{2i-2-2m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified,}\\ \prod_{i=1}^m (1 - tXp^{i-1-m})^{-2} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p,\\ \prod_{i=1}^m (1 - tXp^{i-1-m})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}
\end{aligned}$$

Thus the assertion holds. □

In order to prove Theorems 4.3.1 and 4.3.2, we introduce some notation. For a positive integer r and $d_0 \in \mathbf{Z}_p^\times$ let

$$\zeta_m(d_0, t) = \sum_{T \in \mathcal{F}_{m,p,*}(d_0)} \frac{1}{\alpha_p(T)} t^{\operatorname{ord}(\det T)}.$$

We make the convention that $\zeta_0(d_0, t) = 1$ or 0 according to whether $d_0 \in \mathbf{Z}_p^*$ or not. To obtain an explicit formula of $\zeta_m(d_0, t)$ let $Z_m(u, d)$ be the integral defined as

$$Z_{m,*}(u, d) = \int_{\mathcal{F}_{m,p,*}(d_0)} |\det x|^{s-m} |dx|,$$

where $u = p^{-s}$ and $|dx|$ is the measure on $\mathrm{Her}_m(K_p)$ so that the volume of $\mathrm{Her}_m(\mathcal{O}_p)$ is 1. Then by [S, Theorem 4.2] we obtain the following result.

PROPOSITION 4.3.4

Let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that* K_p *is unramified over* $\mathbf{Q}_p$. *Then*

$$Z_{m,*}(u,d_0) = \frac{(p^{-1},p^{-2})_{[(m+1)/2]}(-p^{-2},p^{-2})_{[m/2]}}{\prod_{i=1}^{m}(1-(-1)^{m+i}p^{i-1}u)}.$$

(b) *Suppose that* $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. *Then*

$$Z_{m,*}(u,d_0) = \frac{\phi_m(p^{-1})}{\prod_{i=1}^{m}(1-p^{i-1}u)}.$$

(c) *Suppose that* K_p *is ramified over* $\mathbf{Q}_p$.

(1) *Let* $p \neq 2$. *Then*

$$\begin{aligned} &Z_{m,*}(u,d_0) \\ &\quad = \frac{1}{2}(p^{-1},p^{-2})_{[(m+1)/2]} \\ &\quad \times \begin{cases} \frac{1}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ (\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)}) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

(2) *Let* $p = 2$, *and let* $f_2 = 2$. *Then*

$$\begin{aligned} &Z_{m,*}(u,d_0) \\ &\quad = \frac{1}{2}(p^{-1},p^{-2})_{[(m+1)/2]} \\ &\quad \times \begin{cases} \frac{u^{(m+1)/2}}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ u^{m/2}p^{-m/2}(\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)}) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

(3) *Let* $p = 2$, *and let* $f_2 = 3$. *Then*

$$\begin{aligned} Z_{m,*}(u,d_0) &= \frac{1}{2}(p^{-1},p^{-2})_{[(m+1)/2]} \\ &\quad \times \begin{cases} \frac{u}{\prod_{i=1}^{(m+1)/2}(1-p^{2i-2}u)} & \text{if } m \text{ is odd,} \\ p^{-m}(\frac{1}{\prod_{i=1}^{m/2}(1-p^{2i-1}u)} + \frac{\chi_{K_p}((-1)^{m/2}d_0)p^{-m/2}}{\prod_{i=1}^{m/2}(1-p^{2i-2}u)}) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

Proof

First suppose that K_p is unramified over $\mathbf{Q}_p$, $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, or K_p is ramified over $\mathbf{Q}_p$ and $p \neq 2$. Then $Z_{m,*}(u,d_0)$ coincides with $Z_m(u,d_0)$ in [S, Theorem 4.2]. Hence the assertion follows from (1) and (2) and the former half of [S, Theorem 4.2(3)]. Next suppose that $p = 2$ and $f_2 = 2$. Then $Z_{m,*}(u,d_0)$ is not treated in [S, Theorem 4.2], but we can prove the assertion (c.2) using the same argument as in the proof of the latter half of [S, Theorem 4.2(3)]. Similarly we can prove (c.3) by using the same argument as in the proof of the former half of [S, Theorem 4.2(3)]. $\square$

COROLLARY

Let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\zeta_m(d_0,t) = \frac{1}{\phi_m(-p^{-1})} \frac{1}{\prod_{i=1}^m (1+(-1)^i p^{-i} t)}.$$

(b) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\zeta_m(d_0,t) = \frac{1}{\phi_m(p^{-1})} \frac{1}{\prod_{i=1}^m (1-p^{-i} t)}.$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*

(1) *Let m be even. Then*

$$\zeta_m(d_0,t) = \frac{p^{m(m+1)f_p/2 - m^2\delta_{2,p}} \kappa_p(t)}{2\phi_{m/2}(p^{-2})} \times \left\{ \frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i+1}t)} + \frac{\chi_{K_p}((-1)^{m/2}d_0) p^{-i_p m/2}}{\prod_{i=1}^{m/2}(1-p^{-2i}t)} \right\},$$

where $i_p = 0$ or 1 according to whether $p=2$ and $f_p=2$, or not, and

$$\kappa_p(t) = \begin{cases} 1 & \text{if } p \neq 2, \\ t^{m/2} p^{-m(m+1)/2} & \text{if } p=2 \text{ and } f_2 = 2, \\ p^{-m} & \text{if } p=2 \text{ and } f_2=3. \end{cases}$$

(2) *Let m be odd. Then*

$$\zeta_m(d_0,t) = \frac{p^{m(m+1)f_p/2 - m^2\delta_{2,p}} \kappa_p(t)}{2\phi_{(m-1)/2}(p^{-2})} \frac{1}{\prod_{i=1}^{(m+1)/2}(1-p^{-2i+1}t)},$$

where

$$\kappa_p(t) = \begin{cases} 1 & \text{if } p \neq 2, \\ t^{(m+1)/2} p^{-m(m+1)/2} & \text{if } p=2 \text{ and } f_2 = 2, \\ tp^{-m} & \text{if } p=2 \text{ and } f_2=3. \end{cases}$$

Proof

First suppose that K_p is unramified over $\mathbf{Q}_p$. Then by a simple computation we have

$$\zeta_m(d_0,t) = \frac{Z_{m,*}(p^{-m}t, d_0)}{\phi_m(p^{-2})}.$$

Next suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then similarly to above

$$\zeta_m(d_0,t) = \frac{Z_{m,*}(p^{-m}t, d_0)}{\phi_m(p^{-1})^2}.$$

Finally suppose that K_p is ramified over $\mathbf{Q}_p$. Then by a simple computation and Lemma 3.1

$$\zeta_m(d_0,t) = \frac{p^{m(m+1)f_p/2 - m^2\delta_{2,p}} Z_{m,*}(p^{-m}t, d_0)}{\phi_m(p^{-1})}.$$

Thus the assertions follow from Proposition 4.3.4. □

PROPOSITION 4.3.5

Let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$K_m(d_0,X,t) = \sum_{r=0}^{m} \frac{p^{-r^2}(tX^{-1})^r \prod_{i=0}^{r-1}(1-(-1)^m(-p)^i X^2)}{\phi_{m-r}(-p^{-1})} \zeta_r(d_0, tX^{-1}).$$

(b) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$K_m(d_0,X,t) = \sum_{r=0}^{m} \frac{p^{-r^2}(tX^{-1})^r \prod_{i=0}^{r-1}(1-p^i X^2)}{\phi_{m-r}(p^{-1})} \zeta_r(d_0, tX^{-1}).$$

(c) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$K_m(d_0,X,t) = \sum_{r=0}^{m/2} \frac{p^{-4i_p r^2}(tX^{-1})^{(m/2+r)i_p} \prod_{i=0}^{r-1}(1-p^{2i}X^2)}{\phi_{(m-2r)/2}(p^{-2})} \zeta_{2r}\big((-1)^{m/2-r}d_0, tX^{-1}\big)$$

if m is even, and

$$K_m(d_0,X,t) = \sum_{r=0}^{(m-1)/2} \frac{p^{-(2r+1)^2 i_p}(tX^{-1})^{((m+1)/2+r)i_p} \prod_{i=0}^{r-1}(1-p^{2i+1}X^2)}{\phi_{(m-2r-1)/2}(p^{-2})} \times \zeta_{2r+1}\big((-1)^{(m-2r-1)/2}d_0, tX^{-1}\big)$$

if m is odd.

Proof

The assertions can be proved by using the Corollary to Lemma 4.2.2 and Proposition 4.2.4 (see [IK2, Proposition 3.1]). □

It is well known that $\#(\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)) = 2$ if $K_p/\mathbf{Q}_p$ is ramified. Hence we can take a complete set $\mathcal{N}_p$ of representatives of $\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$ so that $\mathcal{N}_p = \{1, \xi_0\}$ with $\chi_{K_p}(\xi_0) = -1$.

Proof of Theorem 4.3.1

(a) By the Corollary to Proposition 4.3.4 and Proposition 4.3.5, we have that

$$K_m(d_0,X,t)=\frac{1}{\phi_m(-p^{-1})}\frac{L_m(d_0,X,t)}{\prod_{i=1}^m(1+(-1)^ip^{-i}X^{-1}t)},$$

where $L_m(d_0,X,t)$ is a polynomial in t of degree m. Hence

$$P_m(d_0,X,t)=\frac{1}{\phi_m(-p^{-1})}\frac{L_m(d_0,X,t)}{\prod_{i=1}^m(1+(-1)^ip^{-i}X^{-1}t)\prod_{i=1}^m(1-p^{-2i}X^2t^2)}.$$

We have that

$$\widetilde{F}(B,-X^{-1})=\widetilde{F}(B,X)$$

for any $B\in\widetilde{F}_p^{(0)}(B,X)$. Hence we have that

$$P_m(d_0,-X^{-1},t)=P_m(d_0,X,t),$$

and therefore the denominator of the rational function $P_m(d_0,X,t)$ in t is at most

$$\prod_{i=1}^m\bigl(1+(-1)^ip^{-i}X^{-1}t\bigr)\prod_{i=1}^m\bigl(1-(-1)^ip^{-i}Xt\bigr).$$

Thus

$$P_m(d_0,X,t)=\frac{a}{\phi_m(-p^{-1})\prod_{i=1}^m(1+(-1)^ip^{-i}X^{-1}t)\prod_{i=1}^m(1-(-1)^ip^{-i}Xt)},$$

with some constant a. It is easily seen that we have $a=1$. This proves the assertion.

(b) The assertion can be proved by using the same argument as above.

(c) By the Corollary to Proposition 4.3.4 and Proposition 4.3.5, we have that

$$K_m(d,X,t)$$
$$=\frac{1}{2}\Bigl\{\frac{L^{(0)}(X,t)}{\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)}+\frac{\chi_{K_p}((-1)^{m/2}d_0)L^{(1)}(X,t)}{\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)}\Bigr\}$$

with some polynomials $L^{(0)}(X,t)$ and $L^{(1)}(X,t)$ in t of degree at most m. Thus we have

$$P_m(d,X,t)$$
$$=\frac{1}{2}\Bigl\{\frac{L^{(0)}(X,t)}{\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^m(1-p^{-i}Xt)}$$
$$+\frac{\chi_{K_p}((-1)^{m/2}d_0)L^{(1)}(X,t)}{\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)\prod_{i=1}^m(1-p^{-i}Xt)}\Bigr\}.$$

For $l=0,1$ put

$$P_m^{(l)}(X,t)=\frac{1}{2}\sum_{d\in\mathcal{N}_p}\chi_{K_p}\bigl((-1)^{m/2}d\bigr)^lP_m(d,X,t).$$

Then

$$P_m^{(0)}(X,t)=\frac{L^{(0)}(X,t)}{2\phi_{m/2}(p^{-2})}\frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^m(1-p^{-i}Xt)},$$

and

$$P_m^{(1)}(X,t) = \frac{L^{(1)}(X,t)}{2\phi_{m/2}(p^{-2})} \frac{1}{\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)\prod_{i=1}^{m}(1-p^{-i}Xt)}.$$

Then by the functional equation of Siegel series we have that

$$P_m(d,X^{-1},t) = \chi_{K_p}\big((-1)^{m/2}d\big)P_m(d,X,t)$$

for any $d \in \mathcal{N}_p$. Hence we have that

$$P_m^{(0)}(X^{-1},t) = P_m^{(1)}(X,t).$$

Hence the reduced denominator of the rational function $P_m^{(0)}(X,t)$ in t is at most

$$\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i}Xt),$$

and similarly to (a) we have that

$$P_m^{(0)}(X,t) = \frac{1}{2\phi_{m/2}(p^{-2})\prod_{i=1}^{m/2}(1-p^{-2i+1}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i}Xt)}.$$

Similarly

$$P_m^{(1)}(X,t) = \frac{1}{2\phi_{m/2}(p^{-2})\prod_{i=1}^{m/2}(1-p^{-2i}X^{-1}t)\prod_{i=1}^{m/2}(1-p^{-2i+1}Xt)}.$$

We have

$$P_m(d_0,X,t) = P_m^{(0)}(X,t) + \chi_{K_p}\big((-1)^{m/2}d_0\big)P_m^{(1)}(X,t).$$

This proves the assertion. □

Proof of Theorem 4.3.2
The assertion can also be proved by using the same argument as above. □

THEOREM 4.3.6
Let $d_0 \in \mathbf{Z}_p^*$.

(a) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\hat{P}_m(d_0,X,t) = P_m(d_0,X,t)$$

for any $m>0$.

(b) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$\hat{P}_{2n+1}(d_0,X,t) = P_{2n+1}(d_0,X,t)$$

and

$$\hat{P}_{2n}(d_0,X,t) = \frac{1}{2\phi_n(p^{-2})}\Big\{\frac{t^{ni_p}}{\prod_{i=1}^{n}(1-tp^{-2i+1}X^{-1})(1-tp^{-2i}X)} + \frac{\chi_{K_p}((-1)^n d_0)(t\chi_{K_p}(p))^{ni_p}}{\prod_{i=1}^{n}(1-tp^{-2i}\chi_{K_p}(p)X^{-1})(1-tp^{-2i+1}\chi_{K_p}(p)X)}\Big\}.$$

Proof
The assertion (a) is clear from the definition. We note that $P_m(d_0,X,t)$ does not depend on the choice of π. Suppose that K_p is ramified over $\mathbf{Q}_p$. If $m=2n+1$, then it follows from Theorem 4.3.2(c) that

$$\lambda^*_{m,p}(\pi^i d,X)=\lambda^*_{m,p}(\pi^i,X)$$

for any $d\in\mathbf{Z}_p^*$ and, in particular, we have that

$$\lambda^*_{m,p}(p^i d_0,X)=\lambda^*_{m,p}(\pi^i,X).$$

This proves the assertion. Suppose that $m=2n$. Write $\hat{P}_{2n}(d_0,X,t)$ as

$$\hat{P}_{2n}(d_0,X,t)=\hat{P}_{2n}(d_0,X,t)_{\mathrm{even}}+\hat{P}_{2n}(d_0,X,t)_{\mathrm{odd}},$$

where

$$\hat{P}_{2n}(d_0,X,t)_{\mathrm{even}}=\frac{1}{2}\big\{\hat{P}_{2n}(d_0,X,t)+\hat{P}_{2n}(d_0,X,-t)\big\}$$

and

$$\hat{P}_{2n}(d_0,X,t)_{\mathrm{odd}}=\frac{1}{2}\big\{\hat{P}_{2n}(d_0,X,t)-\hat{P}_{2n}(d_0,X,-t)\big\}.$$

We have

$$\hat{P}_{2n}(d_0,X,t)_{\mathrm{even}}=\sum_{i=0}^{\infty}\lambda^*_{2n,p}(p^{2i}d_0,X,Y)t^{2i}=\sum_{i=0}^{\infty}\lambda^*_{2n,p}(\pi^{2i}d_0,X,Y)t^{2i}$$

and

$$\hat{P}_{2n}(d_0,X,t)_{\mathrm{odd}}=\sum_{i=0}^{\infty}\lambda^*_{2n,p}(p^{2i+1}d_0,X)t^{2i+1}=\sum_{i=0}^{\infty}\lambda^*_{2n,p}(\pi^{2i+1}d_0\pi p^{-1},X)t^{2i+1}.$$

Hence we have

$$\hat{P}_{2n}(d_0,X,t)_{\mathrm{even}}=\frac{1}{2}\big\{P_{2n}(d_0,X,t)+P_{2n}(d_0,X,-t)\big\},$$

and

$$\hat{P}_{2n}(d_0,X,Y,t)_{\mathrm{odd}}=\frac{1}{2}\big\{P_{2n}(d_0\pi p^{-1},X,t)-P_{2n}(d_0\pi p^{-1},X,-t)\big\},$$

and hence we have

$$\hat{P}_{2n}(d_0,X,t)=P_{2n}^{(0)}(d_0,X,t)+\frac{1}{2}\big(1+\chi_{K_p}(\pi p^{-1})\big)\chi_{K_p}\big((-1)^n d_0\big)P_{2n}^{(1)}(d_0,X,t)$$
$$+\frac{1}{2}\big(1-\chi_{K_p}(\pi p^{-1})\big)\chi_{K_p}\big((-1)^n d_0\big)P_{2n}^{(1)}(d_0,X,-t).$$

Assume that $\chi_{K_p}(\pi p^{-1})=1$. Then $\chi(d_0\pi p^{-1})=\chi(d_0)$, and we have that

$$\hat{P}_{2n}(d_0,X,t)=P_{2n}(d_0,X,t).$$

Suppose that $\chi_{K_p}(\pi p^{-1})=-1$. Then $\chi(d_0\pi p^{-1})=-\chi(d_0)$, and we have that

$$\hat{P}_{2n}(d_0,X,t)=P_{2n}^{(0)}(d_0,X,t)+\chi_{K_p}\big((-1)^n d_0\big)P_{2n}^{(1)}(d_0,X,-t).$$

Since $\pi\in N_{K_p/\mathbf{Q}_p}(K_p^{\times})$, we have that $\chi_{K_p}(\pi p^{-1})=\chi_{K_p}(p)$. This proves the assertion. $\square$

COROLLARY

Let $m=2n$ be even. Suppose that K_p is ramified over $\mathbf{Q}_p$. For $l=0,1$ put

$$\hat{P}_{2n}^{(l)}(X,t)=\frac{1}{2}\sum_{d\in\mathcal{N}_p}\chi_{K_p}\big((-1)^n d\big)^l\hat{P}_{2n}(d,X,t).$$

Then we have that

$$\hat{P}_{2n}(d,X,t)=\frac{1}{2}\big(\hat{P}_{2n}^{(0)}(X,t)+\chi_{K_p}\big((-1)^n d\big)\hat{P}_{2n}^{(1)}(X,t)\big),$$

$$\hat{P}_{2n}^{(0)}(X,t)=P_{2n}^{(0)}(X,t),$$

and

$$\hat{P}_{2n}^{(1)}(X,t)=P_{2n}^{(1)}\big(X,\chi_{K_p}(p)t\big).$$

The following result will be used to prove Theorems 2.3 and 2.4.

PROPOSITION 4.3.7

Let $d\in\mathbf{Z}_p^\times$. Then we have that

$$\lambda_{m,p}^*(d,X)=u_p\lambda_{m,p}(d,X).$$

Proof

Let I be the left-hand side of the above equation. Let

$$\mathrm{GL}_m(\mathcal{O}_p)_1=\big\{U\in\mathrm{GL}_m(\mathcal{O}_p)\ \big|\ \overline{\det U}\det U=1\big\}.$$

Then we note that there exists a bijection from $\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)_1$ to $\widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*),\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)$. Hence

$$I=\sum_{A\in\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/\mathrm{GL}_m(\mathcal{O}_p)_1}\frac{\widetilde{F}_p^{(0)}(A,X)}{\alpha_p(A)}.$$

Now for $T\in\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)$, let l be the number of $\mathrm{SL}_m(\mathcal{O}_p)$-equivalence classes in $\widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)$ which are $\mathrm{GL}_m(\mathcal{O}_p)$-equivalent to T. Then it can easily be shown that $l=l_{p,T}$. Hence the assertion holds. □

5. Proof of the main theorem

Proof of Theorem 2.3

For a while put $\lambda_p^*(d)=\lambda_{m,p}^*(d,\alpha_p^{-1})$. Then by Theorem 3.4 and Proposition 4.3.7, we have that

$$L\big(s,I_{2n}(f)\big)=\mu_{2n,k,D}\sum_d\prod_p\big(u_p^{-1}\lambda_p^*(d)\big)d^{-s+k+2n}.$$

Then by Theorems 4.3.1(a), 4.3.1(b), and 4.3.6(a), $\lambda_p^*(d)$ depends only on $p^{\mathrm{ord}_p(d)}$ if $p\nmid D$. Hence we write $\lambda_p^*(d)$ as $\widetilde{\lambda}_p(p^{\mathrm{ord}_p(d)})$. On the other hand, if $p\mid D$, then by Theorems 4.3.1(c) and 4.3.6(b), $\lambda_p^*(d)$ can be expressed as

$$\lambda_p^*(d) = \lambda_p^{(0)}(d) + \chi_{K_p}\big((-1)^n dp^{-\operatorname{ord}_p(d)}\big)\lambda_p^{(1)}(d),$$

where $\lambda_p^{(l)}(d)$ is a rational number depending only on $p^{\operatorname{ord}_p(d)}$ for $l=0,1$. Hence we write $\lambda_p^{(l)}(d)$ as $\widetilde{\lambda}_p^{(l)}(p^{\operatorname{ord}_p(d)})$. Then we have that

$$\begin{aligned}
b_m(f;d) = \sum_{Q\subset Q_D} \prod_{p\,|\,d, p\nmid D} &\Big(u_p^{-1}\widetilde{\lambda}_p(p^{\operatorname{ord}_p(d)}) \prod_{q\in Q} \chi_{K_q}(p^{\operatorname{ord}_p(d)})\Big) \\
\times &\prod_{p\,|\,d, p\,|\,D, p\notin Q} \Big(u_p^{-1}\widetilde{\lambda}_p^{(0)}(p^{\operatorname{ord}_p(d)}) \prod_{q\in Q} \chi_{K_q}(p^{\operatorname{ord}_p(d)})\Big) \\
\times &\prod_{p\,|\,d, p\in Q} \Big(u_p^{-1}\widetilde{\lambda}_p^{(1)}(p^{\operatorname{ord}_p(d)}) \prod_{q\in Q, q\neq p} \chi_{K_q}(p^{\operatorname{ord}_p(d)})\Big) \prod_{q\in Q}\chi_{K_q}\big((-1)^n\big)
\end{aligned}$$

for a positive integer d. We note that for a subset Q of Q_D we have that

$$\chi_Q(m) = \prod_{q\in Q}\chi_{K_q}(m)$$

for an integer m coprime to any $q\in Q$, and

$$\chi_Q'(p) = \chi_{K_p}(p) \prod_{q\in Q, q\neq p} \chi_{K_q}(p)$$

for any $p\in Q$. Hence, by Theorems 4.3.1 and 4.3.6 and the Corollary to Theorem 4.3.6, we have that

$$\begin{aligned}
L\big(s, I_{2n}(f)\big) &= \mu_{2n,k,D} \sum_{Q\subset Q_D} \prod_{p\nmid D} \sum_{i=0}^{\infty} u_p^{-1}\widetilde{\lambda}_p(p^i)\chi_Q(p^i)p^{(-s+k+2n)i} \\
&\quad\times \prod_{p\,|\,D, p\notin Q} \sum_{i=0}^{\infty} u_p^{-1}\widetilde{\lambda}_p^{(0)}(p^i)\chi_Q(p^i)p^{(-s+k+2n)i}\chi_Q\big((-1)^n\big) \\
&\quad\times \prod_{p\in Q}\sum_{i=0}^{\infty} u_p^{-1}\widetilde{\lambda}_p^{(1)}(p^i)\Big(\prod_{q\in Q, q\neq p}\chi_{K_q}(p^i)\Big)p^{(-s+k+2n)i} \\
&= \mu_{2n,k,D}\sum_{Q\subset Q_D}\chi_Q\big((-1)^n\big)\prod_{p\nmid D}\big(u_p^{-1}P_{2n,p}\big(1,\alpha_p^{-1},\chi_Q(p)p^{-s+k+2n}\big)\big) \\
&\quad\times \prod_{p\,|\,D, p\notin Q}\big(u_p^{-1}P_{2n,p}^{(0)}\big(\alpha_p^{-1},\chi_Q(p)p^{-s+k+2n}\big)\big) \\
&\quad\times \prod_{p\in Q}\big(u_p^{-1}P_{2n,p}^{(1)}\big(\alpha_p^{-1},\chi_Q'(p)p^{-s+k+2n}\big)\big).
\end{aligned}$$

Now for $l=0,1$ write $P_{2n,p}^{(l)}(X,t)$ as

$$P_{2n,p}^{(l)}(X,t) = t^{ni_p}\widetilde{P}_{2n,p}^{(l)}(X,t),$$

where $i_p=0$ or 1 according to whether $4\mid D$ and $p=2$, or not. Notice that $u_p = (1-\chi(p)p^{-1})^{-1}$ if $p\nmid D$ and $u_p = 2^{-1}$ if $p\mid D$. Hence we have that

$$L\big(s, I_{2n}(f)\big) = \mu_{2n,k,D} \sum_{Q \subset Q_D} \chi_Q\big((-1)^n\big)$$
$$\times \prod_{p \in Q'_D} p^{(-s+k+2n)n} \Big(\prod_{p \in Q_D, p \notin Q} \chi_Q(p) \prod_{p \in Q} \chi'_Q(p) \Big)^n$$
$$\times \prod_{p \nmid D} \big((1 - \chi(p)p^{-1}) P_{2n,p}\big(1, \alpha_p^{-1}, \chi_Q(p) p^{-s+k+2n}\big)\big)$$
$$\times \prod_{p \mid D, p \notin Q} \big(2 \widetilde{P}^{(0)}_{2n,p}\big(\alpha_p^{-1}, \chi_Q(p) p^{-s+k+2n}\big)\big)$$
$$\times \prod_{p \in Q} \big(2 \widetilde{P}^{(1)}_{2n,p}\big(\alpha_p^{-1}, \chi'_Q(p) p^{-s+k+2n}\big)\big),$$

where $Q'_D = Q_D \backslash \{2\}$ or Q_D according to whether $4 \mid D$ or not. Note that

$$2^{2c_D n(-s+k+2n)} \prod_{p \in Q'_D} p^{(-s+k+2n)n} = D^{(-s+k+2n)n},$$

and

$$\prod_{p \in Q_D, p \notin Q} \chi_Q(p) \prod_{p \in Q} \chi'_Q(p) = 1.$$

Thus the assertion follows from Theorem 4.3.1. □

Proof of Theorem 2.4
The assertion follows directly from Theorems 3.4 and 4.3.2. □

Acknowledgments. The author thanks Professor T. Watanabe and Professor R. Schulze-Pillot for giving him many crucial comments on the mass formula for the unitary group. He also thanks Professor Y. Mizuno, Professor T. Ikeda, and A. Yukie for useful discussions.

References

[BS] S. Böcherer and F. Satō, *Rationality of certain formal power series related to local densities*, Comment. Math. Univ. St. Pauli **36** (1987), 53–86. MR 0892380.

[G] V. A. Gritsenko, *The Maass space for* SU(2,2)*. The Hecke ring, and zeta functions* (in Russian), Trudy Mat. Inst. Steklov. **183** (1990), 68–78, 223–225; English translation in Proc. Steklov Inst. Math. **1991**, 75–86. MR 1092016.

[IK1] T. Ibukiyama and H. Katsurada, *An explicit formula for Koecher-Maaß Dirichlet series for the Ikeda lifting*, Abh. Math. Sem. Univ. Hamburg **74** (2004), 101–121. MR 2112824. DOI 10.1007/BF02941528.

[IK2] ———, "Koecher-Maaß series for real analytic Siegel Eisenstein series" in *Automorphic Forms and Zeta Functions*, World Sci., Hackensack, N.J., 2006, 170–197. MR 2208775. DOI 10.1142/9789812774415_0009.

[IS] T. Ibukiyama and H. Saito, *On zeta functions associated to symmetric matrices, I: An explicit form of zeta functions*, Amer. J. Math. **117** (1995), 1097–1155. MR 1350594. DOI 10.2307/2374973.

[I1] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), 641–681. MR 1884618. DOI 10.2307/3062143.

[I2] ———, *On the lifting of hermitian modular forms*, Compos. Math. **144** (2008), 1107–1154. MR 2457521. DOI 10.1112/S0010437X08003643.

[J] R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. **84** (1962), 441–465. MR 0150128.

[Ka1] H. Katsurada, *On the period of the Ikeda lift for* $U(m, m)$, preprint, arXiv:1102.4393v1 [math.NT].

[Ka2] ———, *Koecher–Maass series of the adelic Hermitian Eisenstein series and the adelic Ikeda lift for* $U(m, m)$, Comment. Math. Univ. St. Pauli. **63** (2014), 137–159.

[Ki1] Y. Kitaoka, *A note on local densities of quadratic forms*, Nagoya Math. J. **92** (1983), 145–152. MR 0726146.

[Ki2] ———, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. **95** (1984), 73–84. MR 0759464.

[Ki3] ———, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Math. **106**, Cambridge Univ. Press, Cambridge, 1993. MR 1245266. DOI 10.1017/CBO9780511666155.

[Ko] H. Kojima, *An arithmetic of Hermitian modular forms of degree two*, Invent. Math. **69** (1982), 217–227. MR 0674402. DOI 10.1007/BF01399502.

[Kr] A. Krieg, *The Maaß spaces on the Hermitian half-space of degree 2*, Math. Ann. **289** (1991), 663–681. MR 1103042. DOI 10.1007/BF01446595.

[M] Y. Mizuno, *An explicit formula of the Koecher-Maass series associated with Hermitian modular forms belonging to the Maass space*, Manuscripta Math. **119** (2006), 159–181. MR 2215966. DOI 10.1007/s00229-005-0613-y.

[R] U. Rehmann, *Klassenzahlen einiger totaldefiniter klassischer Gruppen ueber Zahlkoerpern*, Ph.D. dissertation, Georg-August-Universität Göttingen, Göttingen, Germany, 1971.

[S] H. Saito, *Explicit formula of orbital* p*-adic zeta functions associated to symmetric and Hermitian matrices*, Comment. Math. Univ. St. Pauli **46** (1997), 175–216. MR 1471829.

[Sc] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren Math. Wiss. **270**, Springer, Berlin, 1985. MR 0770063. DOI 10.1007/978-3-642-69971-9.

[Sh1] G. Shimura, *Euler Products and Eisenstein Series*, CBMS Reg. Conf. Ser. Math. **93**, Amer. Math. Soc., Providence, 1997. MR 1450866.

[Su] T. Sugano, *Jacobi forms and the theta lifting*, Comment. Math. Univ. St. Pauli **44** (1995), 1–58. MR 1336417.

[W] A. Weil, *Adeles and Algebraic Groups*, with appendices by M. Demazure and T. Ono, Progr. Math. **23**, Birkhäuser, Boston, 1982. MR 0670072.

Muroran Institute of Technology, 27-1 Mizumoto, Muroran, 050-8585, Japan;
hidenori@mmm.muroran-it.ac.jp