MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVII, Mathematics No. 3, 1953.

On the integral closure of an integral domain

By

Yoshiro MORI

(Received August 8, 1952)

Introduction. By an integral domain we mean a commutative ring $\mathfrak{R}$ which satisfies the following condition: $\mathfrak{R}$ satisfies the ascending chain condition and possesses no zero-divisor $\neq 0$. A local ring is a commutative ring $\mathfrak{R}$ with an unit element in which:

(1) The set $\mathfrak{p}_0$ of all non-units is an ideal in $\mathfrak{R}$;

(2) Every ideal in $\mathfrak{R}$ has a finite basis.

A local ring $\mathfrak{R}$ is called a local domain if the ring $\mathfrak{R}$ possesses no zero-divisor.

Let $\mathfrak{R}$ be an integral domain and K be the field of quotients of $\mathfrak{R}$. It is conjectured by Krull [2, p. 108] that the integral closure $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ in K is an "Endliche diskrete Hauptordnung". If $\mathfrak{R}:\overline{\mathfrak{R}}\neq(0)$, $\overline{\mathfrak{R}}$ is a Noetherian ring and also Krull's conjecture is valid [2, p. 105]. Therefore it only remains that his conjecture should be proved in the case where $\mathfrak{R}:\overline{\mathfrak{R}}=(0)$. When $\mathfrak{R}$ is a 1-dimensional local domain, it was already proved by Krull [1]. Hence it is clear that Krull's conjecture is valid provided that an integral domain $\mathfrak{R}$ is "einartig" [2, p. 109]. The purpose of this paper is to prove that Krull's conjecture is valid in the case where $\mathfrak{R}:\overline{\mathfrak{R}}=(0)$ and $\mathfrak{R}$ is not "einartig".

In the first part of this paper we shall prove that Krull's conjecture is valid if the completion $\mathfrak{R}^*$ of a local domain $\mathfrak{R}$ possesses no nilpotent element. The second part is devoted to the proof of Krull's conjecture in the case in which $\mathfrak{R}^*$ has nilpotent elements, and we shall prove that Krull's conjecture is generally valid in an integral domain. In the third part we discuss the sufficient condition that $\mathfrak{R}:\overline{\mathfrak{R}}\neq(0)$ holds for a local domain.

In this paper we denote the completion of a local ring $\mathfrak{R}$ by $\mathfrak{R}^*$ and the integral closure of an integral domain $\mathfrak{S}$ in the field of quotients of $\mathfrak{S}$ by $\overline{\mathfrak{S}}$.

Numbers in brackets refer to the Bibliography at the end of the paper.

Part I

Let $\mathfrak{R}^*$ be the completion of a local domain $\mathfrak{R}$, then we have the following two possibilities:

(1) $\mathfrak{R}^*$ has no nilpotent element;

(2) $\mathfrak{R}^*$ has nilpotent elements.

First we shall prove, in the case (1), that Krull's conjecture is valid. If $\mathfrak{R}^*$ has no nilpotent element,

$$(0)\mathfrak{R}^* = \mathfrak{M}_1^* \cap \mathfrak{M}_2^* \cap \ldots \cap \mathfrak{M}_i^* \cap \ldots \cap \mathfrak{M}_h^* \text{ [5, p. 254].}$$

where $\mathfrak{M}_i$ is the prime ideal which is not imbedded in any other prime ideal of the zero ideal in $\mathfrak{R}^*$. Let $\mathfrak{K}^*$ be the ring of quotients of $\mathfrak{R}^*$, then we have the following Lemmas.

Lemma 1. $\mathfrak{M}_i^*\mathfrak{K}^*$ is a prime ideal in $\mathfrak{K}^*$ and $\mathfrak{M}_i^*\mathfrak{K}^* \cap \mathfrak{R}^* = \mathfrak{M}_i^*$ $(i=1, 2, 3, \ldots, h)$.

Lemma 2. $\bigcap_{i=1}^{h} \mathfrak{M}_i^*\mathfrak{K}^* = (0)\mathfrak{K}^*$

Lemma 3. $\mathfrak{M}_i^*\mathfrak{K}^*$ is a maximal ideal in $\mathfrak{K}^*$ $(i=1, 2, \ldots, h)$.

Lemma 4. $\mathfrak{K}^* = \mathfrak{K}_1^* + \mathfrak{K}_2^* + \ldots + \mathfrak{K}_i^* + \ldots + \mathfrak{K}_h^*$ (direct sum)

where $\mathfrak{K}_i^* \cong \mathfrak{K}^*/\mathfrak{M}_i^*\mathfrak{K}^*$ $(i=1, 2, \ldots, h)$ [6, p. 43].

If we denote the unit element of $\mathfrak{K}_i^*$ by ε_i^*, it is well known that $\varepsilon_i^*\varepsilon_j^* = \begin{cases} 0 & i \neq j \\ \varepsilon_i^* & i = j \end{cases}$ and $\varepsilon_1^* + \varepsilon_2^* + \ldots + \varepsilon_h^* = 1$ [6, p. 43].

Lemma 5. $\mathfrak{R}^*\varepsilon_i^* \cong \mathfrak{R}^*/\mathfrak{M}_i^*$ $(i=1, 2, \ldots, h)$.

Proof. Let a^* any element of $\mathfrak{R}^*$. Then, by Lemma 4, $a^* = \sum_{i=1}^{h} a_i^*$ where $a_i^* \in \mathfrak{K}_i^*$ and $a_i^* = a^*\varepsilon_i^*$. Hence the correspondence $a^* \rightarrow a^*\varepsilon_i^*$ gives the ring homomorphism of $\mathfrak{R}^*$ onto $\mathfrak{R}^*\varepsilon_i^*$. But since $\mathfrak{K}^*\varepsilon_i^* \cong \mathfrak{K}^*/\mathfrak{M}_i\mathfrak{K}^*$ by Lemma 4, $a^* \equiv 0$ (mod. $\mathfrak{M}_i^*$) by Lemma 1 provided that $a^*\varepsilon_i^* = 0$. Hence by the well-known theorem, we have $\mathfrak{R}^*\varepsilon_i^* \cong \mathfrak{R}^*/\mathfrak{M}_i^*$. This completes the proof.

Lemma 6. If we denote the integral closure of $\mathfrak{R}^*$ in the ring of quotients $\mathfrak{K}^*$ of $\mathfrak{R}^*$ by $\overline{\mathfrak{R}}^*$,

$$\overline{\mathfrak{R}}^* = \overline{\mathfrak{R}}_1^* + \overline{\mathfrak{R}}_2^* + \ldots + \overline{\mathfrak{R}}_i^* + \ldots + \overline{\mathfrak{R}}_h \text{ (direct sum)}$$

where $\qquad \overline{\mathfrak{R}}_i^* = \overline{\mathfrak{R}}^*\varepsilon_i^*$ $(i=1, 2, \ldots, h)$.

Proposition 1. If we put $\mathfrak{R}^*/\mathfrak{M}_i^* = \Omega_i^*$ and denote the integral closure of Ω_i^* in the field of quotients of Ω_i^* by $\bar{\Omega}_i^*$, then $\bar{\Omega}_i^* \cong \overline{\mathfrak{R}}_i^*$ $(i=1, 2, \ldots, h)$.

Proof. If we put $\mathfrak{M}_i^*\mathfrak{K}^* \cap \overline{\mathfrak{R}}^* = \bar{M}_i^*$, it follows that $\bar{M}_i^*$ is the prime ideal of $\bar{\mathfrak{R}}^*$. Similarly to the proof of Lemma 5, we have $\bar{\mathfrak{R}}^*\varepsilon_i^* \simeq \bar{\mathfrak{R}}^*/\bar{M}_i^*$. Hence $\bar{\mathfrak{R}}_i^* \simeq \bar{\mathfrak{R}}^*/\bar{M}_i^*$.

We shall now prove that $\bar{\mathfrak{R}}^*/\bar{M}_i^* = \bar{\Omega}_i^*$. First we prove that $\bar{\mathfrak{R}}^*/\bar{M}_i^* \subseteq \bar{\Omega}_i^*$. For, let $W=a/\pi$ be an element of $\bar{\mathfrak{R}}^*$ where a and $\pi \in \mathfrak{R}^*$ and π is a non-zero-divisor, then

$$W^n + c_1 W^{n-1} + \dots + c_i W^{n-i} + \dots + c_{n-1} W + c_n = 0, \text{ where } c_i \in \mathfrak{R}^*.$$

Let $\tilde{W}$, $\tilde{c}_i$ be the residue classes of W, c_i modulo $\bar{M}_i^*$, then

$$\tilde{W}^n + \tilde{c}_1 \tilde{W}^{n-1} + \dots + \tilde{c}_i \tilde{W}^{n-i} + \dots + \tilde{c}_{n-1} \tilde{W} + \tilde{c}_n = 0, \text{ where } \tilde{c}_i \in \Omega_i^*.$$

On the other hand, $\pi W = a$ in $\bar{\mathfrak{R}}^*$. Hence $\tilde{\pi}\tilde{W} = \tilde{a}$, where $\tilde{\pi}$, $\tilde{a} \in \Omega_i^*$. Therefore $\tilde{W} \in \bar{\Omega}_i^*$. This implies that $\bar{\mathfrak{R}}^*/\bar{M}_i \subseteq \bar{\Omega}_i^*$.

We now prove that $\bar{\mathfrak{R}}^*/\bar{M}_i^* \supseteq \bar{\Omega}_i^*$. In fact, let $\tilde{b}/\tilde{a}$ be an element of $\bar{\Omega}_i^*$, where $\tilde{a}$, $\tilde{b} \in \Omega_i^*$, then $\tilde{n}(\tilde{b}/\tilde{a})^e \in \Omega_i^*$ $(e=1, 2, 3, \dots)$ where $\tilde{n}$ is a certain element $\neq 0$ of Ω_i^*. The above argument implies that $\tilde{n}(\tilde{b})^e = (\tilde{a})^e \tilde{r}_e$, where $\tilde{r}_e \in \Omega_i^*$. Let n_i, a_i, b_i and r_{ie} be elements of $\mathfrak{R}^*$ whose residue classes modulo $\mathfrak{M}_i^*$ are $\tilde{n}$, $\tilde{a}$, $\tilde{b}$ and $\tilde{r}_e$ respectively, then $n_i b_i^e \equiv a_i^e r_{ie}$ (mod. $\mathfrak{M}_i^*$). Let $\lambda_i \not\equiv 0$ $(\mathfrak{M}_i)$ and $\lambda_i \equiv 0$ $(\mathfrak{M}_1^* \cap \mathfrak{M}_2^* \cap \dots \cap \mathfrak{M}_{i-1}^* \cap \mathfrak{M}_{i+1}^* \cap \dots \cap \mathfrak{M}_h^*)$ $(i=1, 2, \dots, h)$. Putting $n = \sum_{i=1}^{h} \lambda_i n_i$, $a = \sum_{i=1}^{h} \lambda_i a_i$, $b = \sum_{i=1}^{h} \lambda_i b_i$ and $r_e = \sum_{i=1}^{h} \lambda_i r_{ie}$, then $nb^e - a^e r^e = 0$. For $nb^e - a^e r_e \equiv \lambda_j n_j (\lambda_j b_j)^e - (\lambda_j a_i)^e \lambda_j r_{je}$ $(\mathfrak{M}_j^*)$, hence $nb^e - a^e r_e \equiv \lambda_j^{e+1} (n_j b_j^e - a_j^e r_{je})$ $(\mathfrak{M}_j^*)$ $(j=1, 2, \dots, h)$. This implies that $nb^e - a^e r_e \equiv 0$ $(\mathfrak{M}_j^*)$ $(j=1, 2, \dots, h)$. Hence $nb^e - a^e r_e = 0$. But since a is a non-zero-divisor in $\mathfrak{R}^*$, we have $n(b/a)^e = r_e$. Hence $b/a \in \bar{\mathfrak{R}}^*$. If we put $b/a = W$, we have $Wa = b$. Hence $\tilde{W}\tilde{a} = \tilde{b}$, where $\tilde{W}$ is the residue class modulo $\bar{M}_i^*$. This implies that $\bar{\Omega}^*_i \subseteq \bar{\mathfrak{R}}^*/\bar{M}_i^*$. Thus the proof is completed.

Colollary. $\bar{\mathfrak{R}}^* \simeq \bar{\Omega}_1^* + \bar{\Omega}_2^* + \dots + \bar{\Omega}_i^* + \dots + \bar{\Omega}_h^*$.

Proposition 2. $\bar{\Omega}_i^*$ is an "Endliche diskrete Hauptordnung".

Proof. Since Ω_i^* is a complete local domain, if $x_1, x_2, \dots, x_m$ be the system of parameters for Ω_i^* [3] and R be the coefficient ring in Ω_i^*, then $\Omega_0 = R\{x_1, x_2, \dots, x_m\}$ is a p-adic ring and Ω_i^* is a finite Ω_0-module [4, Lemma 15, 16]. Hence $\bar{\Omega}_i^*$ is an "Endliche diskrete Hauptordnung" [2, p. 133]. This completes the proof.

Proposition 3. Let $\mathfrak{R}$ be a local domain and $\bar{\mathfrak{R}}$ be the integral closure of $\mathfrak{R}$ in the field of quotients of $\mathfrak{R}$. If no nilpotent element exists in the completion $\mathfrak{R}^*$ of $\mathfrak{R}$, then $\bar{\mathfrak{R}}$ is an "Endliche diskrete Hauptordnung".

Proof. Let $\alpha \in \bar{\mathfrak{R}}$, then, since $\bar{\mathfrak{R}}^*\varepsilon_i^* \cong \bar{\mathfrak{L}}_i^*$ by prop. 1, $\alpha\bar{\mathfrak{R}}^*\varepsilon_i^*$ is an intersection of symbolic powers of associated minimal prime ideals in $\bar{\mathfrak{R}}^*\varepsilon_i^*$ by prop. 2. Now, let $\alpha\bar{\mathfrak{R}}^*\varepsilon_i = \bigcap_{j=1}^{l_i} \bar{\mathfrak{q}}_{ij}^*$ be an irredundant primary decomposition of $\alpha\bar{\mathfrak{R}}^*\varepsilon_i^*$ in $\bar{\mathfrak{R}}^*\varepsilon_i^*$. If we put

$$\bar{Q}_{ij}^* = \bar{\mathfrak{R}}_1^* + \bar{\mathfrak{R}}_2^* + \dots + \bar{\mathfrak{R}}_{i-1}^* + \bar{\mathfrak{q}}_{ij}^* + \bar{\mathfrak{R}}_{i+1}^* + \dots + \bar{\mathfrak{R}}_h^*,$$

then $\bar{Q}_{ij}^*$ is a primary ideal in $\bar{\mathfrak{R}}^*$ by the well-known theorem. Hence $\alpha\bar{\mathfrak{R}}^* = \bigcap_{i,j} \bar{Q}_{ij}^*$. In fact, $\bigcap_{i,j} \bar{Q}_{ij}^* = \bigcap_i (\bigcap_j \bar{Q}_{ij}^*) = \bigcap_i (\bar{\mathfrak{R}}_1^* + \bar{\mathfrak{R}}_2^* + \dots + \bar{\mathfrak{R}}_{i-1}^* + \alpha\bar{\mathfrak{R}}_i^* + \bar{\mathfrak{R}}_{i+1}^* + \dots + \bar{\mathfrak{R}}_h^*) = \alpha\bar{\mathfrak{R}}_1^* + \alpha\bar{\mathfrak{R}}_2^* + \dots + \alpha\bar{\mathfrak{R}}_i^* + \dots + \alpha\bar{\mathfrak{R}}_h^* = \alpha\bar{\mathfrak{R}}^*$. But we see that $\bar{Q}_{ij}^*$ is a symbolic power of prime ideal of $\bar{\mathfrak{R}}^*$. For since it is clear that $\bar{Q}_{ij}^*$ is a primary ideal in $\bar{\mathfrak{R}}^*$, if the associated prime ideal of $\bar{Q}_{ij}^*$ is denoted by $\bar{P}_{ij}^*$, then $\bar{P}_{ij}^*$ is a set of nilpotent elements of $\bar{\mathfrak{R}}^*$ with respect to $\bar{Q}_{ij}^*$. Hence $\bar{P}_{ij}^* = \bar{\mathfrak{R}}_1^* + \bar{\mathfrak{R}}_2^* + \dots + \bar{\mathfrak{p}}_{ij}^* + \dots + \bar{\mathfrak{R}}_h^*$. where $\bar{\mathfrak{p}}_{ij}^*$ is a prime ideal of $\bar{\mathfrak{R}}_i^*$ belonging to the primary ideal $\bar{\mathfrak{q}}_{ij}^*$. Since $\bar{\mathfrak{q}}_{ij}^* = \bar{\mathfrak{p}}_{ij}^{*(e)}$ by Prop. 2, $\bar{Q}_{ij}^* = \bar{P}_{ij}^{*(e)}$. If we put $\bar{Q}_{ij}^* \cap \bar{\mathfrak{R}} = \bar{\mathfrak{q}}_{ij}$, then $\bar{\mathfrak{q}}_{ij}$ is a primary ideal of $\bar{\mathfrak{R}}$ and the prime ideal $\bar{\mathfrak{p}}_{ij}$ belonging to $\bar{\mathfrak{q}}_{ij}$ is a minimal prime ideal in $\bar{\mathfrak{R}}$, and $\alpha\bar{\mathfrak{R}} = \cap \bar{\mathfrak{q}}_{ij}$. For, putting $\alpha A = \beta$, where $A \in \bar{\mathfrak{R}}^*$ and $\beta \in \bar{\mathfrak{R}}$, then $A \in K$ (field of quotients of $\mathfrak{R}$). But since $\bar{\mathfrak{R}}^* \cap K = \bar{\mathfrak{R}}$, $A \in \bar{\mathfrak{R}}$. Hence $\alpha\bar{\mathfrak{R}}^* \cap \bar{\mathfrak{R}} = \alpha\bar{\mathfrak{R}}$ and also $\alpha\bar{\mathfrak{R}} = \cap \bar{\mathfrak{q}}_{ij}$. It is clear that $\bar{\mathfrak{q}}_{ij}$ is a primary ideal belonging to the prime ideal $\bar{P}_{ij}^* \cap \bar{\mathfrak{R}} = \bar{\mathfrak{p}}_{ij}$. Hence $\bar{\mathfrak{p}}_{ij}$ is a prime ideal belonging to $\alpha\bar{\mathfrak{R}}$. If we assume that $\alpha\bar{\mathfrak{R}} = \cap \bar{\mathfrak{q}}_{ij}$ is an irredundant intersection of ideals $\bar{\mathfrak{q}}_{ij}$, we have $(\bar{\mathfrak{p}}_{ij})^{-1} \supset \bar{\mathfrak{R}}$. Hence $\bar{\mathfrak{p}}_{ij}$ is a minimal prime ideal in $\bar{\mathfrak{R}}$. For, if we assume that $\bar{\mathfrak{p}}_{ij}$ is not a minimal prime ideal of $\bar{\mathfrak{R}}$, then $(\bar{\mathfrak{p}}_{ij})^{-1}(\bar{\mathfrak{p}}_{ij}) = \bar{\mathfrak{p}}_{ij}$. Hence, if $x \in (\bar{\mathfrak{p}}_{ij})^{-1}$ and $x \notin \bar{\mathfrak{R}}$, we obtain $x\bar{\mathfrak{p}}_{ij} \equiv 0$ $(\bar{\mathfrak{p}}_{ij})$ and also $x^N\bar{\mathfrak{p}}_{ij} \equiv 0$ $(\bar{\mathfrak{p}}_{ij})$ $(N=1, 2, \dots)$. Hence there is an element $\bar{\rho} (\in \bar{\mathfrak{R}})$ such that $\bar{\rho}x^N \equiv 0$ $(\bar{\mathfrak{R}})$ $(N=1, 2, 3, \dots)$. But since $x \in \mathfrak{K}^*$, it follows that $x = \sum_{i=1}^h x_i$ and $\bar{\rho} = \sum_{i=1}^h \bar{\rho}_i$ by Lemma 4 and Lemma 6, where $x_i \in \mathfrak{K}_i^*$, $\bar{\rho}_i \in \bar{\mathfrak{R}}_i^*$. Hence $(\sum_{i=1}^h \bar{\rho}_i)(\sum_{i=1}^h x_i^N) \equiv 0 (\bar{\mathfrak{R}}^*)$. Therefore $\bar{\rho}_i x_i^N \equiv 0 (\bar{\mathfrak{R}}_i^*)$ $(N=1, 2, 3, \dots)$. But since $\bar{\mathfrak{R}}_i^*$ is an "Endliche diskrete Hauptordnung" by Prop. 1, we have $x_i \in \bar{\mathfrak{R}}_i^*$ $(i=1, 2, \dots, h)$. Therefore $x \in \bar{\mathfrak{R}}^*$ and whence $x \in \bar{\mathfrak{R}}$. This is a contradiction. Therefor $\bar{\mathfrak{p}}_{ij}$ is a minimal prime ideal of $\bar{\mathfrak{R}}$. But since $\bar{\mathfrak{q}}_{ij}$ is a primary component belonging to $\bar{\mathfrak{p}}_{ij}$,

$\bar{\mathfrak{q}}_{ij}$ is a symbolic power of $\bar{\mathfrak{p}}_{ij}$. Hence $\bar{\mathfrak{R}}$ is an "Endliche diskrete Hauptordnung" [2, p. 104]. This completes the proof.

Part II

We shall prove the validity of Krull's conjecture in the case where $\mathfrak{R}^*$ has nilpotent elements. If the radical of $\mathfrak{R}^*$ is denoted by $\mathfrak{l}^*$, it is clear that $\mathfrak{l}^*\mathfrak{K}^*$ is the radical of $\mathfrak{K}^*$ and the radical of $\bar{\mathfrak{R}}^*$ too. For, let $\bar{\mathfrak{l}}^*$ be the radical of $\bar{\mathfrak{R}}^*$, then $\mathfrak{l}^*\mathfrak{K}^* \subseteq \bar{\mathfrak{l}}^*$, since any element of $\mathfrak{l}^*\mathfrak{K}^*$ is integrally dependent on $\mathfrak{R}^*$. But being $\bar{\mathfrak{l}}^*\mathfrak{K}^* \subseteq \mathfrak{l}^*\mathfrak{K}^*$, it follows that $\mathfrak{l}^*\mathfrak{K}^* = \bar{\mathfrak{l}}^*$. Now, let $\bar{l}$ be any nilpotent of $\bar{\mathfrak{R}}^*$ and let α be a non-zero-divisor of $\bar{\mathfrak{R}}^*$, $\bar{l}/\alpha$ is a nilpotent element of $\bar{\mathfrak{R}}^*$. Hence $\bar{l} \in \alpha\bar{\mathfrak{R}}^*$. Therefore, if an ideal $\bar{\mathfrak{A}}^*$ of $\bar{\mathfrak{R}}^*$ has a non-zero-divisor, we have $\bar{\mathfrak{A}}^* \supseteq \bar{\mathfrak{l}}^*$. Therefore there is a 1–1 correspondence such that $\bar{\mathfrak{A}}^*/\mathfrak{l}^* \simeq \bar{\bar{\mathfrak{A}}}^*$ between the ideal $\bar{\bar{\mathfrak{A}}}^*$ of $\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^*$ and the ideal $\bar{\mathfrak{A}}^* \supset \bar{\mathfrak{l}}^*$ of $\bar{\mathfrak{R}}^*$. Putting $\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^* = \bar{\bar{\mathfrak{o}}}^*$, the ring of quotients of $\bar{\bar{\mathfrak{o}}}^*$ is $\mathfrak{K}^*/\mathfrak{l}^*\mathfrak{K}^*$. For, $\bar{\mathfrak{R}}_S/\bar{\mathfrak{l}}^*\bar{\mathfrak{R}}_S \cong (\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^*)_{S/\bar{\mathfrak{l}}^*}$, where S is the set of all non-zero-divisors in $\bar{\mathfrak{R}}^*$ [2, p. 20]. If we set $\mathfrak{R}^*/\mathfrak{l}^* = \mathfrak{o}^*$, since $\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^* \supset \mathfrak{R}^*/\mathfrak{l}^*$, we have that $\bar{\bar{\mathfrak{o}}}^* \supset \mathfrak{o}^*$. But since $(\mathfrak{R}^*/\mathfrak{l}^*)_{S/\mathfrak{l}^*} \cong \mathfrak{R}_S^*/\mathfrak{l}^*\mathfrak{R}_S^* = \mathfrak{K}^*/\mathfrak{l}^*\mathfrak{K}^*$, where S is the set of all non-zero-divisors in $\mathfrak{R}^*$ [2, p. 20], $\mathfrak{o}^* \subset \bar{\bar{\mathfrak{o}}}^* \subset \mathfrak{K}^*/\mathfrak{l}^*\mathfrak{K}$. Now, let $\bar{\mathfrak{o}}^*$ be the integral closure of $\mathfrak{o}^*$ in the ring of quotients of $\mathfrak{o}^*$, then any element $\tilde{A}$ of $\bar{\mathfrak{o}}^*$ is expressible as $\tilde{l}/\tilde{\pi}$ where $\tilde{l}$, $\tilde{\pi}$ are elements of $\mathfrak{o}^*$ and $\tilde{\pi}$ is a non-zero-divisor of $\mathfrak{o}^*$. Hence

$$(\tilde{l}/\tilde{\pi})^m + \tilde{c}_1(\tilde{l}/\tilde{\pi})^{m-1} + \dots + \tilde{c}_{m-1}(\tilde{l}/\tilde{\pi}) + \tilde{c}_m = 0 \text{ where } \tilde{c}_i \in \mathfrak{o}^*.$$

Let c_i, l, π be respectively representatives in $\mathfrak{R}^*$ of the residue classes $\tilde{c}_i$, $\tilde{l}$, $\tilde{\pi}$, then $l^m + c_1 l^{m-1}\pi + c_2 l^{m-2}\pi^2 + \dots + c_{m-1} l\pi^{m-1} + c_m\pi^m \equiv 0$ $(\mathfrak{l}^*)$. Hence $(l/\pi)^m + c_1(l/\pi) + \dots + c_{m-1}(l/\pi) + c_m \equiv 0(\mathfrak{l}^*\mathfrak{K}^*)$. But $\mathfrak{l}^*\mathfrak{K}^*$ being the radical of $\bar{\mathfrak{R}}^*$, it follows that l/π is integrally dependent on $\mathfrak{R}^*$. If we put $l/\pi = A$, we have $l = \pi A$. Hence we obtain $\tilde{l} = \tilde{\pi}\tilde{A}$ in $\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^*$. Therefore $\bar{\mathfrak{o}}^* \subseteq \bar{\bar{\mathfrak{o}}}^*$. Since $\bar{\bar{\mathfrak{o}}}^* \subseteq \bar{\mathfrak{o}}^*$, it follows that $\bar{\mathfrak{o}}^* = \bar{\bar{\mathfrak{o}}}^*$.

If α is an element $\bar{\mathfrak{R}}$, α is a non-zero-divisor in $\bar{\mathfrak{R}}^*$. Hence $\alpha\bar{\mathfrak{R}}^*$ can be expressed as an intersection of finite primary ideals containing the radical $\bar{\mathfrak{l}}^*$ of $\bar{\mathfrak{R}}^*$ by Prop. 3. If $\alpha\bar{\mathfrak{R}}^* = \bigcap_{ij} \bar{Q}_{ij}$ is an irredundant intersection of primary ideals $\bar{Q}_{ij}^*$, we put $\bar{Q}_{ij}^* \cap \bar{\mathfrak{R}} = \bar{\mathfrak{q}}_{ij}$.

Then $a\bar{\bar{\mathfrak{R}}}=\cap \mathfrak{q}_{ij}$. If we assume that $a\bar{\bar{\mathfrak{R}}}=\cap \bar{\mathfrak{q}}_{ij}$ is an irredundant representation, the prime ideal $\bar{\mathfrak{p}}_{ij}$ belonging to the primary ideal $\bar{\mathfrak{q}}_{ij}$ is a minimal prime ideal in $\bar{\bar{\mathfrak{R}}}$. For, if we assume that $\bar{\mathfrak{p}}_{ij}$ is not minimal in $\bar{\mathfrak{R}}$, similarly to the proof of Prop. 3, $(\bar{\mathfrak{p}}_{ij})^{-1}\supset\bar{\mathfrak{R}}$, and $(\bar{\mathfrak{p}}_{ij})^{-1}(\bar{\mathfrak{p}}_{ij})=\bar{\mathfrak{p}}_{ij}$. Hence if $x\notin\bar{\mathfrak{R}}$, and $x\in(\bar{\mathfrak{p}}_{ij})^{-1}$, then $x\bar{\mathfrak{p}}_{ij}\equiv 0(\bar{\mathfrak{p}}_{ij})$ and also $x^N\bar{\mathfrak{p}}_{ij}\equiv 0(\bar{\mathfrak{p}}_{ij})$ $(N=1, 2, 3,\ldots)$. Therefore there is an element $\bar{\rho}$(in $\bar{\mathfrak{R}}$) such that $\bar{\rho}x^N\equiv 0(\dot{\bar{\mathfrak{R}}})$ $(N=1, 2, 3,\ldots)$. As $x\in\mathfrak{K}^*$, if $\tilde{x}$, $\tilde{\rho}$ are the residue classes of x, $\bar{\rho}$ mod. $\mathfrak{l}^*\mathfrak{K}^*$, $\tilde{\rho}\tilde{x}^N\equiv 0(\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^*)$: Hence by Prop. 3, $\tilde{x}\in\bar{\mathfrak{R}}^*/\bar{\mathfrak{l}}^*$. Therefore $x\in\bar{\mathfrak{R}}^*$. This implies that $x\in\bar{\mathfrak{R}}$. This is a contradiction. Hence $\bar{\mathfrak{p}}_{ij}$ is a minimal prime ideal. Similarly to the proof of Prop. 3, we have that $\bar{\mathfrak{R}}$ is an "Endliche diskrete Hauptordnung". Therefore we have the following theorem from the above argument and Prop. 3.

Theorem 1. Let $\bar{\mathfrak{R}}$ be the integral closure of a local domain $\mathfrak{R}$ in the field of quotients of $\mathfrak{R}$, then $\bar{\mathfrak{R}}$ is an "Endliche diskrete Hauptordnung".

Let $\mathfrak{S}$ be an integral domain, then $\bigcap_{\mathfrak{p}_0}\mathfrak{S}_{\mathfrak{p}_0}=\mathfrak{S}$ (where $\mathfrak{p}_0$ runs over all maximal ideals of $\mathfrak{S}$). But since $\mathfrak{S}_{\mathfrak{p}_0}$ is a local domain, $(\bar{\mathfrak{S}}_{\mathfrak{p}_0})=\bigcap_{\mathfrak{p}\in\mathfrak{p}_0}(\bar{\mathfrak{S}}_{\mathfrak{p}})$ (where $\mathfrak{p}$ runs over any minimal prime ideal of $\mathfrak{S}_{\mathfrak{p}_0}$) by theorem 1, provided that $(\bar{\mathfrak{S}}_{\mathfrak{p}_0})$ is the integral closure of $\mathfrak{S}_{\mathfrak{p}_0}$ and $(\bar{\mathfrak{S}}_{\mathfrak{p}})$ is the integral closure of $\mathfrak{S}_{\mathfrak{p}}$. Hence, since $\bar{\mathfrak{S}}=\cap(\bar{\mathfrak{S}}_{\mathfrak{p}_0})$, we have $\bar{\mathfrak{S}}=\bigcap_{\mathfrak{p}}(\bar{\mathfrak{S}}_{\mathfrak{p}})$ (where $\mathfrak{p}$ runs over any minimal prime ideal). This implies that $\bar{\mathfrak{S}}$ is an "Endliche diskrete Hauptordnung" [2, p. 109]. Thus we have the following

Theorem 2. Let $\mathfrak{S}$ be an integral domain and $\bar{\mathfrak{S}}$ be the integral closure of $\mathfrak{S}$ in the field of quotients of $\mathfrak{S}$, then $\bar{\mathfrak{S}}$ is an "Endliche diskrete Hauptordnung".

Part III

In a local domain $\mathfrak{R}$, we shall discuss the sufficient condition that $\mathfrak{R}:\bar{\mathfrak{R}}\neq(0)$. If a local domain is 1-dimensional, namely "einartig", $\mathfrak{R}:\bar{\mathfrak{R}}\neq(0)$ if and only if the completion $\mathfrak{R}^*$ of $\mathfrak{R}$ has no nilpotent element [1]. But, if $\mathfrak{R}$ is not "einartig", that is, n-

If π is a prime ideal in $\mathfrak{S}$, we denote the quotient ring of $\mathfrak{S}$ with respect to π by $\mathfrak{S}_{\pi}$.

dimensional ($n \geqq 2$), we do not know whether the above argument be valid. Therefore, when $\mathfrak{R}$ is n-dimensional ($n \geqq 2$) and the completion $\mathfrak{R}^*$ of $\mathfrak{R}$ has no nilpotent element, we discuss whether $\mathfrak{R}: \overline{\mathfrak{R}} \neq (0)$ be valid.

If $\mathfrak{R}^*$ has no nilpotent element, $(0)\mathfrak{R}^* = \bigcap_{i=1}^{h} \mathfrak{M}_i^*$, where $\mathfrak{M}_i^*$ is the prime ideal which is not imbedded in any other prime ideal of the zero ideal in $\mathfrak{R}^*$. The $\overline{\mathfrak{R}}^* = \bar{\Omega}_1^* + \bar{\Omega}_2^* + \dots + \bar{\Omega}_i^* + \dots + \bar{\Omega}_h^*$ by Corollary of Prop. 1, where $\bar{\Omega}_i^*$ is the integral closure of $\mathfrak{R}^*/\mathfrak{M}_i^* = \Omega_i^*$.

Now we shall prove that $\mathfrak{R}: \overline{\mathfrak{R}} \neq (0)$ if a local domain $\mathfrak{R}$ satisfies one of the following conditions:

(1) $\mathfrak{R}$ and its residue field $\mathfrak{R}/\mathfrak{p}_0 = \Gamma$ have different characteristics,

(2) $\mathfrak{R}$ and its residue field $\mathfrak{R}/\mathfrak{p}_0 = \Gamma$ have same characteristic p (including $p=0$) and $[\Gamma : \Gamma^{(p)}]$ is finite,

(3) $\mathfrak{R}$ and its residue field $\mathfrak{R}/\mathfrak{p}_0 = \Gamma$ have same characteristic $p > 0$ and $\overline{W}_i^*$ (integral closure of W_i^*) is a finite module over W_i^* where the complete local domain W_i^* is a ring finite extension of Ω_i^* by p-th roots of finite elements of Γ ($i=1, 2, 3, \dots, h$).

Since $\bar{\Omega}_i^*$ is a finite module extension over Ω_i^* in the above cases (1), (2), (3) respectively, we have $\Omega_i^* : \bar{\Omega}_i^* \neq (0)$ ($i=1, 2, \dots, h$). Now if $r^*/\pi^* \in \overline{\mathfrak{R}}^*$, where r^*, $\pi^* \in \mathfrak{R}$, and π^* is a non-zero-divisor in $\mathfrak{R}^*$, $\tilde{r}_i^*/\tilde{\pi}_i^* \in \bar{\Omega}_i^*$ by Prop. 1, where $\tilde{\pi}_i^*$, $\tilde{r}_i$ are residue classes of π^* and r^* modulo $\mathfrak{M}_i^*$. Therefore $\tilde{f}_i^*.\ \tilde{r}_i^*/\tilde{\pi}_i^* = \tilde{\lambda}_i^* \in \Omega_i^*$ ($i=1, 2, \dots, h$) where $\tilde{f}_i^* \in \Omega_i^* : \bar{\Omega}_i^*$. Now let representatives in $\mathfrak{R}^*$ of $\tilde{f}_i^*$, $\tilde{\lambda}_i^*$ be f_i^*, λ_i^*, and $\tau_i^* \not\equiv 0(\mathfrak{M}_i^*)$ but $\tau_i^* \equiv 0(\mathfrak{M}_1^* \cap \mathfrak{M}_2^* \cap \dots \cap \mathfrak{M}_{i-1}^* \cap \mathfrak{M}_{i+1}^* \cap \dots \cap \mathfrak{M}_h^*)$ ($i=1, 2, \dots, h$). If we set $F^* = \sum_{i=1}^{h} \tau_i^* f_i^*$ and $\lambda^* = \sum_{i=1}^{h} \tau_i^* \lambda_i^*$, then $F^* r^* - \pi^* \lambda^* \equiv 0$. For, $F^* r^* - \pi^* \lambda^* \equiv \tau_i^* f_i^* r_i^* - \pi_i^* \tau_i^* \lambda_i^* \equiv \tau_i^* (f_i^* r_i^* - \pi_i^* \lambda_i^*) \equiv 0 \pmod{\mathfrak{M}_i^*}$ ($i=1, 2, \dots, h$). Thus $F^* r^* - \pi^* \lambda^* = 0$. Namely $F^*(r^*/\pi^*) = \lambda^* \in \mathfrak{R}^*$ as π^* is a non-zero-divisor in $\mathfrak{R}^*$. Since r^*/π^* is any element of $\overline{\mathfrak{R}}^*$ and F^* is a fixed element in $\mathfrak{R}^*$, we have $F^* \overline{\mathfrak{R}}^* \equiv 0(\mathfrak{R}^*)$. But F^* being a non-zero-divisor in $\mathfrak{R}^*$, we can conclude that $\mathfrak{R}^* : \overline{\mathfrak{R}}^* \neq (0)$.

Now assume that $\mathfrak{R} : \overline{\mathfrak{R}} = (0)$, namely $\mathfrak{R} \subset \mathfrak{R}_1 = \mathfrak{R}[A_1] \subset \mathfrak{R}_2 = \mathfrak{R}_1[A_2] \subset \dots \subset \mathfrak{R}_i = \mathfrak{R}_{i-1}[A_i] \subset \dots \subset \overline{\mathfrak{R}}$, then $A_1 \notin \mathfrak{R}^*$. In fact, if we assume that $A_1 = b_1/a_1$ (where a_1, $b_1 \in \mathfrak{R}$) $\in \mathfrak{R}^*$, then $a_1 r^* = b_1$, where $r^* \in \mathfrak{R}^*$.

Hence $a_1 r = b_1$ and $r \in \mathfrak{R}$. This is a contradiction. Namely $\mathfrak{R}^* \subset \mathfrak{R}^*[A_1]$. Next, assume that $A_2 = b_2/a_2$ (where $a_2, b_2 \in \mathfrak{R}) \in \mathfrak{R}^*[A_1]$, then $b_2/a_2 = \sum_{i=1}^{G_1} c_i^* (b_1/a_1)^i$, namely $b_2 a_1^{G_1} = a_2(\sum_1^{G_1} c_i^* b_1^i a_1^{G_1-i})$, where $c_i^* \in \mathfrak{R}^*$ $(i=1, 2, \ldots, G_1)$. Hence $b_2 a_1^{G_1} = a_2(\sum c_i a_1^{G_1-i} b_1^i)$ where $c_i \in \mathfrak{R}$. This implies that $A_2 = b_2/a_2 = \sum_1^{G_1} c_i (b_1/a_1)^i$. This contradicts the assumption $\mathfrak{R}_2 \supset \mathfrak{R}_1$. Therefore $A_2 \notin \mathfrak{R}^*[A_1]$. Continuing in this way, $\mathfrak{R}^* \subset \mathfrak{R}^*[A_1] \subset \mathfrak{R}^*[A_1, A_2] \subset \ldots \subset \mathfrak{R}^*[A_1, A_2, \ldots, A_{i-1}] \subset \mathfrak{R}^*[A_1, A_2, \ldots, A_{i-1}, A_i] \subset \ldots$, which contradicts the above proposition $\mathfrak{R}^* : \overline{\mathfrak{R}}^* \neq (0)$. Namely $\mathfrak{R} : \overline{\mathfrak{R}} \neq (0)$. Thus we have the following

Proposition 4. If a local domain $\mathfrak{R}$ satisfies one of above conditions (1), (2), (3) and the completion $\mathfrak{R}^*$ of $\mathfrak{R}$ has no nilpotent element, then $\mathfrak{R} : \overline{\mathfrak{R}} \neq (0)$.

Bibliography

1) W. Krull, Ein Satz über primäre integritätbereich, Math. Ann. Vol. 103 (1930) pp. 540-565.

2) W. Krull Idealtheorie. Ergebnisse der Math. und ihrer Grenz, IV_3, Berlin (1935).

3) C. Chevalley, on the theory of local rings, Ann. of Math. Vol. 44 (1943), pp. 690-708.

4) I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc., Vol. 59 (1946) pp. 54-106.

5) Y. Akizuki, Bemerkungen über den Aufbau des Nullideals, proc. of phy.—Math. Soc. of Jap., III_5. Vol. 14 (1932) pp. 253-262.

6) Van der Waerden, Moderne Algebra. springer (Berlin), 1940.