J. Math. Kyoto Univ. (JMKYAZ)
31-4 (1991) 909-930

On theta series and the splitting of $S_2(\Gamma_0(q))$

By

Ken-ichi SHIOTA*

Introduction

Let q be a prime number, and $\mathcal{O}$ be a maximal order in the (q, ∞)-quaternion algebra $\mathfrak{D}$ over the rational number field $\boldsymbol{Q}$. The class number H of $\mathcal{O}$ is equal to the dimension of the space $M_2(\Gamma_0(q))$ of modular forms on $\Gamma_0(q)$ of weight 2, and there is associated to $\mathcal{O}$, a system of theta series $\{\vartheta_{ij}\}_{1\leq i,j\leq H}$ in $M_2(\Gamma_0(q))$ (see § 1). Let $W_j=\langle\vartheta_{1j}, \vartheta_{2j}, \cdots, \vartheta_{Hj}\rangle_{\boldsymbol{C}}$ denote the $\boldsymbol{C}$-linear span of the j-th column of $\{\vartheta_{ij}\}_{1\leq i,j\leq H}$.

In 1935, E. Hecke [He] observed for small levels q that:

(I) $$\dim W_j=H \qquad \text{for each } j,$$

and conjectured that (I) might be valid for any prime q. If (I) is true, we can of course conclude that

(II) $$\{\vartheta_{ij}\}_{1\leq i,j\leq H} \text{ spans } M_2(\Gamma_0(q)).$$

As is well known, M. Eichler proved (II) by means of trace formula ([Ei1]), and there have been much development along this line. While Hecke's original conjecture (I) has its own significance—linear independence of some natural family of theta series, it can not be true literally. Because, for some j, W_j lies apriori in the (-1)-eigenspace of the Atkin-Lehner involution, of which dimension is given by the type number T of $\mathfrak{D}$ (see Theorem 1.8). Thus arranging the numbering of columns properly, it is naturally asked whether the following (I′) holds or not.

(I′) $$\dim W_j=T \qquad \text{for} \quad 1\leqq j\leqq 2T-H;$$
$$\dim W_j=H \qquad \text{for} \quad 2T-H<j\leqq H.$$

For this problem, A. Pizer [Pi3] made an algorithm to compute ϑ_{ij}'s and a table of them up to $q=97$. Examining Pizer's table, M. Ohta found a degeneration (i.e. $\dim W_3=\dim W_4=5<H$) at $q=67$, where $H=6$ and $T=4$, which was the only one known example (one, since $W_3=W_4$ in fact).

It is worthwhile to study those degenerations of W_j's more extensively:

(i) if the degenerations occurs very rarely, it must characterize the prime q in some way;

(ii) if the degeneration occurs rather commonly, it should be useful to get a

* This research is partially supported by JSPS (No. 62790128).
Received December 20, 1989

splitting of $M_2(\Gamma_0(q))$ (and the space of cusp forms $S_2(\Gamma_0(q))$) as a Hecke algebra module over $\boldsymbol{Q}$.

Using Pizer's algorithm with some modification (cf. § 9), the author have computed ϑ_{ij}'s up to $q=997$, and it turned out that the latter (ii) is the case. The purpose of this paper is to report a few remarkable facts on such splittings of $S_2(\Gamma_0(q))$ and some new examples obtained in the way.

Denoting by $S_2^{\pm}(\Gamma_0(q))$ the (± 1)-eigenspaces of the Atkin-Lehner involution, we define the *obedientness* for an irreducible component (we call it *a factor*) of $S_2(\Gamma_0(q))$ as a $\boldsymbol{Q}$-rational Hecke algebra module:

(a) a factor F of $S_2^{+}(\Gamma_0(q))$ is said to be *obedient* if $F \subseteqq W_j$ for all $2T-H<j\leqq H$, *disobedient* otherwise;

(b) a factor F of $S_2^{-}(\Gamma_0(q))$ is said to be *obedient* if $F \subseteqq W_j$ for all $1\leqq j\leqq H$, *disobedient* otherwise.

That all factors in $S_2(\Gamma_0(q))$ are *obedient* is nothing but (I′), however, with this terminology we can state the following facts holding for all our examples:

(0) For any *disobedient* factor F of $S_2(\Gamma_0(q))$, there exist two indices j and k such that $W_j=F\oplus W_k$.

(1) $S_2^{+}(\Gamma_0(q))$ has a 1-dimensional *obedient* factor if and only if $S_2^{+}(\Gamma_0(q))$ itself is 1-dimensional.

(2) A 1-dimensional factor of $S_2^{-}(\Gamma_0(q))$ is *obedient* if and only if the strong Weil curve parametrized by it has a rational division point (see § 6).

(3) A factor of $S_2^{-}(\Gamma_0(q))$ is *obedient* if and only if a common eigenform f of all Hecke operators in it satisfies the congruence $a(f, n)\equiv a(E, n) \bmod \mathfrak{l}$ for all $n\geqq 1$. Here $a(f, n)$ denotes the n-th Fourier coefficient of f, E the Eisenstein series, and $\mathfrak{l}$ a prime ideal in the field of Fourier coefficients of f (see § 7).

Also we found some interesting examples:

(4) For $q=151$, we have the relation $\vartheta_{4,10}=\vartheta_{5,10}$, which gives a first (as far as we know) example of a pair of mutally inequivalent quadratic form of rank 4 over $\boldsymbol{Z}$, belonging to the same spinor genus, and associating to the same theta series (see § 4).

(5) For $q=307$, we have $\dim W_j<T$ for all $1\leqq j\leqq 2T-H$ and $\dim W_j<H$ for all $2T-H<j\leqq H$, i.e. the conjecture of Hecke is not true even in its weakest form (see § 5).

Acknowledgements. The author would like to express his sincere thanks to Professors H. Hijikata, H. Yoshida and H. Saito for their valuable advice and encouragement, and to Professor K. Doi for turning his attention to this subject.

§ 1. Notations and preliminaries

Notation 1.1. For a field K and a positive integer n, K^n denotes the set of all column vectors of size n with entries in K. The j-th entry of $x\in K^n$ is denoted by x_j. Similarly, the (i, j)-th entry of a matrix x is denoted by x_{ij}. The transposed matrix of a matrix x is writen by ${}^t x$. The identity matrix of size n is written by 1_n.

Notation 1.2. Throughout the paper, q denotes an odd prime number. Denote by $M_2(\Gamma_0(q))$ (resp. $S_2(\Gamma_0(q))$) the space of all elliptic modular (resp. cusp) forms of weight 2 with respect to $\Gamma_0(q)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}(2, \boldsymbol{Z})\,\middle|\, c\equiv 0 \bmod q\right\}$, and by $M_2^{\pm}(\Gamma_0(q))$ (resp. $S_2^{\pm}(\Gamma_0(q))$) its (± 1)-eigenspaces under the Atkin-Lehner involution $f\mapsto f\left|\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\right.$. The Fourier expansion of $f\in M_2(\Gamma_0(q))$ is written as

$$f(z)=\sum_{n=0}^{\infty} a(f, n)\mathrm{e}(nz),$$

where $\mathrm{e}(z)=\exp(2\pi\sqrt{-1}z)$ for $z\in\boldsymbol{C}$ such that $\mathrm{Im}(z)>0$.

Put $\mathfrak{N}$ the set of all the common eigenforms $f\in M_2(\Gamma_0(q))$ of all Hecke operators $\mathrm{T}(n)$ $(n=1, 2, \cdots)$ normalized so that $a(f, 1)=1$, and $\mathfrak{N}^0=\mathfrak{N}\cap S_2(\Gamma_0(q))$ the set of *new-forms*. Then $\mathfrak{N}=\mathfrak{N}_0\cup\{E\}$ with the Eisenstein series

$$E(z)=\frac{q-1}{24}+\sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d\mid n\\ q\nmid d}} d\Big)\mathrm{e}(nz).$$

For $f\in\mathfrak{N}$, put $K_f=\boldsymbol{Q}(a(f, n)\mid n=0, 1, \cdots)$ which we view as a (totally real) subfield of $\boldsymbol{C}$.

The *conjugate* of $f\in\mathfrak{N}$ with respect to $\sigma\in\mathrm{Aut}(\boldsymbol{C})$ is an element of $\mathfrak{N}$ defined by $f^{\sigma}(z)=\sum_{n=0}^{\infty} a(f, n)^{\sigma}\mathrm{e}(nz)$. We fix a complete set $\mathfrak{N}^{00}$ of representatives of conjugate classes of $\mathfrak{N}^0$. For $f_1, f_2, \cdots, f_n\in\mathfrak{N}^{00}\cup\{E\}$, denote by $\langle f_1, f_2, \cdots, f_n\rangle_{\mathscr{H}}$ the $\boldsymbol{Q}$-rational Hecke-submodule of $M_2(\Gamma_0(q))$ generated by them, i.e. the $\boldsymbol{C}$-vector subspace generated by all the conjugates of them.

By a *factor* of $M_2(\Gamma_0(q))$, we mean an irreducible component of $M_2(\Gamma_0(q))$ as a $\boldsymbol{Q}$-rational Hecke-module. The set of all factors of $S_2(\Gamma_0(q))$ is often identified with $\mathfrak{N}^{00}$ by the correspondence: $\langle f\rangle_{\mathscr{H}}\leftrightarrow f$.

Notation 1.3. Let $\mathscr{D}$ be the (q, ∞)-quaternion algebra over $\boldsymbol{Q}$ i.e. the one characterized by

$$\mathscr{D}\otimes\boldsymbol{Q}_l\cong\begin{cases}\text{the unique division quaternion algebra over } \boldsymbol{Q}_l & \text{if } l=q \text{ or } \infty;\\ \mathrm{M}(2, \boldsymbol{Q}_l) & \text{otherwise},\end{cases}$$

where we understand $\boldsymbol{Q}_\infty=\boldsymbol{R}$. The main involution of $\mathscr{D}/\boldsymbol{Q}$ is denoted by $x\mapsto\bar{x}$, and the norm map by $x\mapsto\mathrm{N}(x)=x\cdot\bar{x}$.

For a $\boldsymbol{Z}$-lattice $\mathfrak{a}$ in $\mathscr{D}$, the *left* (resp. *right*) *order* of $\mathfrak{a}$ is the order $\{x\in\mathscr{D}\mid x\mathfrak{a}\subseteq\mathfrak{a}$ (resp. $\mathfrak{a}x\subseteq\mathfrak{a})\}$. For an order $\mathcal{O}$ in $\mathscr{D}$, a $\boldsymbol{Z}$-lattice $\mathfrak{a}$ in $\mathscr{D}$ is said to be a *left* (resp. *right*) *$\mathcal{O}$-ideal* if the left (resp. right) order of $\mathfrak{a}$ is equal to $\mathcal{O}$. Two left $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent* if there exists an element x of $\mathscr{D}^\times$ such that $\mathfrak{a}=\mathfrak{b}x$, and then we write $\mathfrak{a}\sim\mathfrak{b}$. The number H of left $\mathcal{O}$-ideal classes is equal for all maximal orders $\mathcal{O}$ in $\mathscr{D}$, and called the *class number* of $\mathcal{O}$.

When two maximal orders $\mathcal{O}$ and $\mathcal{O}'$ in $\mathscr{D}$ are isomorphic as rings, or equivalently, when they are conjugate by an element in $\mathcal{D}^\times$, we write $\mathcal{O}\cong\mathcal{O}'$. The number T of isomorphism classes of maximal orders in $\mathcal{D}$ is called the *type number* of $\mathcal{D}$.

For two $\boldsymbol{Z}$-lattices $\mathfrak{a}$ and $\mathfrak{b}$ such that the right order of $\mathfrak{a}$ is equal to the left order of $\mathfrak{b}$, denote by $\mathfrak{a}\cdot\mathfrak{b}$ the $\boldsymbol{Z}$-lattice $\langle xy \mid x\in\mathfrak{a},\ y\in\mathfrak{b}\rangle_{\boldsymbol{Z}}$. The left (resp. right) order of $\mathfrak{a}\cdot\mathfrak{b}$ is that of $\mathfrak{a}$ (resp. $\mathfrak{b}$). The norm of $\mathfrak{a}$, written by $\mathrm{N}(\mathfrak{a})$, is the positive generator of the fractional ideal $\langle \mathrm{N}(x) \mid x\in\mathfrak{a}\rangle_{\boldsymbol{Z}}$ in $\boldsymbol{Q}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are as above we have $\mathrm{N}(\mathfrak{a}\cdot\mathfrak{b})=\mathrm{N}(\mathfrak{a})\mathrm{N}(\mathfrak{b})$. The inverse of $\mathfrak{a}$, written by $\mathfrak{a}^{-1}$, is the $\boldsymbol{Z}$-lattice $\{x\in\mathcal{D} \mid \mathfrak{a}x\mathfrak{a}\subseteq\mathfrak{a}\}$. Then we have

(1.1) the left (resp. right) order of $\mathfrak{a}^{-1}$=the right (resp. left) order of it,

$$\bar{\mathfrak{a}}=\mathrm{N}(\mathfrak{a})\mathfrak{a}^{-1}, \tag{1.2}$$

$$\mathfrak{a}^{-1}\cdot\mathfrak{a}\ (\text{resp. } \mathfrak{a}\cdot\mathfrak{a}^{-1})=\text{the right (resp. left) order of } \mathfrak{a}. \tag{1.3}$$

Notation 1.4. Hereafter we fix a maximal order $\mathcal{O}$ in $\mathcal{D}$ and a complete set $\{\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_H\}$ of representatives of left $\mathcal{O}$-ideal classes, and put $\mathfrak{a}_{ij}=\mathfrak{a}_j^{-1}\cdot\mathfrak{a}_i$, $\mathcal{O}_j=\mathfrak{a}_{jj}$ and $e_j=\#\mathcal{O}_j^\times$ for each $1\leqq i, j\leqq H$. Then $\mathcal{O}_j$ is the right order of $\mathfrak{a}_j$, any maximal order in $\mathcal{D}$ is isomorphic to some $\mathcal{O}_j$, and $\{\mathfrak{a}_{1j}, \mathfrak{a}_{2j}, \cdots, \mathfrak{a}_{Hj}\}$ gives a complete set of left $\mathcal{O}_j$-ideal classes. For $q\geqq 5$, e_j is 2, 4 or 6, more precisely,

$$\#\{j \mid e_j=4\}=\begin{cases} 0, & \text{if } q\equiv 1 \bmod 4; \\ 1, & \text{if } q\equiv 3 \bmod 4, \end{cases} \tag{1.4}$$

$$\#\{j \mid e_j=6\}=\begin{cases} 0, & \text{if } q\equiv 1 \bmod 3; \\ 1, & \text{if } q\equiv 2 \bmod 3, \end{cases} \tag{1.5}$$

Recall the mass formula of Eichler-Deuring:

$$\frac{q-1}{24}=\sum_{j=1}^{H}\frac{1}{e_j}. \tag{1.6}$$

Put the theta series associated to $\mathfrak{a}_{ij}$ as

$$\vartheta_{ij}(z)=\frac{1}{e_j}\sum_{x\in\mathfrak{a}_{ij}}\mathrm{e}\left(\frac{\mathrm{N}(x)}{\mathrm{N}(\mathfrak{a}_{ij})}z\right),$$

and denote by Θ the $H\times H$-matrix with the (i, j)-th entry ϑ_{ij}. Define the *Brandt matrices* $B(n)\in\mathrm{M}(2, \boldsymbol{Q})$ by $\Theta=\sum_{n=0}^{\infty}B(n)\mathrm{e}(nz)$. Note that

$$B(0)=\begin{pmatrix} \frac{1}{e_1} & \cdots & \frac{1}{e_H} \\ \vdots & & \vdots \\ \frac{1}{e_1} & \cdots & \frac{1}{e_H} \end{pmatrix}, \qquad B(1)=1_H, \tag{1.7}$$

and that all the entries of $B(n)$ with $n\geqq 1$ are non-negative integers.

The *Basis Problem*, together with the theory of newforms of Atkin-Lehner, tells

Theorem 1.5 (Eichler). *We have*

(1) $\vartheta_{ij}\in M_2(\Gamma_0(q))$,

(2) $M_2(\Gamma_0(q))=\langle\vartheta_{ij}\,|\,1\leqq i,\ j\leqq H\rangle_C$,

(3) $H=\dim_C M_2(\Gamma_0(q))$,

(4) $(\vartheta_{ij}\,|\,\mathrm{T}(n))_{1\leq i,j\leq H}=B(n)\Theta$, *for all* $n\geqq 1$,

(5) $B(n)$*'s* $(n=0, 1, \cdots)$ *are simultaneously diagonalizable.*

Notation 1.6. The main object in this article is

$$W_j=\langle\vartheta_{1j},\ \vartheta_{2j},\ \cdots,\ \vartheta_{Hj}\rangle_C \qquad (j=1, 2, \cdots, H),$$

which is a $\boldsymbol{Q}$-rational Hecke-submodule of $M_2(\Gamma_0(q))$ by Theorem 1.5.(4).

In order to state one of properties of W_j, one requires

Definition 1.7. For a maximal order $\mathcal{O}'$ in $\mathcal{D}$, a two-sided $\mathcal{O}'$-ideal is a left $\mathcal{O}'$-ideal whose right order is also $\mathcal{O}'$. A principal two-sided $\mathcal{O}'$-ideal is one in the form $\mathcal{O}'x$ with some $x\in\mathcal{D}^\times$. We say that $\mathcal{O}'$ is of *type* I (resp. *type* II) if there exists no (resp. just one) class of non-principal two-sided $\mathcal{O}'$-ideals. Any maximal order $\mathcal{O}'$ in $\mathcal{D}$ is either of type I or II, and it is of type I (resp. type II) if and only if $\#\{j\,|\,\mathcal{O}'\cong\mathcal{O}_j\}=1$ (resp. 2). If we write $\mathcal{O}_j\cong\mathcal{O}_k$, we understand that $\mathcal{O}_j$ is of type II and $j\neq k$.

Theorem 1.8 (Eichler [Eil, p. 169], or cf. [Po], [Pi2]). *Assume that* $\mathcal{O}_j$ *is of type* I. *Then we have*

(1) *if* $\mathcal{O}_i\cong\mathcal{O}_k$, *then* $\vartheta_{ij}=\vartheta_{kj}$,

(2) $\vartheta_{ij}\in M_2^-(\Gamma_0(q))$ *for all* $1\leqq i\leqq H$,

(3) $T=\dim_C M_2^-(\Gamma_0(q))$,

(4) $\dfrac{H}{2}<T\leqq H$; *and* $T=H$ *if and only if* $q\leqq 31$ *or* $q=41, 47, 59,$ *or* 71.

Thus W_j is a vector subspace of H-dimensional vector space $M_2(\Gamma_0(q))$ with H generators, and if further $\mathcal{O}_j$ is of type I, it is a vector subspace of T-dimensional vector space $M_2^-(\Gamma_0(q))$ with essentially T generators. From this and numerical examples for small levels, it was conjectured that W_j is *trivial* in the sense that each W_j is equal to $M_2^-(\Gamma_0(q))$ or $M_2(\Gamma_0(q))$ according as $\mathcal{O}_j$ is of type I or of type II ([He, Satz 53], [Pi78]). But it is false in general, and what is more important, several $\boldsymbol{Q}$-rational Hecke-submodules of $M_2(\Gamma_0(q))$ are obtained as W_j.

Definition 1.9. For a factor F of $S_2(\Gamma_0(q))$, we say that F is *obedient* if the following condition is satisfied, or *disobedient* otherwise:

(1) in the case $F\subseteqq S_2^-(\Gamma_0(q))$, $F\subseteqq W_i$ for all $1\leqq j\leqq H$;

(2) in the case $F\subseteqq S_2^+(\Gamma_0(q))$, $F\subseteqq W_j$ for all j such that $\mathcal{O}_j$ is of type II.

To understand the meaning of it, we see the following example which is first noticed by M. Ohta (see [HPS]).

Example 1.10. In the case $q=67$, we have

$$H=6, \quad T=4,$$

$$\mathfrak{N}^{00}=\{f_A, f_B, f_C\},$$

$$S_2^+(\Gamma_0(q))=\langle f_A\rangle_{\mathcal{H}}, \quad S_2^-(\Gamma_0(q))=\langle f_B, f_C\rangle_{\mathcal{H}},$$

$$\dim_C\langle f_A\rangle_{\mathcal{H}}=2, \quad \dim_C\langle f_B\rangle_{\mathcal{H}}=1, \quad \dim_C\langle f_C\rangle_{\mathcal{H}}=2,$$

and with a suitable numbering,

$$\mathcal{O}_1 \text{ and } \mathcal{O}_2 \text{ are of type I}, \quad \mathcal{O}_3\cong\mathcal{O}_4, \quad \mathcal{O}_5\cong\mathcal{O}_6,$$

$$W_1=W_2=M_2^-(\Gamma_0(q)), \quad W_3=W_4=\langle E, f_A, f_C\rangle_{\mathcal{H}}, \quad W_5=W_6=M_2(\Gamma_0(q)).$$

Thus the factor $\langle f_B\rangle_{\mathcal{H}}$ is *disobedient*, and the splitting of $S_2(\Gamma_0(q))$ can be explained by the theta series.

The problem we will consider is

Problem 1.11. How many factors of $S_2(\Gamma_0(q))$ are *disobedient*? Can we find some tendency for the *obedientness*?

§2. Table I

2.1. Table I lists the following data for all prime levels $q<1000$ such that $H\geqq 1$.

The first three columns indicate the level q, the class number $H=\dim_C M_2(\Gamma_0(q))$, the type number $T=\dim_C M_2^-(\Gamma_0(q))$. The fourth and the fifth columns describe the splitting of $S_2^+(\Gamma_0(q))$ and of $S_2^-(\Gamma_0(q))$ respectively, by the dimensions of factors. The factors with the dimension in brackets [ ] are *obedient* ones, while the others are *disobedient*. For the last column, see §7.

Observation 2.2. In the range of this table, any *disobedient* factor F can be expressed as $W_j=F\oplus W_k$ with suitable two indices j and k.

§3. Eigenvectors of Brandt matrices

In this section, following the idea of Prof. H. Saito, we describe a method to determine the factors belonging to each W_j from the diagonalization of Θ. This clarifies the situation, and enables us to save much run time of computing.

At first, we recall some fundamental relations between the theta series.

Table I

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
11	2	2		[1]	5
17	2	2		[1]	2
19	2	2		[1]	3
23	3	3		[2]	11
29	3	3		[2]	7
31	3	3		[2]	5
37	3	2	[1]	[1]	3
41	4	4		[3]	2, 5
43	4	3	[1]	[2]	7
47	5	5		[4]	23
53	5	4	[1]	[3]	13
59	6	6		[5]	29
61	5	4	[1]	[3]	5
67	6	4	[2]	1+[2]	no / 11
71	7	7		[3+3]	5 / 7
73	6	4	[2]	[1+2]	2 / 3
79	7	6	[1]	[5]	13
83	8	7	[1]	[6]	41
89	8	7	[1]	[1+5]	2 / 11
97	8	5	[3]	[4]	2
101	9	8	[1]	[7]	5
103	9	7	[2]	[6]	17
107	10	8	[2]	[7]	53
109	9	6	[3]	1+[4]	no / 3
113	10	7	[3]	[1+2+3]	2 / 2 / 7
127	11	8	[3]	[7]	3, 7
131	12	11	[1]	[10]	5, 13
137	12	8	[4]	[7]	2, 17
139	12	9	[3]	1+[7]	no / 23
149	13	10	[3]	[9]	37
151	13	10	[3]	3+[6]	no / 5

Table I (continued)

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
157	13	8	[5]	[7]	13
163	14	8	1+[5]	[7]	3
167	15	13	[2]	[12]	83
173	15	11	[4]	[10]	43
179	16	13	[3]	1+[11]	no / 89
181	15	10	[5]	[9]	3, 5
191	17	15	[2]	[14]	5, 19
193	16	9	2+[5]	[8]	2
197	17	11	1+[5]	[10]	7
199	17	13	[4]	2+[10]	no / 3, 11
211	18	12	3+[3]	[2+9]	5 / 7
223	19	13	2+[4]	[12]	37
227	20	15	[2+3]	2+2+[10]	no / no / 113
229	19	12	1+[6]	[11]	19
233	20	13	[7]	[1+11]	2 / 29
239	21	18	[3]	[17]	7, 17
241	20	13	[7]	[12]	2, 5
251	22	18	[4]	[17]	5
257	22	15	[7]	[14]	2
263	23	18	[5]	[17]	131
269	23	17	1+[5]	[16]	67
271	23	17	[6]	[16]	3, 5
277	23	13	1+[9]	3+[9]	no / 23
281	24	17	[7]	[16]	2, 5, 7
283	24	15	[9]	[14]	47
293	25	17	[8]	[16]	73
307	26	16	[10]	1+1+1+1+[2+9]	no/no/no/no/3/17
311	27	23	[4]	[22]	5, 31
313	26	15	[11]	2+[12]	no / 2, 13
317	27	16	[11]	[15]	79
331	28	17	1+[3+7]	[16]	5, 11

Table I (continued)

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
337	28	16	[12]	[15]	2, 7
347	30	20	1+2+[7]	[19]	173
349	29	18	[11]	[17]	29
353	30	19	[11]	[1+3+14]	2 / 2 / 2, 11
359	31	25	1+1+[4]	[24]	179
367	31	20	[11]	[19]	61
373	31	18	1+[12]	[17]	31
379	32	19	[13]	[18]	3, 7
383	33	25	2+[6]	[24]	191
389	33	22	2+[3+6]	1+[20]	no / 97
397	33	18	2+[13]	2+[5+10]	no / 11 / 3
401	34	22	[12]	[21]	2, 5
409	34	21	[13]	[20]	2, 17
419	36	27	[9]	[26]	11, 19
421	35	20	[15]	[19]	5, 7
431	37	29	1+4+[3]	1+3+[24]	no / no / 5, 43
433	36	21	[15]	1+3+[16]	no / no / 2, 3
439	37	26	2+[9]	[25]	73
443	38	24	1+1+[12]	1+[22]	no / 13, 17
449	38	24	[14]	[23]	2, 7
457	38	21	2+[15]	[20]	2, 19
461	39	27	2+3+[7]	[26]	5, 23
463	39	23	[16]	[22]	7, 11
467	40	27	1+[12]	[26]	233
479	41	33	[8]	[32]	239
487	41	24	[17]	2+3+[2+16]	no / no / 3 / 3
491	42	30	2+[10]	[29]	5, 7
499	42	24	2+[16]	[23]	83
503	43	32	1+[10]	1+1+3+[26]	no / no / no / 251
509	43	29	[14]	[28]	127
521	44	30	[14]	[29]	2, 5, 13

Table I (continued)

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
523	44	27	2+[15]	[26]	3, 29
541	45	25	[20]	[24]	3, 5
547	46	26	2+[18]	[25]	7, 13
557	47	28	1+[18]	1+[26]	no / 139
563	48	33	3+[3+9]	1+[31]	no / 281
569	48	32	[16]	[31]	2, 71
571	48	29	3+[6+10]	1+1+2+2+4+[18]	no/no/no/no/no/5,19
577	48	26	[22]	2+3+[2+18]	no / no / 3 / 2
587	50	32	5+[13]	[31]	293
593	50	31	1+[18]	2+[1+27]	no / 2 / (2?), 37
599	51	38	2+[11]	[37]	13, 23
601	50	30	[20]	[29]	2, 5
607	51	32	5+7+[7]	[31]	101
613	51	28	[5+18]	[27]	3, 17
617	52	29	[23]	[28]	2, 7, 11
619	52	31	[21]	[30]	103
631	53	33	[20]	[32]	3, 5, 7
641	54	34	[20]	[33]	2, 5
643	54	30	[24]	1+[28]	no / 107
647	55	39	2+[14]	[38]	17, 19
653	55	31	7+[17]	[30]	163
659	56	39	1+[16]	1+[37]	no / 7, 47
661	55	32	[23]	2+[29]	no / 5, 11
673	56	31	[25]	2+[4+24]	no / 7 / 2
677	57	36	1+2+[18]	[35]	13
683	58	34	[24]	2+[31]	no / 11, 31
691	58	34	[24]	[33]	5, 23
701	59	38	[21]	1+[36]	no / 5, 7
709	59	32	[27]	1+[30]	no / 59
719	61	46	[5+10]	[45]	359
727	61	37	[24]	[36]	11

Table I (continued)

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
733	61	34	2+[25]	1+[32]	no / 61
739	62	36	3+[23]	1+[34]	no / 3, 41
743	63	42	[21]	[41]	7, 53
751	63	39	[24]	[38]	5
757	63	34	[29]	[33]	3, 7
761	64	42	2+[20]	[41]	2, 5, 19
769	64	37	[27]	[36]	2
773	65	39	2+[24]	[38]	193
787	66	38	[28]	[37]	131
797	67	41	1+[25]	2+[38]	no / 199
809	68	42	2+[24]	[41]	2, 101
811	68	41	1+[26]	[40]	3, 5
821	69	42	[27]	[41]	5, 41
823	69	39	[30]	[38]	137
827	70	42	1+3+[24]	[41]	7, 59
829	69	40	1+[28]	[39]	3, 23
839	71	52	[19]	[51]	419
853	71	38	[33]	[37]	71
857	72	44	[28]	[43]	2, 107
859	72	43	[29]	[42]	11, 13
863	73	47	4+[22]	[46]	431
877	73	39	2+[32]	[38]	73
881	74	47	[27]	[46]	2, 5, 11
883	74	40	[34]	[39]	3, 7
887	75	52	2+[21]	[51]	443
907	76	41	[35]	[40]	151
911	77	54	9+[14]	[53]	5, 7, 13
919	77	48	2+[27]	[47]	3, 17
929	78	48	2+[28]	[47]	2, 29
937	78	44	[34]	[43]	2, 3, 13
941	79	51	[28]	[50]	5, 47

Table I (continued)

q	H	T	$S_2^+(\Gamma_0(q))$	$S_2^-(\Gamma_0(q))$	l
947	80	45	[35]	[44]	11, 43
953	80	48	[32]	[47]	2, 7, 17
967	81	46	[35]	[45]	7, 23
971	82	56	[26]	[55]	5, 97
977	82	46	[36]	[45]	2, 61
983	83	55	[28]	[54]	491
991	83	50	[33]	[49]	3, 5, 11
997	83	45	1+4+5+[5+23]	1+1+[42]	no / no / 83

Proposition 3.1. *We have*
(1) $e_j\vartheta_{ij}=e_i\vartheta_{ji}$ *for each* $1\leqq i, j\leqq H$,
(2) $\sum_{j=1}^{H}\vartheta_{ij}=E$ *for each* $1\leqq i\leqq H$,
(3) $\sum_{j=1}^{H}\vartheta_{jj}=\sum_{f\in\mathfrak{N}}f$,
(4) *if* $\mathcal{O}_i\cong\mathcal{O}_j$ *and* $\mathcal{O}_k\cong\mathcal{O}_l$, *then* $\vartheta_{ik}=\vartheta_{jl}$.

Proof. (1) is the simplist case of [Ei2, II, Theorem 2]. That the left hand side of (2) is independent of i is shown in [Pi3, Lemma 2.18], therefore we denote it by g. We see easily that $g|\mathrm{T}(n)=\left(\sum_{j=1}^{H}B(n)_{ij}\right)g$ for all $n\geqq 1$ by Theorem 1.5.(4), and that $a(g, 0)=\sum_{j=1}^{H}\frac{1}{e_j}>0$ and $a(g, 1)=1$ by (1.7). Hence $g=E$. (3) is immediate from Theorem 1.5.(5). By [Pil, Lemma 2.18], we may assume that $\mathcal{O}_i=\mathcal{O}_j$ and $\mathcal{O}_k=\mathcal{O}_l$. Then $\{\mathfrak{a}_{1k}, \mathfrak{a}_{2k}, \cdots, \mathfrak{a}_{Hk}\}$ is also a complete set of representatives for left $\mathcal{O}_l$-ideal classes, hence, considering the right orders of them, we see that there are two possibilities:

(a) $\mathfrak{a}_{ik}\sim\mathfrak{a}_{jl}$ and $\mathfrak{a}_{il}\sim\mathfrak{a}_{jk}$; or (b) $\mathfrak{a}_{ik}\sim\mathfrak{a}_{il}$ and $\mathfrak{a}_{jl}\sim\mathfrak{a}_{jk}$.

Note that $e_i=e_j=e_k=e_l=2$. The case (a) is obvious (cf. [Pi2, Lemma 2.7]), therefore we treat the case (b). Then we have $\vartheta_{ik}=\vartheta_{il}$ and $\vartheta_{jk}=\vartheta_{jl}$. Interchanging i, j with k, l, we have also that $\vartheta_{ki}=\vartheta_{kj}$ or ϑ_{lj}. The latter case is also obvious, while in the former, $\vartheta_{ik}=\vartheta_{ki}=\vartheta_{kj}=\vartheta_{jk}=\vartheta_{jl}$ holds. q.e.d.

Notation 3.2. For each $f\in\mathfrak{N}^0$ and $1\leqq j\leqq H$, denote $v(f, j)$ the element of C^H with the i-th entry $\frac{\langle\vartheta_{ij}, f\rangle}{\langle f, f\rangle}$ where $\langle\ ,\ \rangle$ denotes the Petersson inner product. Note that

$$\vartheta_{ij}=\frac{1}{e_j}\frac{24}{q-1}E+\sum_{f\in\mathfrak{N}^0}\frac{\langle\vartheta_{ij}, f\rangle}{\langle f, f\rangle}f \tag{3.1}$$

Proposition 3.3. *We have*

(1) $\Theta v(f, j)=f\cdot v(f, j)$,

(2) *for each* $f\in\mathfrak{N}^0$, *there exists an index* j *such that* $v(f, j)\neq 0$, *i.e.,* $v(f, j)$ *is an eigenvector of* Θ *corresponding to* f,

(3) $v(f^{\sigma}, j)=v(f, j)^{\sigma}$ *for all* $\sigma\in\mathrm{Aut}(\boldsymbol{C})$, *especially* $v(f, j)\in(K_f)^H$.

Proof. (1) is immediate from Theorem 1.5.(4). From Proposition 3.1.(3), we get $\sum_{j=1}^{H}\frac{\langle\vartheta_{jj}, f\rangle}{\langle f, f\rangle}=1$, and this implies (2). (3) is derived from the $\boldsymbol{Q}$-rationality of ϑ_{ij} and E and the uniqueness of the expression (3.1). q.e.d.

Notation 3.4. From the above, we can take and fix a system $\{v(f)\}_{f\in\mathfrak{N}^0}$ of eigenvectors of Θ so that

(1) $\Theta v(f)=f\cdot v(f)$,

(2) each $v(f, j)$ is a constant (in $K_f^{\times}$) multiple of $v(f)$,

(3) $v(f^{\sigma})=v(f)^{\sigma}$ for all $\sigma\in\mathrm{Aut}(\boldsymbol{C})$.

Further, we put $v(E)={}^t(1, 1, \cdots, 1)$. Then Theorem 3.1.(2) is read as $\Theta v(E)=E\cdot v(E)$, which is (1) for $f=E$. Thus, numbering $\mathfrak{N}=\{f_1, f_2, \cdots, f_H\}$ and putting $Q=(v(f_1), v(f_2), \cdots, v(f_H))$, we get the diagonalization of Θ as

$$Q^{-1}\Theta Q=\begin{pmatrix} f_1 & & \\ & \ddots & \\ & & f_H \end{pmatrix}. \tag{3.2}$$

Note that $\{v(f)\}_{f\in\mathfrak{N}}$ is a basis of $\boldsymbol{C}^H$.

Definition 3.5. An element $r\in\boldsymbol{C}^H$ is called a *relation for* W_j if $\sum_{i=1}^{H}r_i\vartheta_{ji}=0$ holds. Note that $W_j=\langle\vartheta_{j1}, \vartheta_{j2}, \cdots, \vartheta_{jH}\rangle_{\boldsymbol{C}}$ by Proposition 3.1.(1), and we use ϑ_{ji} in this definition. The vector space over $\boldsymbol{C}$ consisting of all relations for W_j is denoted by R_j.

Lemma 3.6. *We have*

(1) $R_j=\langle v(f)\,|\,f\in\mathfrak{N}$ *such that* $v(f)_j=0\rangle_{\boldsymbol{C}}$,

(2) $\dim_{\boldsymbol{C}}W_j=\#\{f\in\mathfrak{N}\,|\,v(f)_j\neq 0\}$,

(3) $W_j=\langle f\in\mathfrak{N}\,|\,v(f)_j\neq 0\rangle_{\boldsymbol{C}}$.

Proof. That $v(f)\in R_j$ is immediate from Remark 3.4.(1). Write $r\in R_j$ as $r=\sum_{f\in\mathfrak{N}}c_f v(f)$ with $c_f\in\boldsymbol{C}$. Then

$$0=\sum_{i=1}^{H}r_i\vartheta_{ji}=(\Theta r)_j=\sum_{f\in\mathfrak{N}}(c_f v(f)_j)\cdot f\,,$$

hence $c_f=0$ for all f such that $v(f)_j\neq 0$. (2) is derived from (1). Being a Hecke-submodule, W_j has a basis consisting of elements of $\mathfrak{N}$. If $v(f)_j=0$, then $\langle\vartheta_{ij}, f\rangle=0$ for all $1\leqq i\leqq H$, hence $f\notin W_j$. Therefore $W_j\subseteqq\langle f\in\mathfrak{N}\,|\,v(f)_j\neq 0\rangle_{\boldsymbol{C}}$, and the equality holds by (2). q.e.d.

Remark 3.7. For each $f\in\mathfrak{N}$, two $\boldsymbol{Q}$-rational bases of $\langle v(f)^{\sigma}\mid\sigma\in\mathrm{Aut}(\boldsymbol{C})\rangle_{C}$ are obtained as follows. Taking arbitrary basis $\{\omega_1, \omega_2, \cdots, \omega_g\}$ of $K_f/\boldsymbol{Q}$ where $g=[K_f:\boldsymbol{Q}]$, $\{\mathrm{Tr}_{K_f/\boldsymbol{Q}}(\omega_i v(f))\}_{1\leq i\leq g}$ is one. Writing $v(f)=\sum_{i=1}^{g}\omega_i v^{(i)}$ with $v^{(i)}\in\boldsymbol{Q}^H$, $\{v^{(i)}\}_{1\leq i\leq g}$ is another. The former is useful for numerical computation, the latter for explanation (cf. Example 4.1, 9.4).

§ 4. The case $q=151$

Let us apply the method described in § 3 to

Example 4.1. In the case $q=151$, we have

$$H=13, \qquad T=10,$$

$$\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_6 \text{ and } \mathcal{O}_9 \text{ are of type I},$$

$$\mathcal{O}_7\cong\mathcal{O}_8, \quad \mathcal{O}_{10}\cong\mathcal{O}_{11}, \quad \mathcal{O}_{12}\cong\mathcal{O}_{13},$$

$$e_1=4, \quad e_j=2 \text{ for } 2\leqq j\leqq 13,$$

$$S_2^{+}(\Gamma_0(q))=\langle f_A\rangle_{\mathcal{H}}, \quad S_2^{-}(\Gamma_0(q))=\langle f_B, f_C\rangle_{\mathcal{H}},$$

$$\dim_C\langle f_A\rangle_{\mathcal{H}}=3, \quad \dim_C\langle f_B\rangle_{\mathcal{H}}=3, \quad \dim_C\langle f_C\rangle_{\mathcal{H}}=6,$$

and

$$B(2)=\begin{pmatrix}
1&2&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&1&0&0&0&0&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&0&0&1\\
0&0&0&0&1&0&1&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&1&0\\
0&0&0&0&0&1&0&0&0&1&0&0&1\\
0&0&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&0&1&0&0&1&1&0
\end{pmatrix}.$$

The irreducible decomposition of the characteristic polynomial of $B(2)$ over $\boldsymbol{Q}$ is $(X-3)F_A(X)F_B(X)F_C(X)$ where

$$F_A(X)=X^3+2X^2-X-1, \quad F_B(X)=X^3-5X+3,$$

$$F_C(X)=X^6+2X^5-6X^4-8X^3+11X^2+2X-3.$$

$\xi=a(f_A, 2)$ is a root of F_A, and

$$v(f_A)=(\xi^2+\xi-1)\,{}^t(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0)$$

$$+\xi\,{}^t(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1)$$

$$+{}^t(0,0,0,0,0,0,1,-1,0,0,0,0,0).$$

Thus the relations between the theta series caused by f_A are

$$\begin{cases} \vartheta_{7j}=\vartheta_{8j}, \\ \vartheta_{10j}=\vartheta_{11j}, \qquad \text{for } j=1,2,\cdots,6 \text{ and } 9, \\ \vartheta_{12j}=\vartheta_{13j}, \end{cases}$$

which are already known by Theorem 1.8(1). $f_A \in W_j$ if and only if $j=7, 8, 10, 11, 12, 13$, hence f_A is *obedient*.

$\eta=a(f_B, 2)$ is a root of F_B, and

$$\begin{aligned} v(f_B)=&(\eta^2-2){}^t(0,0,0,1,-1,0,0,0,0,0,0,0,0) \\ &+(\eta-1){}^t(0,1,-1,0,-1,0,1,1,-1,0,0,0,0) \\ &+{}^t(2,0,-1,0,-1,-1,0,0,0,0,0,1,1). \end{aligned}$$

Thus the relations between the theta series caused by f_B are

$$\begin{cases} \vartheta_{4j}=\vartheta_{5j}, \\ \vartheta_{2j}+\vartheta_{7j}+\vartheta_{8j}=\vartheta_{3j}+\vartheta_{5j}+\vartheta_{9j}, \qquad \text{for } j=10, 11. \\ \vartheta_{1j}+\vartheta_{12j}+\vartheta_{13j}=\vartheta_{3j}+\vartheta_{5j}+\vartheta_{6j}, \end{cases}$$

Since $f_B \notin W_j$ for $j=10, 11$, f_B is *disobedient*.

That all the entries of $v(f_C)$ are non-zero is known by their approximate values, hence f_C is *obedient*.

Remark 4.2. The relation $\vartheta_{4,10}=\vartheta_{5,10}$ is an example of inequivalent two rational quadratic forms of rank 4, belonging to the same spinor genus, and associating to the same theta series. The Minkowski-reduced matrices corresponding to the norm forms of $\mathfrak{a}_{4,10}$ and $\mathfrak{a}_{5,10}$ are

$$\begin{pmatrix} 6 & 2 & -1 & 1 \\ 2 & 12 & 5 & 4 \\ -1 & 5 & 16 & 6 \\ 1 & 4 & 6 & 28 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 0 & -2 & -3 \\ 0 & 12 & 3 & 4 \\ -2 & 3 & 14 & 2 \\ -3 & 4 & 2 & 28 \end{pmatrix}$$

respectively.

That those norm forms are inequivalent over $\boldsymbol{Z}$ is also derived from the well-known

Proposition 4.3. *Let $\mathcal{B}$ a division quaternion algebra over a field k of characteristic $\neq 2$. Viewing $(\mathcal{B}, \mathrm{N}_{\mathcal{B}/k})$ as a quadraic space over k of rank 4, its orthogonal group $\mathrm{O}(\mathcal{B}, \mathrm{N}_{\mathcal{B}/k})$ is generated by the canonical involution of $\mathcal{B}/k$ and the subgroup*

$$\{x \mapsto \alpha x \beta^{-1} \mid \alpha, \beta \in \mathcal{B}^\times \text{ such that } \mathrm{N}_{\mathcal{B}/k}(\alpha)=\mathrm{N}_{\mathcal{B}/k}(\beta)\}$$

of index 2.

By virtue of [Pi2, Lemma 2.7] and since $\mathrm{N}(\mathcal{D}^\times)=\boldsymbol{Q}_+^\times$, we may assume $\mathrm{N}(\mathfrak{a}_{ij})=1$ for all i, j. Then the norm form $\dfrac{\mathrm{N}(x)}{\mathrm{N}(\mathfrak{a}_{ij})}=\mathrm{N}(x)$ $(x \in \mathfrak{a}_{ij})$ just corresponds to the lattice $\mathfrak{a}_{ij}$

in $(\mathfrak{D}, \mathrm{N})$. Suppose that there exists an isometry in $\mathrm{O}(\mathfrak{D}, \mathrm{N})$ which maps $\mathfrak{a}_{4,10}$ onto $\mathfrak{a}_{5,10}$. Then either $\mathfrak{a}_{4,10}$ or $\overline{\mathfrak{a}_{4,10}}$ must be written as $\alpha\mathfrak{a}_{5,10}\beta$ with some $\alpha, \beta \in \mathfrak{D}^{\times}$. Comparing the right orders, by (1.1-2) we have $\mathcal{O}_4 \cong \mathcal{O}_5$ or $\mathcal{O}_{10} \cong \mathcal{O}_5$, both of which are impossible.

Further, in the adelic language (cf. [Pi77, § 2], and § 8 below), we can take each $\mathfrak{a}_j$ so that $\mathfrak{a}_j = \mathcal{O} y_j$ with some $y_j \in \mathfrak{D}_A^{\times}$ such that $\mathrm{N}(y_j)=1$. This and Proposition 4.3 implies that the norm forms of all $\mathfrak{a}_{ij}$ belong to the same spinor genus (cf. [OM]). Thus our example is different from the ones mentioned in [SP, Remark 1].

§ 5. A conjecture of Hecke

A conjecture of Hecke [He, Satz 53], stating that all W_j's are equal to $M_2(\Gamma_0(q))$, has been weakened gradually as seen in § 1. As the last version of it, one can ask

Problem 5.1. Is there at least one index j such that W_j is *trivial* in the sense in § 1 ?

But we found a counter example even to this:

Example 5.2. Let $q=307$. Then $H=26$, $T=16$. Denoting $\mathfrak{D}=\boldsymbol{Q}[I, J]=\boldsymbol{Q}\cdot 1+\boldsymbol{Q}\cdot I+\boldsymbol{Q}\cdot J+\boldsymbol{Q}\cdot K$ with $I^2=-1$, $J^2=-307$, $K=I\cdot J=-J\cdot I$, we can take a complete set of representatives of left $\mathcal{O}$-ideal classes $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{26}$ with the maximal order $\mathcal{O}=\frac{1}{2}\mathfrak{a}_1$ as in Table II.

There are four $\boldsymbol{Q}$-rational newforms f_B, f_C, f_D, f_E in $S_2^-(\Gamma_0(q))$, and for each j, at least one of the j-th entries of $v(f_B)$, $v(f_C)$, $v(f_D)$ or $v(f_E)$ is zero, hence all W_j's are *non-trivial*.

§ 6. Observation on the 1-dimensional factors

First recall the following fact on elliptic curves:

Theorem 6.1 (Setzer [Se], Miyawaki [Mi]). *For an odd prime number $q \geqq 11$ and a positive integer $n \geqq 2$, assume that there exists an elliptic curve defined over $\boldsymbol{Q}$ of q-power conductor having a $\boldsymbol{Q}$-rational division point of order n. Then* $(q, n)=(11, 5)$, $(17, 2)$, $(17, 4)$, $(19, 3)$, $(37, 3)$ *or* $(64+u^2, 2)$ *with some* $u \in \boldsymbol{Z}$.

In the last case, there exist just two such curves (called Setzer-Neumann curves), one of which is of conductor q, the other of q^2.

One should notice that, in Table I, an *obedient* 1-dimensional factor appears only at the levels q listed in Theorem 6.1 and the levels where $S_2^+(\Gamma_0(q))$ is of dimension 1. More precisely, together with the table of Mestre [Me], we see that

Table II

j	generators of $\mathfrak{a}_j$				$N(\mathfrak{a}_j)$	$v(f_B)_j$	$v(f_C)_j$	$v(f_D)_j$	$v(f_E)_j$
1	$1+J$	$I+K$	$2J,$	$2K$	4	0	2	2	2
2	$1+J+2K,$	$I+2J+K,$	$4J,$	$4K$	8	0	−1	1	1
3	$1+J+6K,$	$I+2J+K,$	$8J,$	$8K$	16	0	−1	0	0
4	$1+J+2K,$	$I+6J+K,$	$8J,$	$8K$	16	0	−1	0	0
5	$1+J+14K,$	$I+2J+K,$	$16J,$	$16K$	32	1	1	0	−1
6	$1+J+6K,$	$I+10J+K,$	$16J,$	$16K$	32	−1	0	−1	0
7	$1+J+10K,$	$I+6J+K,$	$16J,$	$16K$	32	−1	0	−1	0
8	$1+J+2K,$	$I+14J+K,$	$16J,$	$16K$	32	1	1	0	−1
9	$1+J+14K,$	$I+18J+K,$	$32J,$	$32K$	64	0	0	0	−1
10	$1+17J+22K,$	$I+10J+17K,$	$32J,$	$32K$	64	0	1	−1	0
11	$1+17J+10K,$	$I+22J+17K,$	$32J,$	$32K$	64	0	1	−1	0
12	$1+J+18K,$	$I+14J+K,$	$32J,$	$32K$	64	0	0	0	−1
13	$1+33J+46K,$	$I+18J+33K,$	$64J,$	$64K$	128	−1	−1	0	−1
14	$1+33J+14K,$	$I+50J+33K,$	$64J,$	$64K$	128	0	0	0	0
15	$1+17J+54K,$	$I+10J+17K,$	$64J,$	$64K$	128	1	−1	0	0
16	$1+17J+10K,$	$I+54J+17K,$	$64J,$	$64K$	128	1	−1	0	0
17	$1+33J+50K,$	$I+14J+33K,$	$64J,$	$64K$	128	0	0	0	0
18	$1+97J+110K,$	$I+18J+97K,$	$128J,$	$128K$	256	−1	0	0	0
19	$1+33J+78K,$	$I+50J+33K,$	$128J,$	$128K$	256	0	0	1	0
20	$1+33J+14K,$	$I+114J+33K,$	$128J,$	$128K$	256	0	0	−1	1
21	$1+81J+118K,$	$I+10J+81K,$	$128J,$	$128K$	256	0	0	1	0
22	$1+81J+10K,$	$I+118J+81K,$	$128J,$	$128K$	256	0	0	1	0
23	$1+33J+114K,$	$I+14J+33K,$	$128J,$	$128K$	256	0	0	−1	1
24	$1+33J+50K,$	$I+78J+33K,$	$128J,$	$128K$	256	0	0	1	0
25	$1+97J+238K,$	$I+18J+97K,$	$256J,$	$256K$	512	−1	1	1	0
26	$1+97J+110K,$	$I+146J+97K,$	$256J,$	$256K$	512	1	0	−1	1

Observation 6.2. For all prime levels $q<1000$, the following facts hold.

(1) A 1-dimensional factor of $S_2^+(\Gamma_0(q))$ is *obedient* if and only if $S_2^+(\Gamma_0(q))$ itself is 1-dimensional.

(2) A 1-dimensional factor of $S_2^-(\Gamma_0(q))$ is *obedient* if and only if the strong Weil curve parametrized by it is one of those in Theorem 6.1.

Remark 6.3. Note that, if q, n and $\langle f\rangle_{\mathcal{K}}$ are as in (2) above, we get the congruence $1-a(f, p)+p\equiv 0 \bmod n$ for all prime numbers $p\neq q$.

The calculation was done also for the levels $q=1153$, 1289, 1433, 1913, 2089 and 2273, for which a Setzer-Neumann curve of conductor q exists. The facts stated in Observation 6.2 still holds for these levels, and further we notice

Observation 6.4. For a $\boldsymbol{Q}$-rational newform $f\in\mathfrak{N}^0$of level q ($q\leqq 2273$) corresponding to a Setzer-Neumann curve, we can take $v(f)$ so that $\frac{v(f)_j}{e_j}$ is a *small odd* integer for *all* $1\leqq j\leqq H$. For example,

$$\left(\frac{v(f)_j}{e_j}\right)_{1\leq j\leq H}={}^t(1,\ 1,\ -1,\ -1,\ 1,\ -3,\ -3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -1,\ -1,\ 3,\ -1,\ -3,\ 1)$$

when $q=233$. For other cases we get

$$\left\{\frac{v(f)_j}{e_j}\,\middle|\,1\leqq j\leqq H\right\}=\begin{cases}\{\pm 1\} & \text{if } q=73,\ 89,\ 113,\\ \{\pm 1,\ -3\} & \text{if } q=353,\\ \{\pm 1,\ \pm 3\} & \text{if } q=593,\ 1153,\\ \{\pm 1,\ \pm 3,\ \pm 5\} & \text{if } q=1289,\ 1433,\\ \{\pm 1,\ \pm 3,\ \pm 5,\ 7\} & \text{if } q=1913,\ 2273,\\ \{\pm 1,\ \pm 3,\ -5\} & \text{if } q=2089.\end{cases}$$

For a meaning of this, see Remark 8.4.

§7. Observation on the factors of $S_2(\Gamma_0(q))$

In view of Remark 6.3, an observation on a congruence for $\mathrm{N}_{K_f/\boldsymbol{Q}}(1-a(f, p)+p)$ (p: prime) is suggested by Prof. H. Yoshida.

At first, recall the congruence of Brumer-Doi in the case of weight 2 ([DM, Theorem 7.5.4], see also [Ma, Theorem 1, Theorem 4 and Table]).

Lemma 7.1. *Let l be a prime number dividing the numerator of $\frac{q-1}{12}$. Then, for each $1\leqq j\leqq H$ such that $W_j\neq\langle E\rangle_C$, there exist $f\in W_j\cap\mathfrak{N}^0$ and a prime ideal $\mathfrak{l}$ in K_f over l such that*

(7.1) $$a(f, n)\equiv a(E, n) \mod \mathfrak{l} \quad \textit{for all } n\geqq 1.$$

Note that

$$a(E, p)=\begin{cases}1+p & \text{if } p\neq q,\\ 1 & \text{if } p=q,\end{cases}$$

for a prime number p, hence (7.1) implies

(7.2) $$\mathrm{N}_{K_f/\boldsymbol{Q}}(1-a(f, p)+p)\equiv 0 \mod l \quad \text{for all prime number } p\neq q$$

Proof. Take ϑ_{ij} which is not a constant multiple of E. By (3.1), we have

$\left(\frac{e_j}{2}\right)\left(\frac{q-1}{12}\right)\vartheta_{ij}=E-g$ with some $g\in W_j\cap S_2(\Gamma_0(q))$. All the Fourier coefficients of g are l-integers in $\boldsymbol{Q}$, and from the assumption, we have $a(g, n)\equiv a(E, n) \bmod l$ for all $n\geqq 1$. Further, by Theorem 1.5.(5), we see that $a(g|\mathrm{T}(p), n)\equiv a(E, p)a(g, n) \bmod l$ holds for all prime numbers p and $n\geqq 1$. Thus $g \bmod l$ is a *common eigenform mod l* of all Hecke operators, hence the existence of f and $\mathfrak{l}$ as above follows from [DS, Lemma 6.11]. q. e. d.

7.2. Though we may state Lemma 7.1 with $\frac{q-1}{24}e_j$ instead of $\frac{q-1}{12}$, the set of of such l's is the same by (1.4-5).

Observation 7.3. In the range of Table I, the following facts hold.

(1) A congruence of type (7.1) (hence of type (7.2)) holds for a factor $F=\langle f\rangle_{\mathcal{H}}$ in $S_2^-(\Gamma_0(q))$ if and only if F is *obedient*.

(2) For the factors $\langle f\rangle_{\mathcal{H}}$ in $S_2^+(\Gamma_0(q))$ of dimension $\leqq 10$, it is checked that any congruence of type (7.2) does not hold.

7.4. The sixth column of Table I lists the prime numbers l such that a congruence of type (7.1) holds for a factor $\langle f\rangle_{\mathcal{H}}$ in $S_2^-(\Gamma_0(q))$ and a prime ideal $\mathfrak{l}$ in K_f over l. When $S_2^-(\Gamma_0(q))$ has at least two factors, those l's are separated by slash / in the same order as factors. For example, in the case $q=199$, *no* such congruence holds for the factor of dimension 2, while it holds for the factor of dimension 10 with $l=3$ and also with $l=11$.

The possible l's can be calculated from the numerical data at least for the factors of small dimension, thus all the *no*'s in Table I are proved. Then most of the congruences listed in Table I are shown by Lemma 7.1, but in some cases, further discussion (e.g. [DO, Lemma 2.1]) is needed. For the 27-dimensional factor in $S_2^-(\Gamma_0(593))$, $l=2$ seems possible, but is not proved. Note that we are not saying that those l's are all.

§ 8. Automorphic forms on $\mathfrak{D}_A^\times$

We recall a proposition in [Yo1, § 7].

Notation 8.1. Denote by $\mathfrak{D}_A^\times$ the adelization of $\mathfrak{D}^\times$, and put $\mathcal{K}_p=(\mathcal{O}\otimes\boldsymbol{Z}_p)^\times$ for each prime number p, $\mathcal{K}=\prod_p \mathcal{K}_p\times\boldsymbol{H}^\times$, where $\boldsymbol{H}$ is the Hamiltonian quaternion. An automorphic form φ on $\mathfrak{D}_A^\times$ is defined to be a $\boldsymbol{C}$-valued function on $\mathfrak{D}_A^\times$ which is left $\mathfrak{D}^\times$-, right $\mathcal{K}$-invariant, and we denote the space of all such φ's by $\mathcal{S}$. For a prime number $p\neq q$, the Hecke operator $\mathrm{T}'(p)$ acting on $\mathcal{S}$ is defined by

$$[\varphi|\mathrm{T}'(p)](x)=\sum_s \varphi(xh_s)$$

where $\mathcal{K}_p\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\mathcal{K}_p=\bigcup_s h_s\mathcal{K}_p$ is the coset decomposition under the fixed identification $\mathfrak{D}\otimes\boldsymbol{Q}_p\cong\mathrm{M}(2, \boldsymbol{Q}_p)$. We have $\dim_{\boldsymbol{C}}\mathcal{S}=H$, and we can take a basis $\{\varphi_1, \varphi_2, \cdots, \varphi_H\}$ of $\mathcal{S}$ consisting of common eigenfunctions of all Hecke operators $\mathrm{T}'(p)$ (p: prime, $\neq q$). Let $\mathfrak{D}_A^\times=\bigcup_{j=1}^{H}\mathfrak{D}^\times y_j\mathcal{K}$ be the double coset decomposition with $y_j\in\mathfrak{D}_A^\times$ such that $\mathrm{N}(y_j)=1$, and put

$$F_{ij}=\sum_{k=1}^{H}(\varphi_i(y_k))\vartheta_{jk} \qquad (j=1, 2, \cdots, H).$$

Then we have

Proposition 8.2 (Yoshida).
(1) $F_{ij}\in M_2(\Gamma_0(q))$.
(2) $F_{ij}|\mathrm{T}(p)=\lambda_i(p)F_{ij}$ *where* $\lambda_i(p)$ *is the eigenvalue of* φ_i *with respect to* $\mathrm{T}'(p)$.
(3) $a(F_{ij}, 1)=\varphi_i(y_j)$, *hence* $F_{ij}\neq 0$ *if and only if* $\varphi_i(y_j)\neq 0$.
(4) $\dim_C W_j=\dim_C\langle F_{ij}|1\leqq i\leqq H\rangle_C=\#\{i|\varphi_i(y_j)\neq 0\}$.

Note that our ϑ_{ij} is $\dfrac{1}{e_j}\vartheta_{ij}$ in [Yo1].

Remark 8.3. Taking φ_1 to be a constant function, we get another proof of Proposition 3.1.(2) with (1.6).

Remark 8.4. If we replace φ_i with its suitable constant multiple, we get

$$\varphi_i(y_j)=v(f_i)_j \qquad \text{for all } 1\leqq i,\ j\leqq H$$

with a suitable numbering $\mathfrak{N}=\{f_1, f_2, \cdots, f_H\}$.

In fact, by Proposition 8.2(2-3), each F_{ij} $(1\leqq j\leqq H)$ is a constant multiple of one element in $\mathfrak{N}$, say f_i, and we can put $F_{ij}=c_{ij}f_i$ with some $c_{ij}\in\boldsymbol{C}$. Denote by Φ (resp. C) the $H\times H$-matrix with the (i, j)-th entry $\varphi_i(y_j)$ (resp. c_{ij}) and by D the $H\times H$-diagonal matrix with the i-th diagonal entry f_i. Then we have $\Phi^t\Theta=DC$, hence $\Phi^tQ^{-1}D=DC^tQ^{-1}$ by (3.2). The linear independence of f_j's over $\boldsymbol{C}$ implies that Φ^tQ^{-1} is a diagonal matrix, therefore the assertion follows.

Thus $v(f)$ plays an important role in constructing a newform f from theta series:

$$f=\left(\frac{e_j}{v(f)_j}\right)\sum_{i=1}^{H}\left(\frac{v(f)_i}{e_i}\right)\vartheta_{ij}, \tag{8.1}$$

where j is any index such that $v(f)_j\neq 0$.

§ 9. Remarks on the computation

In the computation of Brandt matrices, the author basically applied the algorithm given by A. Pizer [Pi3], except for

Theorem 9.1 (Hijikata). *For a given maximal order* $\mathcal{O}$ *in* $\mathfrak{D}$, *one can get a complete set of representatives of left* $\mathcal{O}$*-ideal classes in the following manner.*

Fix arbitrary prime number $l\neq q$. *Define a sequence* $X_0, X_1, \cdots$ *of sets of left* $\mathcal{O}$*-ideals inductively so that*
(1) $X_0=\{\mathcal{O}\}$,
(2) X_n *consists of left* $\mathcal{O}$*-ideals* $\mathfrak{b}$ *such that*

$$\begin{cases} \mathrm{N}(\mathfrak{b})=l^n, \\ \mathfrak{b} \textit{ is a sublattice of some } \mathfrak{a}\in X_{n-1}, \end{cases} \tag{9.1}$$

(3) *any left $\mathcal{O}$-ideal* $\mathfrak{b}$ *satisfying* (9.1) *is equivalent to some element in* $\bigcup_{i=0}^{n} X_i$,

(4) *any two elements in* $\bigcup_{i=0}^{n} X_i$ *are inequivalent.*

Repeat this procedure until $X_{n+1}=\emptyset$, *then* $\bigcup_{i=1}^{n} X_i$ *gives a completet set of representatives of left* $\mathcal{O}$-*ideal classes.*

Remark 9.2. If $\mathfrak{a}$ and $\mathfrak{b}$ are as in Theorem 9.1.(2), then the exponent of the additive group $\mathfrak{a}/\mathfrak{b}$ is l (but $[\mathfrak{a}:\mathfrak{b}]\neq l$ in general). All sublattices $\mathfrak{b}$ of $\mathfrak{a}$ with the quotient group of exponent l are obtained in the same manner as in [Pi3, p. 369]. Such $\mathfrak{b}$ is a left $\mathcal{O}$-ideal if and only if $\mathcal{O}\mathfrak{b}=\mathfrak{b}$, and then $\mathrm{N}(\mathfrak{b})$ is calculated by use of (1.1-2). Thus all $\mathfrak{b}$ satisfying (9.1) can be determined.

9.3. In order to diagonalize Θ, it is enough to diagonalize one $B(n)$, or certain ($\boldsymbol{R}$-) linear combination of $B(n)$'s, whose eigenvalues are all distinct. Putting $U=\begin{pmatrix}\sqrt{e_1} & & \\ & \ddots & \\ & & \sqrt{e_H}\end{pmatrix}$, we see that $U^{-1}B(n)U$ is a real symmetric matrix by Proposition 3.1.(1), hence its diagonalization can be computed by "Jacobi method".

It is convenient to normalize the eigenvectors of Θ so that, for each of them, the non-zero entry with minimal index is 1. Then they give the approximate values of $\{v(f)\}_{f\in\mathfrak{N}}$.

In order to apply Lemma 3.6, it is necessary to prove that each entry of eigenvectors with approximate value 0 is exactly 0. This is achieved by Remark 3.8 and

Remark 9.4. For $f\in\mathfrak{N}^0$, take an integer n such that $K_f=\boldsymbol{Q}(a(f,n))$ i.e. $a(g,n)$'s are distinct for all conjugates g of f. Let $F(X)$ be the minimal polynomial of $a(f,n)$ over $\boldsymbol{Q}$, then we have

$$\langle v(f)^{\sigma}\mid\sigma\in\mathrm{Aut}(\boldsymbol{C})\rangle_{C}=\{x\in\boldsymbol{C}^{H}\mid F(B(n))x=0\}.$$

9.5. If $\mathcal{O}_j\cong\mathcal{O}_k$, then we have

$$v(f)_j=\mp v(f)_k \tag{9.2}$$

according as $f\in S_2^{\pm}(\Gamma_0(q))$ (see [Pi1, Theorem 3.2]). The types of $\mathcal{O}_j$'s are determined by this.

9.6. The representation matrices of the Hecke operators acting on $S_2^{\pm}(\Gamma_0(q))$ are computed by use of Proposition 3.1.(1, 4) and the above remark. The characteristic polynomials of them are computed by "Frame method", and factorizing them over $\boldsymbol{Q}$ we know the splitting of $S_2^{\pm}(\Gamma_0(q))$.

9.7. We can also calculate the values of $s_{f,ij}=\dfrac{\langle\vartheta_{ij},f\rangle}{\langle f,f\rangle}$ (at least approximately). In fact, putting $c=\left\{\sum_{i=1}^{H}\dfrac{1}{e_j}(v(f)_i)^2\right\}^{-1}$, Prop. 3.1 implies that $s_{f,ij}=\dfrac{c}{e_j}v(f)_i v(f)_j$.

Note that $\vartheta_{ij}-\vartheta_{kj}=\sum_{f\in\mathfrak{N}^0}s_f f$ with $s_f=s_{f,ij}-s_{f,kj}$. By the same argument as in Lemma 7.1, some prime ideals $\mathfrak{l}$ in K_f dividing the denominator of s_f can be *congru-*

ence primes in the sense of [DO]. For instance, in Table I, $l=3$ for $q=487$ is proved with this argument.

Added in proof: By K. Hashimoto [Ha], the linear dependence of ϑ_{jj}'s ($1\leqq j\leqq H$) and of theta series attached to two other kinds of lattice in D is studied, and certain relations among them are observed.

DEPARTMENT OF INFORMATION SCIENCE
FACULTY OF SCIENCE
KOCHI UNIVERSITY

References

[DM] K. Doi, T. Miyake, Modular forms and number theory (in Japanese), Kinokuniya, 1976.

[DO] K. Doi, M. Ohta, On some congruences between cusp forms on $\Gamma_0(N)$, Lecture Notes in Math. 601, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[DS] P. Deligne, J.-P. Serre, Formes modulaires de poids 1, Ann. scient. Éc. Norm. Sup. 4^e série, t. **7** (1974), 507-530.

[Ei1] M. Eichler, Über die Darstellbarkeit von Thetafunktionen durch Thetareihen, J. reine angew. Math., **195** (1956), 156-171.

[Ei2] M. Eichler, The basis problem for modular forms and the traces of the Hecke operators, Lecture Notes in Math. 320, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[Ha] K. Hashimoto, Finding linear relations of theta series attached to the maximal orders of quaternion algebras, preprint.

[He] E. Hecke, Analytische Arithmetik der positiven quadratischen Formen, Mathematische Werke, 789-918.

[HPS] H. Hijikata, A. Pizer, T. Shemanske, The basis problem for modular forms on $\Gamma_0(N)$, Memoirs of the Amer. Math. Soc., Vol. 82, No. 418, 1989.

[Ki] Y. Kitaoka, Lectures on Siegel modular forms and representation by quadratic forms, Bombay, 1986.

[Ma] B. Mazur, Modular curves and the Eisenstein ideal, IHES Publ. Math., **47** (1977), 33-186.

[Me] J.-F. Mestre, Courbes de Weil fortes de conducteur premier $N<14000$ (Table), unpublished.

[Mi] I. Miyawaki, Elliptic curves of prime power conductor with $\boldsymbol{Q}$-rational points of finite order, Osaka J. Math., **10** (1973), 309-323.

[OM] O. T. O'Meara, Introduction to Quadratic Formes, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[Pi1] A. Pizer, The Action of the Canonical Involution on Modular Forms of Weight 2 on $\Gamma_0(M)$, Math. Ann., **226** (1977), 99-116.

[Pi2] A. Pizer, A note on a conjecture of Hecke, Pacific J. Math., **79** (1978), 541-547.

[Pi3] A. Pizer, An Algorithm for Computing Modular Forms on $\Gamma_0(N)$, J. of Algebra, **64** (1980), 340-390.

[Po] P. Ponomarev, A Correspondence between Quaternary Quadratic Forms, Nagoya Math J. **62** (1976), 125-140.

[Se] B. Setzer, Elliptic curves of prime conductor, J. London Math. Soc. (2), **10** (1975), 367-378.

[Si] J. H. Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[SP] R. Schulze-Pillot, A linear relation between theta series of degree and weight 2, Lecture Notes in Math. 1380, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[Yo1] H. Yoshida, Siegel's Modular Forms and the Arithmetic of Quadratic Forms, Inv. Math., **60** (1980), 193-248.

[Yo2] H. Yoshida, On Siegel modular forms obtained from theta series, J. reine angew. Math., **352** (1984), 184-219.